Math 417
Problem Set 8
Due: Not to be turned in

1. Let G be a group and let $G' = \langle S \rangle$, where $S = \{x^{-1}y^{-1}xy \,|\, x, y \in G\}$. The subgroup G' is called the *derived subgroup* or the *commutator subgroup* of G.

   (a) Prove that G' is a normal subgroup of G. Hint: It is enough to show that $gsg^{-1} \in G'$ for every $s \in S$.

   (b) Prove that G/G' is abelian.

   (c) Let N be any normal subgroup of G. Prove that if G/N is abelian, then $G' \leq N$.

   (d) Prove that if H is a subgroup of G with $G' \leq H$, then H is a normal subgroup of G.

2. Let n be a positive integer and let k be a divisor of n. Prove that $\mathbb{Z}_n / \langle k \rangle \cong \mathbb{Z}_k$.

3. Express $\mathbb{Z}_{165}$ as an external direct product of cyclic groups in four different ways.

4. How many abelian groups (up to isomorphism) are there of order $75600 = 2^4 3^3 5^2 7$?

5. Find all finite groups that have exactly two conjugacy classes.

6. Suppose you know the following information about the abelian group G:

   - $|G| = 16$.
   - G has an element of order 8.
   - G has at least two elements of order 2.

7. In $\mathbb{Z}$, let $H = \langle 5 \rangle$ and let $K = \langle 7 \rangle$. Prove that $\mathbb{Z} = H + K$. Is it true that $\mathbb{Z} = H \times K$? Why or why not?

8. Prove that $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} / \langle (3,3,3) \rangle \cong \mathbb{Z}_3 \oplus \mathbb{Z} \oplus \mathbb{Z}$.