Math 417
Problem Set 1
Solutions

Work all of the following problems. Remember, you are encouraged to work together on Problem Sets, but each student must turn in his or her own write-up. Be sure to adhere to the Rules and Expectations outlined in the Course Information Sheet.

1 Traditional Problems

1. Let $S = \mathbb{R}^2$ be the set of ordered pairs of real numbers and, for $(a, b), (c, d) \in S$, define $(a, b) \sim (c, d)$ if $4a + 3b = 4c + 3d$. Show that $\sim$ is an equivalence relation on S. There is a nice geometric description of the equivalence classes. What is it?

   Solution: Clearly $(a, b) \sim (a, b)$ since $4a + 3b = 4a + 3b$. If $(a, b) \sim (c, d)$, then $4a + 3b = 4c + 3d$ and so $4c+3d = 4a+3b$, which means that $(c, d) \sim (a, b)$. Finally, if $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $4a + 3b = 4c + 3d$ and $4c + 3d = 4e + 3f$, and so $4a + 3b = 4e + 3f$, i.e., $(a, b) \sim (e, f)$. Thus $\sim$ is an equivalence relation on S. The equivalence class of the point (a, b) is the set of points on the line $4x + 3y = m$, where $m = 4a + 3b$. In other words, the equivalence classes are the lines of slope $-\frac{4}{3}$.

2. Let G be a group and let $g \in G$. Define a function $\phi_g : G \to G$ by $\phi_g(x) = gxg^{-1}$ for all $x \in G$. Show that ϕ_g is one-to-one and onto. (Recall that a function f is *one-to-one* if, whenever $f(a) = f(b)$ we must have $a = b$. Recall that a function $f : S \to T$ is *onto* if, for each $t \in T$ there is an element $s \in S$ such that $f(s) = t$.)

   Solution: Suppose $\phi_g(x) = \phi_g(y)$. Then $gxg^{-1} = gyg^{-1}$ and so $g^{-1}(gxg^{-1})g = g^{-1}(gyg^{-1})g$, i.e., $(g^{-1}g)x(g^{-1}g) = (g^{-1}g)x(g^{-1}g)$, i.e., $1_Gx1_G = 1_Gy1_G$, i.e., $x = y$. Thus ϕ_g is one-to-one. Now let $z \in G$ and set $x = g^{-1}zg$. Then $x \in G$ by closure and we have $\phi(x) = \phi(g^{-1}zg) = g(g^{-1}zg)g^{-1} = (gg^{-1})z(gg^{-1}) = 1_Gz1_G = z$. Hence ϕ is onto.

3. Let G be a group. For a, $b \in G$, define $a \sim b$ if there is some $x \in G$ with $a = xbx^{-1}$. Prove that $\sim$ is an equivalence relation on G.

   Solution: We need to show three things:

   (a) $a \sim a$ for all $a \in G$: This is true, since if we set $x = 1_G$, then $xax^{-1} = 1_Ga1_G^{-1} = a$.

   (b) If $a \sim b$ then $b \sim a$ for all $a, b \in G$: This is true since if $a \sim b$, then there is some $x \in G$ with $a = xbx^{-1}$. But then $x^{-1}ax = x^{-1}(xbx^{-1})x = (x^{-1}x)b(x^{-1}x) = 1_Gb1_G = b$. Since $(x^{-1})^{-1} = x$, we have $b = yay^{-1}$ where $y = x^{-1}$, and hence $b \sim a$.

   (c) If $a \sim b$ and $b \sim c$, then $a \sim c$ for all $a, b, c \in G$: To see this is true, suppose $a \sim b$ and $b \sim c$. Then there are elements x and y with $a = xbx^{-1}$ and $b = ycy^{-1}$. Substituting and recalling that $(xy)^{-1} = y^{-1}x^{-1}$ gives $a = x(ycy^{-1})x^{-1} = (xy)c(y^{-1}x^{-1}) = zcz^{-1}$, where $z = xy$. Hence $a \sim c$ as desired.

4. Let G be a group such that $g^2 = e$ for all $g \in G$. Prove that G is abelian.

   Solution: Let $a, b \in G$. Then $abab = (ab)^2 = e$ since $ab \in G$. Multiplying on the left by a gives us $bab = a^2bab = ae = a$. Multiplying on the right by b gives $ba = bab^2 = ab$. Thus, $ab = ba$ and the group is abelian.

2 Computer Problems

Start by reading through Chapter -1 and Chapter 0 (but stop at the bottom of page 10) of the lab manual. You do not need to turn anything in from these chapters, but you'll probably want to do all the problems just to get a feel for how to use GAP. Then do (to turn in) the following problem:

1. As mentioned in the lab manual, there is a nice combinatorial formula for the sum of the first n integers:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

There is a similarly nice formula for the sum of the first n cubes. The goal of this problem is to use GAP to conjecture the formula, which you will then prove using induction.

(a) Define in GAP a function that, given a positive integer n, outputs the sum of the first n cubes. For example, if your function is called f, then $f(1) = 1^3 = 1$, $f(2) = 1^3 + 2^3 = 9$ and $f(3) = 1^3 + 2^3 + 3^3 = 36$. Hint: You'll want to use the GAP command "Sum". You can find out how to use this command by typing `?Sum` at a GAP prompt.

Solution: We have:

```
gap> sumcubes := n->Sum([1..n], x->x^3);
function( n ) ... end
```

We'll check this by using the function to compute the sum of the first 3 cubes. From above, we should get 36.

```
gap> sumcubes(3);
36
```

Good.

(b) Compute, using your function, the sum of the first n cubes for at least 4 different values of $n \geq 4$. Then use this output to conjecture a nice formula for the sum of the first n cubes.

Solution: Let's just do $n = 4$, $n = 5$, $n = 6$, $n = 7$:

```
gap> sumcubes(4);
100
gap> sumcubes(5);
225
gap> sumcubes(6);
441
gap> sumcubes(7);
784
```

In each case, it looks like the sum of the first n cubes is the square of the sum of the first n integers. So our conjectured formula is

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2 = \left(\frac{n(n+1)}{2}\right)^2.$$

(c) Test your conjecture on a value of n that is larger than any of the examples you previously computed.

Solution: Let's test it for $n = 100$. We have:

```
gap> sumcubes(100);
25502500
gap> ((100*101)/2)^2;
25502500
```

The formula works!

(d) Use induction to prove that your conjecture holds for all $n \geq 1$.

Solution: We know the formula holds for $n \leq 7$. So let $n \geq 8$ and assume the formula works for $n-1$. Then we have

$$\begin{aligned}
\sum_{i=1}^{n} i^3 &= \sum_{i=1}^{n-1} i^3 + n^3 \\
&= \left(\frac{(n-1)n}{2}\right)^2 + n^3 \\
&= \frac{n^4 - 2n^3 + n^2}{4} + n^3 = \frac{n^4 + 2n^3 + n^2}{4} \\
&= \frac{n^2(n+1)^2}{4} \\
&= \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}$$

as desired.