Math 417
Problem Set 2
Due: Thursday, February 4

Work all of the following problems. Remember, you are encouraged to work together on Problem Sets, but each student must turn in his or her own write-up. Be sure to adhere to the Rules and Expectations outlined in the Course Information Sheet.

1 Traditional Problems

1. Prove that if H is a group of order 3, then H is cyclic.

2. Let G be a group that has exactly 8 elements of order 3. How many subgroups of order 3 does G have? Explain carefully.

3. Let G be a group of even order. Prove that G has an element of order 2. (Hint: Divide the elements of G into those which are their own inverses and those which are not their own inverses. What is the parity of the latter set?)

4. Let G be a group and let n be a positive integer. Define the subset G^n of G by

$$G^n = \{g^n \mid g \in G\}.$$

   (a) Prove that if G is abelian, then G^n is a subgroup of G.

   (b) Give an example of a nonabelian group G and a positive integer n such that G^n is not a subgroup of G. Explain.

5. Let H and K be subgroups of the group G. Prove that $H \cap K$ is a subgroup of G.

2 Computer Problems

1. Use the GAP commands `ulist` and `cyclic` (see Chapter 3 of the lab manual) to determine whether the group $U(\mathbb{Z}_n)$ (or, using the book's notation, $U(n)$) is cyclic for various values of n of the form p^a where p is an odd prime and a is a positive integer. Do enough examples so that you feel comfortable making a conjecture, carefully state your conjecture, and then do a few more examples to make sure you believe it. (You do not need to prove your conjecture.) Then use GAP to decide whether your conjecture applies also to the group $U(\mathbb{Z}_{2^a})$ where a is a positive integer. Explain.