

# A THEOREM OF GRUSON

BRIAN JOHNSON

## 1. GRUSON'S THEOREM

This is based on a proof of the result given in [4].

Let  $A$  be a commutative ring,  $M$  a finitely generated  $A$ -module, and

$$F \xrightarrow{\phi} A^n \longrightarrow M \longrightarrow 0$$

a presentation of  $M$ , where  $F$  is a free module of arbitrary rank. Let  $U$  denote the matrix (with  $n$  rows but possibly infinitely many columns) representing  $\phi$  with respect to some fixed bases for  $F$  and  $A^n$ . For  $t \geq 1$  we let  $I_t(\phi)$  denote the ideal of  $A$  generated by all  $t$ -sized minors  $U$ . (This ideal is independent of the choice of bases.) By convention,  $I_t(\phi) = R$  for  $t \leq 0$  and  $I_t(\phi) = 0$  if  $t$  exceeds  $n$  or the rank of  $F$ . For an integer  $i$ , we let  $F_i(M) := I_{n-i}(\phi)$ . It is a result of Fitting that  $F_i(M)$  does not depend on the presentation of  $M$  (e.g., [3, Theorem 1, pg. 58]). If the ring  $A$  is not clear from the context, we write  $F_i^A(M)$  for  $F_i(M)$ .

A projective  $A$ -module is said to have *constant rank*  $r$  if  $P_q \cong A_q^r$  for every prime ideal  $q$ .

**Lemma 1.1.** *Suppose  $E$  is a finitely generated  $A$ -module. Then  $E$  is projective of constant rank  $r$  if and only if  $F_{r-1}(E) = 0$  and  $F_r(E) = A$ .*

*Proof.* See [1, Proposition 1.4.9] for the case  $E$  is finitely presented. We now prove the general case. If  $E$  is finitely generated and projective, then it is finitely presented. Conversely, let

$$F \xrightarrow{\phi} A^n \longrightarrow E \longrightarrow 0$$

be a presentation for  $E$ , and let  $U$  be a matrix representing  $\phi$  with respect to some choice of bases. Since  $F_r(E) = A$ , there exists a submatrix  $B$  of  $U$  with  $n$  rows and a finite number of columns such that  $I_{n-r}(B) = I_{n-r}(U) = A$ . We claim that  $\text{im } B = \text{im } U$ , which would imply  $E$  is finitely presented. Let  $B'$  be the matrix obtained by adjoining to  $B$  an arbitrary column of  $U$ . It suffices to show  $\text{im } B = \text{im } B'$ . Since  $\text{im } B \subseteq \text{im } B'$ , it suffices to prove this locally at any maximal ideal  $m$ . So assume  $A$  is local. Certainly, we have  $I_{n-r}(B) = I_{n-r}(B') = A$  and  $I_{n-r+1}(B) = I_{n-r+1}(B') = 0$  (since

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$I_{n-r+1}(U) = 0$ ). Let  $C$  and  $C'$  be the cokernels of  $B$  and  $B'$ , respectively. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{im } B & \longrightarrow & A^n & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow i & & \downarrow = & & \downarrow \alpha & & \\ 0 & \longrightarrow & \text{im } B' & \longrightarrow & A^n & \longrightarrow & C' & \longrightarrow & 0, \end{array}$$

where  $i$  is the natural inclusion map. By the finitely presented case,  $C$  and  $C'$  are both free of rank  $r$ . By the Snake Lemma,  $\alpha$  is surjective and hence an isomorphism. Again by the Snake Lemma,  $\text{coker } i \cong \ker \alpha = 0$  and hence  $i$  is surjective. Thus,  $\text{im } B = \text{im } B'$ .  $\square$

**Lemma 1.2.** *Suppose  $A$  is a commutative ring and  $P$  is a finitely generated projective  $A$ -module. If  $P$  has constant (positive) rank then there exists  $d$  such that  $P^d \cong A \oplus Q$  for some  $A$ -module  $Q$ .*

*Proof.* Since  $P$  is finitely generated and projective,  $P$  is finitely presented. If  $A$  is Noetherian and finite dimensional, we are done by a theorem of Serre [2, Corollary 2.5] which we prove in Section 2. If not, consider a presentation of  $P$ :

$$A^m \xrightarrow{\phi} A^n \longrightarrow P \longrightarrow 0.$$

Let  $U = (u_{ij})$  be a matrix representing  $\phi$ . Let  $T$  be the prime subring of  $A$ , and let  $D_1, \dots, D_s$  be the  $r \times r$  minors of  $U$  (with  $r$  as above). Then there exists  $b_1, \dots, b_s \in A$  such that  $b_1 D_1 + \dots + b_s D_s = 1$ . Set

$$S = T[\{u_{ij}\}, b_1, \dots, b_s] \subseteq A.$$

Now, since  $T = \mathbb{Z}$  or  $\mathbb{Z}/n\mathbb{Z}$ ,  $T$  is Noetherian and finite dimensional. Thus,  $S$  is also Noetherian and finite dimensional. Let  $M = \text{coker}(S^m \xrightarrow{U} S^n)$ . Then  $F_r^S(M) = S$  and  $F_{r-1}^S(M) = 0$  (where the superscript  $S$  simply denotes the fact we are considering the Fitting invariants as ideals of  $S$ ). By 1.1,  $M$  is a projective  $S$ -module. Then by the Noetherian case,

$$M^d \cong S \oplus N,$$

for some  $d$ . Applying  $A \otimes_S -$  to

$$S^m \longrightarrow S^n \longrightarrow M \longrightarrow 0,$$

we see  $A \otimes_S M \cong P$ . But

$$P^d \cong A \otimes_S M^d \cong (A \otimes_S S) \oplus (A \otimes_S N) \cong A \oplus (A \otimes_S N),$$

and we are done.  $\square$

**Theorem 1.3** (Gruson). *Let  $E$  be a finitely generated faithful module over the commutative ring  $A$ . Then every  $A$ -module admits a finite filtration of submodules whose factors are quotients of direct sums of copies of  $E$ .*

*Proof.* We first claim it is enough to show this for the ring  $A$  itself. Indeed, suppose  $0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_k = A$  is a sequence of ideals of  $A$  such that for each  $i = 1, \dots, k$  there is a surjection

$$\phi_i : E^{(L_i)} \longrightarrow I_i/I_{i-1}.$$

Then suppose  $M$  is any  $A$ -module with presentation  $A^{(L)} \xrightarrow{\phi} M \longrightarrow 0$ . consider the series of submodules of  $M$  given by  $\phi(I_i^{(L)})$ . First, notice that

$$0 = \phi(I_0^{(L)}) \subseteq \phi(I_1^{(L)}) \subseteq \cdots \subseteq \phi(I_k^{(L)}) = \phi(A^{(L)}) = M$$

is indeed a filtration of  $M$ . Then it will suffice to show there is a surjection  $E^{(K)} \twoheadrightarrow \phi(I_i^{(L)})/\phi(I_{i-1}^{(L)})$ . Define a map

$$\psi_i : I_i^{(L)}/I_{i-1}^{(L)} \longrightarrow \phi(I_i^{(L)})/\phi(I_{i-1}^{(L)})$$

by  $x + I_{i-1}^{(L)} \mapsto \phi(x) + \phi(I_{i-1}^{(L)})$ . One checks easily that this is a well-defined homomorphism. Note that every element of  $\phi(I_i^{(L)})/\phi(I_{i-1}^{(L)})$  is of the form  $\phi(x) + \phi(I_{i-1}^{(L)})$  for some  $x \in I_i^{(L)}$ , so  $\psi_i$  is surjective. Thus, we have the desired surjection

$$E^{(K)} = (E^{(L_i)})^{(L)} \twoheadrightarrow I_i^{(L)}/I_{i-1}^{(L)} \twoheadrightarrow \phi(I_i^{(L)})/\phi(I_{i-1}^{(L)}).$$

We now proceed in the case of the ring  $A$  by using the Fitting invariants  $F_i := F_i(E)$  of the module  $E$ . Let

$$F \xrightarrow{\phi} A^n \longrightarrow E \longrightarrow 0$$

be a free presentation of  $E$  and let  $U = (u_{ij})$  be a matrix representing  $\phi$ . Define  $r$  to be the smallest integer such that  $F_r \neq 0$ , and  $m$  to be the smallest integer such that  $F_m = A$ . Note first that as  $F_0 \subseteq \text{Ann}_A E$ , we must have  $F_0 = 0$  since  $E$  is faithful. This implies that  $r > 0$ .

We'll now use induction on  $d = m - r$ . If  $d = 0$ , Lemma 1.1 implies  $E$  is projective of constant rank, and we use Lemma 1.2 to get  $\ell$  such that  $E^\ell \cong A \oplus Q$ . Since every  $A$ -module is the quotient of a free module, we can clearly map a large enough direct sum of copies of  $E$  onto any  $A$ -module we please, and we're done in this case. Suppose  $d > 0$  (i.e.,  $m > r$ ). Let  $I = \text{Ann}_A(E/F_r E)$  and consider the  $A/I$ -module  $E/IE$ . Suppose  $\bar{x} \in \text{Ann}_{A/I}(E/IE)$ . Then  $xE \subseteq IE$ , and as  $I = \{a \in A \mid aE \subseteq F_r E\}$ , certainly  $IE \subseteq F_r E$ . But this implies that  $xE \subseteq F_r E$ , whence  $x \in I$ . Thus  $\bar{x} = \bar{0}$ , and  $E/IE$  is a faithful  $A/I$ -module. Also, notice that since  $F_r \subseteq I$ , we must have  $F_r^{A/I}(E/IE) = 0$ . Indeed, given the presentation for  $E$  above, we may mod out by  $I$  to obtain a presentation for  $E/IE$ , and we find the Fitting invariants are just  $F_i^{A/I}(E/IE) = \frac{F_i(E)+I}{I}$ . Thus,  $F_r^{A/I}(E/IE) = 0$  and  $F_m^{A/I}(E/IE) = A/I$ , and we may apply the induction hypothesis to the  $A/I$ -module  $E/IE$  to get a finite filtration of  $A/I$  modules whose factors

are quotients of direct sums of  $E/IE$ , say

$$I = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_k = A,$$

and the factors remain unchanged, as  $A_i/A_{i-1} \cong (A_i/I)/(A_{i-1}/I)$ . As these factors are quotients of direct sums of  $E/IE$ , we may compose with the natural map  $E \twoheadrightarrow E/IE$ , and we see that they are, in fact, quotients of direct sums of  $E$  as well. Now, all that remains is to find a finite filtration of  $I$  in which the factors have the desired property. We first claim that  $I^n \subseteq F_r$ . To see this, we use the fact that  $\text{Ann}_{A/F_r}(E/F_r E)^n \subseteq F_0^{A/F_r}(E/F_r E)$ . But  $F_0^{A/F_r}(E/F_r E) = \bar{0}$  as  $F_0(E) = 0$ . Thus,  $I^n = \text{Ann}_A(E/F_r E)^n \subseteq F_r$ . Then we have the following filtration:

$$F_r \subseteq F_r + I^{n-1} \subseteq F_r + I^{n-2} \subseteq \cdots \subseteq F_r + I = I.$$

Note that  $I(F_r + I^{n-i}) \subseteq F_r + I^{n-i+1}$ , so  $\frac{F_r + I^{n-i}}{F_r + I^{n-(i-1)}}$  is an  $A/I$ -module for each  $i$ . As above, these can be seen to be quotients of direct sums of  $E$ . So now it only remains to show that we may write  $F_r$  as a quotient of a direct sum of copies of  $E$ . Let  $s := n - r$  and let  $D$  be the determinant of the  $s \times s$  submatrix of  $U$  given by  $(u_{i_k, j_l})$ , where  $1 \leq k, l \leq s$  and  $i_k, j_l \in \{1, \dots, n\}$ . We will find a linear form on  $E$  such that its image contains  $D$ . Choose  $t \in \{1, \dots, n\}$  such that  $t \neq i_k$  for any  $k$  (this is possible as  $r > 0$ ). Consider the following linear form on  $A^n$ :

$$\underline{x} := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \det \begin{pmatrix} u_{i_1, j_1} & \cdots & u_{i_1, j_s} & x_1 \\ \vdots & \vdots & \vdots & \vdots \\ u_{i_s, j_1} & \cdots & u_{i_s, j_s} & x_s \\ u_{t, j_1} & \cdots & u_{t, j_s} & x_t \end{pmatrix}.$$

Now, as  $s + 1 = n - r + 1 = n - (r - 1)$ , and  $F_{r-1} = 0$ , we have that the  $(s + 1) \times (s + 1)$  minors of  $U$  must be zero. Thus, this form is trivial on the image of  $U$ . Indeed, if  $\underline{x} \in \text{im } U$ , then  $\underline{x}$  is an  $A$ -linear combination of the columns of  $U$ , and so the determinant will be an  $A$ -linear combination of  $(s + 1) \times (s + 1)$  minors of  $U$ . By passing to the quotient mod the image of  $U$ , we obtain a linear form on  $E = \text{coker } U$ . That is, let  $e_i$  be the standard  $i$ th basis vector. We have a map  $\phi_D : E \rightarrow R$ , and note that for all  $i$ ,  $\phi(\bar{e}_i)$  is an  $(n - r) \times (n - r)$  minor of  $U$ , so  $\phi_D : E \rightarrow F_r(E)$ . Finally, note

$$e_t \mapsto \det \begin{pmatrix} u_{i_1, j_1} & \cdots & u_{i_1, j_s} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ u_{i_s, j_1} & \cdots & u_{i_s, j_s} & 0 \\ u_{t, j_1} & \cdots & u_{t, j_s} & 1 \end{pmatrix} = D,$$

so we can map a copy of  $E$  onto each of the generators of  $F_r$ .  $\square$

A more general version of Gruson's Theorem is the following:

**Corollary 1.4.** *Let  $A$  be a commutative ring,  $M$  a finitely generated  $A$ -module, and  $I = \text{Ann}_A M$ . Then for every  $A$ -module  $L$  such that  $I^n L = 0$  for some  $n$ , there exists a finite filtration of submodules of  $L$  such that the factor modules are quotients of direct sums of copies of  $M$ .*

*Proof.* Consider the filtration

$$0 = I^n L \subseteq I^{n-1} L \subseteq \cdots \subseteq I^0 L = L.$$

Each of the factors in this filtration is an  $A/I$ -module. Since  $M$  is a finitely generated faithful  $A/I$ -module, each of the factors has a finite filtration whose factor modules are quotients of direct sums of  $M$ . Patching these filtrations together gives the desired finite filtration of  $L$ .  $\square$

A special case of this result is the following:

**Corollary 1.5.** *Let  $A$  be a Noetherian ring and  $L$  and  $M$  finitely generated  $A$ -modules. Suppose  $\text{Supp}_A L \subseteq \text{Supp}_A M$ . Then there exists a finite filtration of  $L$  whose factors are quotients of direct sums of copies of  $M$ .*

The following is a frequently used consequence of Gruson's Theorem:

**Corollary 1.6.** *Let  $A$  be a commutative ring,  $E$  a finitely generated faithful  $A$ -module, and  $M$  an arbitrary  $A$ -module. Then  $E \otimes_A M = 0$  if and only if  $M = 0$ . That is,  $E \otimes_A -$  is a faithful functor.*

*Proof.* Suppose  $E \otimes_A M = 0$ . It suffices to show that  $N \otimes_A M = 0$  for any  $A$ -module  $N$ . Let  $\lambda(N)$  denote the length of the shortest filtration of  $N$  such that the factor modules of the filtration are homomorphic images of direct sums of copies of  $E$ . If  $\lambda(N) = 1$  then  $N$  is the homomorphic image of  $E^{(L)}$  for some  $L$ . Since  $E \otimes_A M = 0$ , certainly  $E^{(L)} \otimes_A M = 0$  and hence  $N \otimes_A M = 0$ . Suppose  $\lambda(N) > 1$ . Then there exists a short exact sequence  $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$  such that  $\lambda(N')$  and  $\lambda(N'')$  are less than  $\lambda(N)$ . By induction,  $N' \otimes_A M = 0 = N'' \otimes_A M$ . By the right exactness of tensor products, we see that  $N \otimes_A M = 0$ .  $\square$

## 2. A THEOREM OF SERRE

This will be a proof of a theorem of Serre based on [2].

Throughout this section  $R$  denotes a commutative Noetherian ring.

**Definition 2.1.** Suppose  $M$  is a finitely generated  $R$ -module,  $M' \subseteq M$  a submodule, and  $p \in \text{Spec } R$ . We say  $M'$  is  $w$ -fold basic in  $M$  at  $p$  if

$$\mu((M/M')_p) \leq \mu(M_p) - w,$$

where  $\mu(M)$  denotes the minimum number of generators of  $M$ . Equivalently, we have

$$\dim_{k(p)} \frac{M'_p + pM_p}{pM_p} \geq w,$$

where  $k(p) := R_p/pR_p$ .

**Lemma 2.2.** *Let  $M$  be a finitely generated  $R$ -module, let  $p_1, \dots, p_n \in \text{Spec } R$ , and suppose the submodule  $M' = \sum_{i=1}^w Rm_i$  is  $w_i$ -fold basic in  $M$  at  $p_i$  for each  $i = 1, \dots, n$ . Then there exist  $r_1, \dots, r_{w-1} \in R$  such that  $N = \sum_{i=1}^{w-1} R(m_i + r_i m_w)$  is  $\min\{w-1, w_i\}$ -fold basic in  $M$  at  $p_i$  for each  $i = 1, \dots, n$ .*

*Proof.* We use induction on  $n$ , the number of primes. The case  $n = 0$  is vacuous. Suppose  $n > 0$ , and reorder the primes so that  $p_n$  is minimal among the  $p_i$ . That is, we have

$$p_1 \cap \dots \cap p_{n-1} \not\subseteq p_n.$$

By induction, we may choose  $r_1, \dots, r_{w-1}$  such that the lemma holds for  $p_1, \dots, p_{n-1}$ . Now, if  $N$  is  $\min\{w-1, w_n\}$ -fold basic in  $M$  at  $p_n$ , we are done. Suppose not. We claim this implies that  $w-1 \geq w_n$ . To see this, note that since  $M'$  is  $w$ -generated, we must have  $w_n \leq w$ . Assuming  $w_n = w$ , we get

$$\min\{w-1, w_n\} = \min\{w-1, w\} = w-1.$$

If  $M'$  is  $w$ -fold basic at  $p_n$ , it follows that  $N$  must be  $(w-1)$ -fold basic at  $p_n$  (by an elementary linear independence argument). Thus, if  $N$  is *not*  $\min\{w-1, w_n\}$ -fold basic at  $p_n$ , we must have that  $w_n < w$ , implying  $w_n \leq w-1$ , and hence  $\min\{w-1, w_n\} = w_n$ . Choose  $r \in (p_1 \cap \dots \cap p_{n-1}) \setminus p_n$ . We claim that

$$m_1 + r_1 m_w, \dots, m_{j-1} + r_{j-1} m_w, m_j + (r_j + r) m_w, \dots, m_{w-1} + r_{w-1} m_w$$

generate a  $w_n$ -fold basic submodule in  $M$  at  $p_n$  for some  $j$ . First, since  $N$  is not  $\min\{w-1, w_n\}$ -fold basic (recalling this implies  $w_n \leq w-1$ ), we have

$$\dim_{k(p_n)} \frac{N_{p_n} + p_n M_{p_n}}{p_n M_{p_n}} \leq w-1.$$

In particular, any set of  $w-1$  elements of  $N$  must be linearly dependent in  $(M/p_n M)_{p_n}$ . Thus, for some  $j \in \{2, \dots, w-1\}$ ,  $m_j + r_j m_w$  is in the space spanned by the previous  $j-1$  elements of the same form, say

$$(1) \quad m_j + r_j m_w = \sum_{i=1}^{j-1} s_i (m_i + r_i m_w)$$

in  $(M/p_n M)_{p_n}$ . We'll show this  $j$  satisfies our claim. Note that  $\{m_i + r_i m_w\}_{i=1}^{w-1} \cup \{m_w\}$  generate  $M'$ , which is  $w_n$ -fold basic in  $M$  at  $p_n$ . Consider the submodule  $L$  generated by

$$m_1 + r_1 m_w, \dots, m_{j-1} + r_{j-1} m_w, m_j + (r_j + r) m_w, \dots, m_{w-1} + r_{w-1} m_w.$$

Note that as  $r \in p_1 \cap \dots \cap p_{n-1}$ , we have that  $(L + p_i M)/p_i M = (N + p_i M)/p_i M$  for  $i \in \{1, \dots, n-1\}$ . Hence,  $L$  is  $\min\{w-1, w_i\}$ -fold basic in

$M$  at  $p_i$  for  $i = 1, \dots, n-1$ . We claim that  $m_w \in L_{p_n}$ , and therefore that  $(M')_{p_n} = L_{p_n}$ . By (1) we can write

$$rm_w = m_j + (r_j + r)m_w - \sum_{i=1}^{j-1} s_i(m_i + r_i m_w).$$

Further,  $r$  is a unit in  $k(p)$ , because  $r \notin p_n$ , so we can write  $m_w$  as a linear combination of the generators of  $L_{p_n}$ . Therefore,  $L_{p_n} = (M')_{p_n}$ , and the basicity of  $M'$  at  $p_n$  implies that  $L$  is also  $w_n$ -fold basic in  $M$  at  $p_n$ .  $\square$

**Definition 2.3.** Suppose  $M$  is a finitely generated  $R$ -module and  $t$  is a nonnegative integer. Define

$$J_t(M) = \left\{ \sum_{M'} \text{Ann}(M/M') \mid M' \subseteq M \text{ can be generated by } t \text{ elements} \right\}.$$

**Remark 2.4.** We note some properties.

(1) We have a chain

$$\text{Ann}(M) = J_0(M) \subseteq J_1(M) \subseteq \dots \subseteq J_n(M) \subseteq \dots$$

(2) If  $\mu(M) \leq n$ , Then  $J_n(M) = J_{n+1}(M) = \dots = R$ .

(3) If  $p$  is a prime, then  $\mu(M_p) > t$  if and only if  $p \supseteq J_t(M)$ .

**Lemma 2.5.** *Let the submodule  $M'$  be  $w$ -fold basic in  $M$  at each prime of height less than or equal to  $k$ . Then  $M'$  is  $w$ -fold basic in  $M$  at all but finitely many primes of height  $k+1$ .*

*Proof.* Since  $M$  is finitely generated,  $J_t(M/M') \neq R$  for only finitely many  $t$ . Each of these has only finitely many primes minimal over it. We claim that if  $\text{ht } p = k+1$  and  $p$  is not minimal over any  $J_t(M/M')$ , then  $M'$  is  $w$ -fold basic in  $M$  at  $p$ . Suppose  $p$  is such a prime. Consider two cases.

Case 1:  $p \not\supseteq J_0(M/M')$ . Since  $J_0(M/M') = \text{Ann}(M/M')$ , this implies that  $(M/M')_p = 0$ , so  $\mu((M/M')_p) = 0$ . Thus, to show that  $M'$  is  $w$ -fold basic at  $p$ , we only need to show that  $0 \leq \mu(M_p) - w$ . But, if  $q \subsetneq p$  is a height  $k$  prime, we know

$$\begin{aligned} 0 &\leq \mu((M/M')_q) \\ &\leq \mu(M_q) - w \\ &\leq \mu(M_p) - w, \end{aligned}$$

and  $M'$  is  $w$ -fold basic at  $p$ .

Case 2:  $p \supseteq J_0(M/M')$ . In this case, there exists some  $t$  such that  $J_t(M/M') \subseteq p$ , but  $J_{t+1}(M/M') \not\subseteq p$ . Since  $p$  is *not* minimal over any  $J_i(M/M')$ , there is some prime  $q$  with height less than  $k+1$  such that  $p \supseteq q \supseteq J_t(M/M')$ . Then we have

$$\mu((M/M')_p) = \mu((M/M')_q) = t+1$$

by part (3) of Remark 2.4. We also have  $\mu(M_p) \geq \mu(M_q)$  as noted earlier, and finally

$$\mu(M_q) - \mu((M/M')_q) \geq w$$

by assumption. Combining the previous observations, we have

$$\mu(M_p) - \mu((M/M')_p) \geq \mu(M_q) - \mu((M/M')_q) \geq w,$$

and  $M'$  is  $w$ -fold basic in  $M$  at  $p$ .  $\square$

**Definition 2.6.** Let  $M$  be an  $R$ -module and  $N = Rn_1 + \cdots + Rn_s$  a submodule of  $M$ . Then  $\{n_1, \dots, n_s\}$  (the set of generators) is called *basic up to height  $k$*  if  $N$  (the submodule they generate) is  $\min\{s, k - \text{ht } p + 1\}$ -fold basic in  $M$  at each prime  $p$  such that  $\text{ht } p \leq k$ . If  $k = \dim R$ , we say  $N$  is *basic in  $M$* . Further, if  $N$  is basic in  $M$  and  $N = Rx$ , we call  $x$  a *basic element* of  $M$ .

**Lemma 2.7.** *Let  $N = Rm_1 + \cdots + Rm_n \subseteq M$  be basic up to height  $k$  with  $n > 1$ . Then there exist elements  $r_1, \dots, r_{n-1} \in R$  such that  $R(m_1 + r_1m_n) + \cdots + R(m_{n-1} + r_{n-1}m_n)$  is basic up to height  $k$ .*

*Proof.* By assumption, if  $p$  is of height  $j - 1$ , where  $1 \leq j \leq k$ , then  $Rm_1 + \cdots + Rm_n$  is  $\min\{n, k - j + 2\}$ -fold basic in  $M$  at  $p$ . Thus, by Lemma 2.5,  $Rm_1 + \cdots + Rm_n$  is  $\min\{n, k - j + 2\}$ -fold basic in  $M$  at  $q$  for all but finitely many primes  $q$  of height  $j$ . Note that we may replace  $\{m_1, \dots, m_n\}$  by  $\{m_1 + r_1m_n, \dots, m_{n-1} + r_{n-1}m_n\}$  (for any  $r_1, \dots, r_n$ ) and sacrifice at worst one ‘‘basicity,’’ since adjoining  $m_r$  to the latter set of generators gives a generating set for  $N$ . Thus,  $R(m_1 + r_1m_n) + \cdots + R(m_{n-1} + r_{n-1}m_n)$  is at least  $\min\{n - 1, k - j + 1\}$ -fold basic at  $q$ , which is what we require. Now, we are only concerned with finitely many  $j$ , and for each of these  $j$ , there are only finitely many primes of height  $j$  where  $Rm_1 + \cdots + Rm_n$  is  $\min\{n, k - j + 1\}$ -fold basic, rather than the  $\min\{n, k - j + 2\}$ -fold basic obtained from the primes of height  $j - 1$ . So, for all of these finitely many primes, we may use Lemma 2.2 to find  $r_1, \dots, r_{n-1} \in R$  such that, at these primes,  $R(m_1 + r_1m_n) + \cdots + R(m_{n-1} + r_{n-1}m_n)$  is  $\min\{n - 1, \min\{n, k - j + 1\}\}$ -fold basic in  $M$  (for the primes of height  $j$  in the finite set). But

$$\min\{n - 1, \min\{n, k - j + 1\}\} = \min\{n - 1, k - j + 1\},$$

so we are done.  $\square$

**Theorem 2.8.** *If  $M$ , or more precisely, a set of generators for  $M$ , is basic up to height  $k$ , then there exists  $m \in M$  such that  $Rm$  is basic up to height  $k$ .*

*Proof.* Suppose  $M = Rm_1 + \cdots + Rm_n$ . If  $n > 1$ , we need to show that we can replace  $\{m_1, \dots, m_n\}$  by a set with one fewer element but still basic up to height  $k$ . This is possible by the previous lemma.  $\square$

**Definition 2.9.** If  $R$  is an integral domain with quotient field  $K$ , then we define the rank of  $M$  to be

$$\text{rank}_R M := \dim_K K \otimes_R M.$$

If  $R$  is not a domain, then we define the rank of  $M$  to be

$$\text{rank}_R M := \min\{\text{rank}_{R/p} M/pM \mid p \in \text{Min } R\}.$$

**Remark 2.10.** If  $P$  is projective of constant rank  $r$  then  $\text{rank}_R P = r$ .

**Theorem 2.11** (Serre, 1958). *Let  $R$  be a Noetherian ring of dimension  $d$ , and let  $P$  be a finitely generated projective  $R$ -module of rank greater than  $d$ . Then  $P$  has a free summand.*

*Proof.* Since  $\mu(P) \geq \text{rank}_{R/q}(P/qP)$  for any  $q \in \text{Spec } R$ , we see that  $\mu(P) \geq \text{rank}_R(P)$ . By assumption,  $\text{rank}_R(P) > d$ , so we have that  $\mu(P) \geq d+1$ . We claim that any generating set for  $P$  is basic. That is, if  $Rx_1 + \cdots + Rx_n = P$ , we claim that  $\{x_1, \dots, x_n\}$  is basic. To prove this, we need to show that  $\{x_1, \dots, x_n\}$  is basic up to height  $d$ , or that  $P$  is  $\min\{n, d-j+1\}$ -fold basic at each prime of height  $j$ . We've already shown that  $n \geq d+1$ , and since  $j \geq 0$ , it follows that  $\min\{n, d-j+1\} = d-j+1$ . Let  $q \in \text{Spec } R$  have height  $j$ . To show that  $P$  is  $(d-j+1)$ -fold basic in  $P$  at  $q$ , we need to show that

$$\mu((P/P)_q) \leq \mu(P_q) - (d-j+1).$$

Clearly  $\mu((P/P)_q) = 0$ , so it is sufficient to show that  $\mu(P_q) \geq d-j+1$ . Since any height  $j$  prime must contain a height 0 prime (possibly itself if  $j = 0$ ), say  $q'$ , we have the following:

$$\mu(P_q) \geq \mu(P_{q'}) \geq d+1 \geq d-j+1,$$

where the middle inequality follows from our assumption on rank  $P$ . Thus,  $\{x_1, \dots, x_n\}$  is a basic set, and by Theorem 2.8 there exists a single basic element  $m \in P$ . That  $\{m\}$  is basic implies  $Rm$  is  $\min\{1, d-j+1\}$ -fold basic in  $P$  at each prime of height  $j$ . As  $j \leq d$ , we see  $\min\{1, d-j+1\} = 1$ . I.e.,

$$\mu((P/Rm)_q) \leq \mu(P_q) - 1.$$

That is,  $m$  is part of a minimal generating set for  $P_q$ . Since  $P_q$  is a free  $R_q$ -module, this implies that  $m$  is part of a basis for  $P_q$ , and hence the inclusion  $R_q m \hookrightarrow P_q$  splits. As this holds for every prime  $q$ , we see that the inclusion  $Rm \hookrightarrow P$  splits and  $Rm \cong R$ .  $\square$

**Corollary 2.12.** *Let  $R$  be a Noetherian ring of dimension  $d$  and  $P$  a projective module of positive rank. Then  $P^n$  has a free summand for all  $n > d$ .*

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