Subgraph Replacements in Enumeration of Tilings

Tri Lai

The Department of Mathematics
Indiana University
Bloomington, IN 47405

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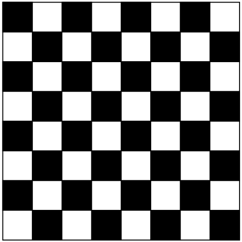
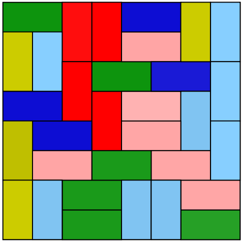
Terminology





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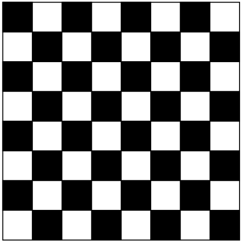

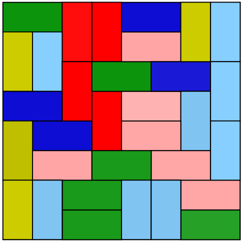

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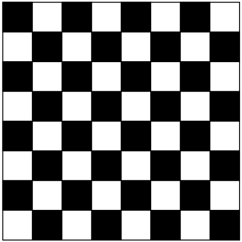
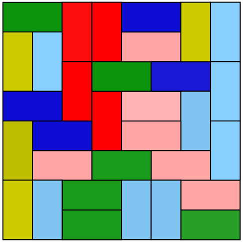
 

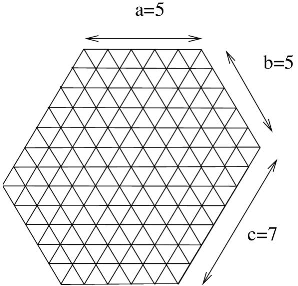
- A lattice divides the plane into elementary regions, called **cells**. A (lattice) **region** is a connected union of cells.
- A **tile** is a union of two cells that share an edge.
- A **tiling** of a region is a covering of the region by tiles so that there are no gaps or overlaps. Denote by $T(R)$ the number of tilings of the region R.

Classic Results

MacMahon 1900

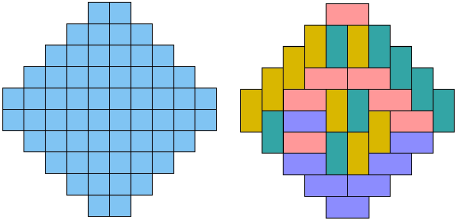
The number of (lozenge) tilings of a hexagon of sides a, b, c, a, b, c on the triangular lattice is

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{t=1}^{c}\frac{i+j+t-1}{i+j+t-2}$$



Classic Results

Elkies, Kuperberg, Larsen and Propp 1991
The Aztec diamond region of order n has $2^{n(n+1)/2}$ (domino) tilings.



Douglas' Theorem

- The square lattice with every second southwest-to-northeast diagonal drawn in.

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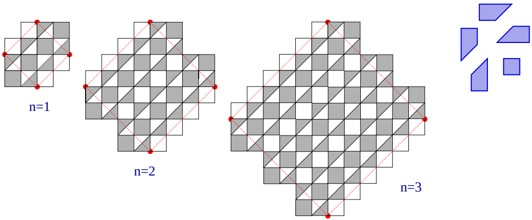
- **C. Douglas 1996** The region of order n has $2^{2n(n+1)}$ tillings.



Propp's Problem

- In 1999, Propp listed 32 open problems in the field of exact enumeration of tilings.

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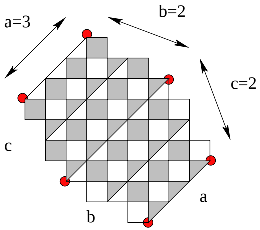
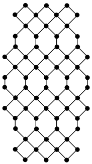
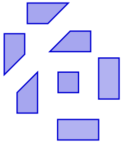
- Problem 16 asks for a formula of the number of tilings of a certain quasi-hexagon of sides a, b, c, a, b, c on the square lattice with every third southwest-to-northeast diagonal drawn in.







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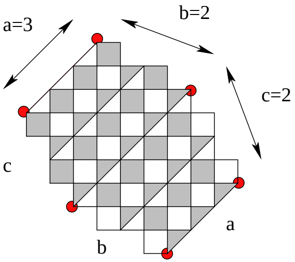

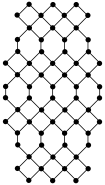

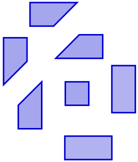

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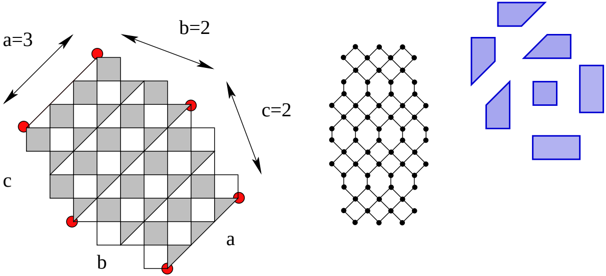
Propp's Problem



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- In general case, it does not give a round number of tilings.

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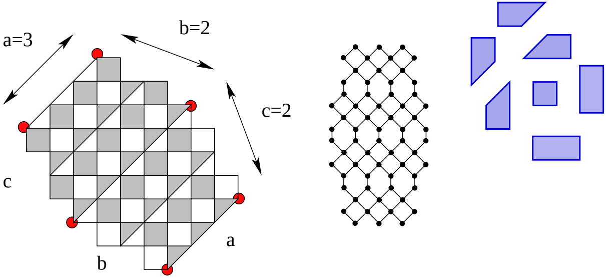
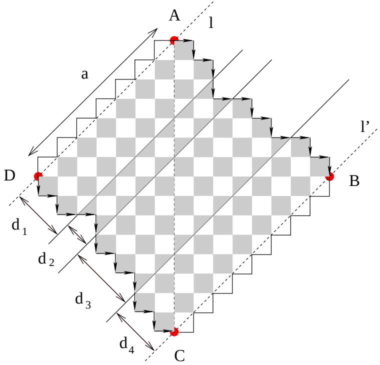


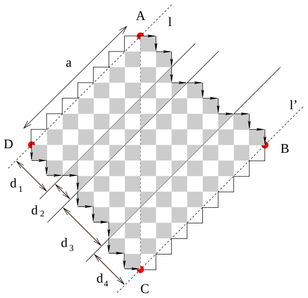
- Ben Wieland showed that if $a = b = c$ then the number of tilings is a power of 2.
- In general case, it does not give a round number of tilings.
- There are $17920 = 2^9.5.7$ tilings in the figure.

Definition of regions



Denoted by $D_a(d_1, d_2, \ldots, d_k)$.
$D_7(4, 2, 5, 4)$

Definition of regions



- A **regular cell** is a unit square or a triangle pointing away from the base.
- The **height** is the number of rows of black regular cells.
- The **width** is the number of cells in the bottom row of cells, i.e. $|BC|/\sqrt{2}$.

Main results

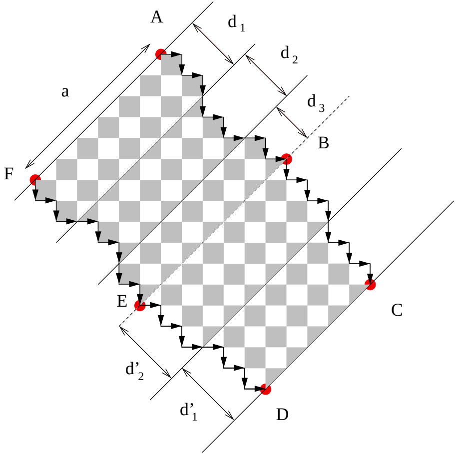
Theorem (L. 2012)

If the height and the width are equal, then

$$T(D_a(d_1, d_2, \dots, d_k)) = 2^{C-h(h+1)/2}, \tag{1}$$

where $h =$*height, and* $C = \#$ *black regular cells.*

Definition of regions



Denoted by $H_a(d_1,\ldots,d_k;d'_1,\ldots,d'_l)$.
$H_6(4,4,3;5,5)$

Definition of regions

- A **regular cell** is a unit square or a triangle pointing away from the line l.
- The **upper height** is the number of rows of black regular cells above l.
- The **lower height** is the number of rows of white regular cells below l.
- The **width** is the number of cells in the bottom row of the upper part, i.e. $|BE|/\sqrt{2}$.

Main results

Theorem (L. 2012)

If $h_1 = h_2 < w$, then

$$T(H_a(d_1,\dots,d_k;\ d'_1,\dots,d'_l)) = 2^{C_1 - \frac{h_1(2w-h_1+1)}{2} + C_2 - \frac{h_2(2w-h_2+1)}{2}} \times \prod_{i=1}^{h_1}\prod_{j=1}^{w-h_1}\prod_{t=1}^{h_1} \frac{i+j+t-1}{i+j+t-2}, \quad (2)$$

where h_1 is upper height, h_2 is lower height, w is the width, $C_1 = \#$ black regular cells above l, and $C_2 = \#$ white regular cells below l.

Perfect matchings and dual graph

- A **perfect matching** of G is a set of edges such that each vertex is adjacent to exactly one selected edge.
- Denote by $M(G)$ the number of perfect matchings of the graph G.

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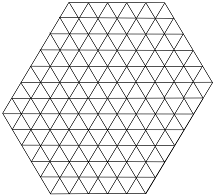
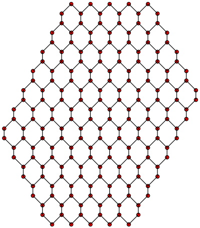
- Denote by $M(G)$ the number of perfect matchings of the graph G.
- The **dual graph** G of a region R is the graph whose vertices are cells in R and whose edges connect two adjacent cells.

Perfect matchings and dual graph





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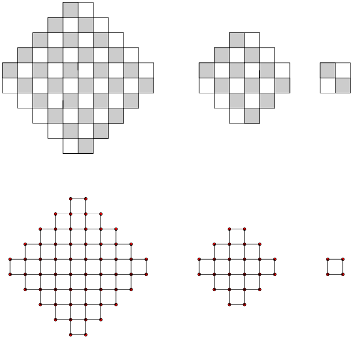
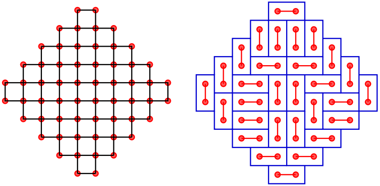


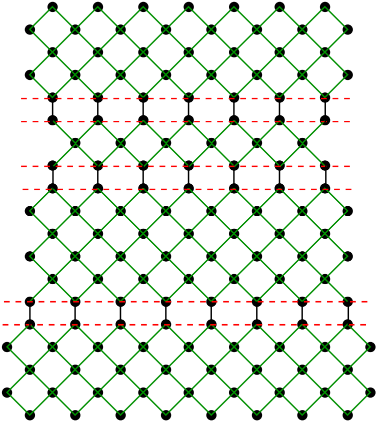
Bijection between tilings and perfect matchings



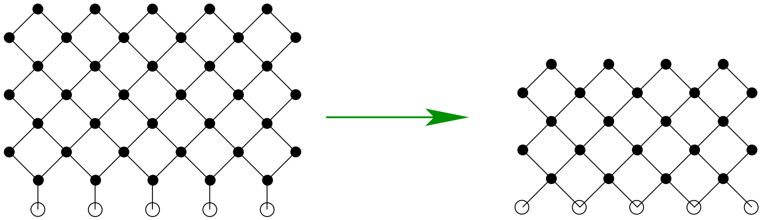
We have a bijection between the tilings of R and the perfect matchings of G

Structure of dual graphs

The dual graph of a general Douglas region consists of layers connected by vertical edges.

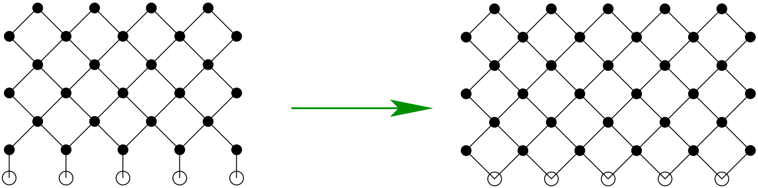


The transformation T_1



$$M(G) = 2^{\#\text{ rows of diamonds on left}} M(G').$$

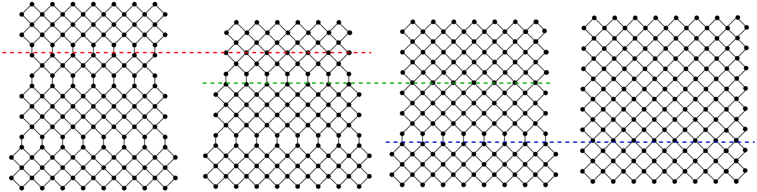
Fundamental transformation T_2



$$M(G) = 2^{-(\#\text{ rows of diamonds on right})} M(G').$$

Proof of Theorem 1

The transformations preserve the condition *"the height and the width are equal"*.



- $M(G_1) = 2^2 M(G_2)$

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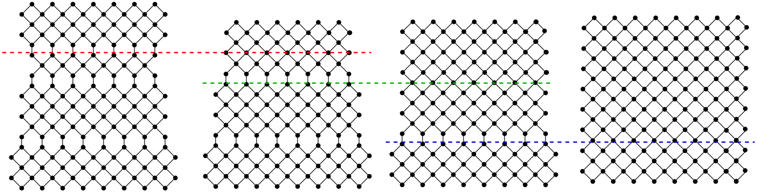

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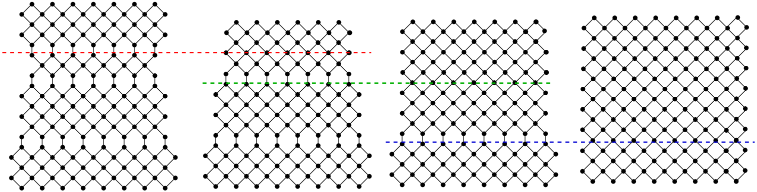
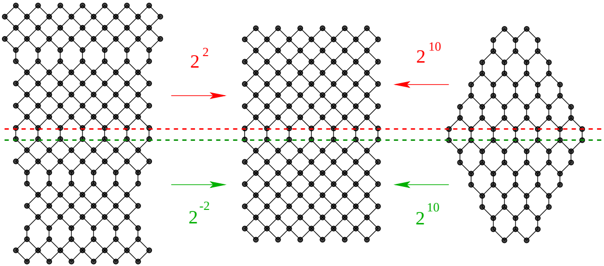


- $M(G_1) = 2^2 M(G_2)$
- $M(G_2) = 2^{-3} M(G_3)$
- $M(G_3) = 2^{-6} M(G_4) = 2^{-6} T(AD_8)$

Proof of Theorem 2



G_1 is the dual graph of $H_7(4,6;2,4,3)$

- $M(G_1) = 2^{C_1 - h_1 w} 2^{C_2 - h_2 w} M(G_2)$

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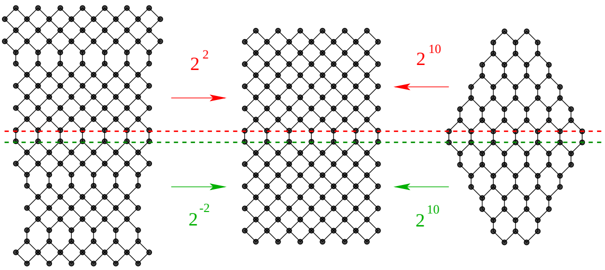

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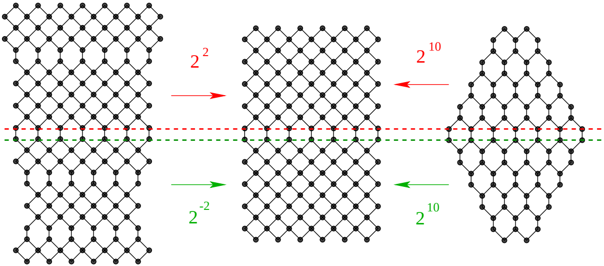

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- $M(G_1) = 2^{C_1 - h_1 w + h_1(h_1-1)/2} 2^{C_2 - h_2 w + h_2(h_2-1)/2} M(G_3)$

Future projects

- Weighted versions of the problems.

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- Find the number of tilings of a "quasi-polygon", say quasi-octagon.

Questions?

Thank you !

Reference

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- C. Douglas, *An illutrative study of the enumeration of tilings: Conjecture discovery and proof techniques*.
- H. Helfgott and I. Gessel, *Enumeration of tillings of diamonds and hexagons with defects*, Electron. J. Combin. **6** (1999), R16.
- J. Propp, *Emnumeration of matchings: Problems and progress*, New perspectives in algebraic combinatorics, Cambidge University Press (1999).
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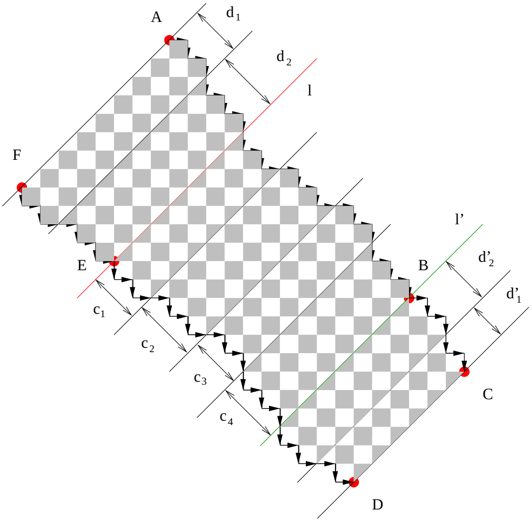
Appendix I: Sum formula for the asymmetric case



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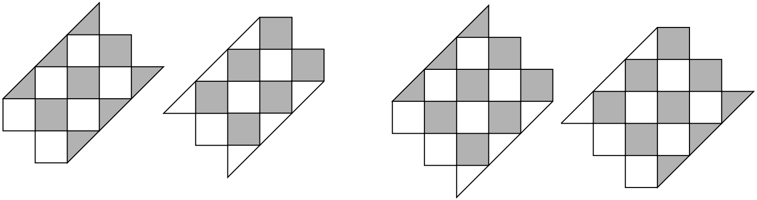


$$\phi_H(j) = \begin{cases} 1 & \text{if } j\text{th middle layer is right-odd} \\ -1 & \text{if } j\text{th middle layer is left-odd} \\ 1 & \text{otherwise,} \end{cases} \quad (3)$$

Appendix I: Sum formula for the asymmetric case

Define the *slope* by

$$\Phi(H) = \sum_{j=1}^{t} \phi_H(j). \tag{4}$$

$$h_0 = \sum_{j=1}^{t} (c_i - \phi_H(j))/2. \tag{5}$$

Theorem

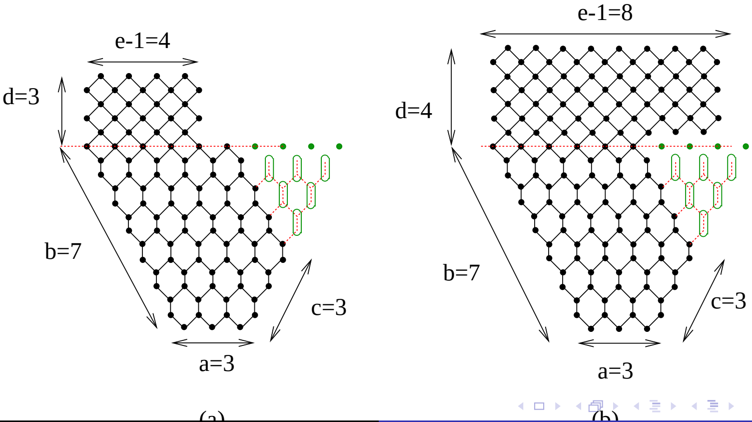
If $h_1 = h_2 = h < w$ and $h_0 > 0$ then

$$M(H) = 2^{C+C'-h(2a+2m-2n-h+1)-h_0(h_0-1)/2} \times M\left(C^{h_0,h+h_0-1}_{a+m-n-h,h_0+\Phi(H)+2h,h}\right) \tag{6}$$

Appendix I: Sum formula for the asymmetric case

Define

$$V_{(a,b,a)}(r_1,\ldots,r_a)=\prod_{1\leq i<j\leq a}\frac{r_j-r_i}{j-i}. \tag{7}$$



Appendix I: Sum formula for the asymmetric case

Lemma

(a). *If* $e \neq c+d-1$, *then* $M(C^{d,e}_{a,b,c}) = 0$.
(b). *If* $d \leq a$, *then*

$$M\left(C^{d,c+d-1}_{a,b,c}\right) = 2^{\frac{d(d-1)}{2}} \times \sum V_{(b,a,b)}(A \cup \{a+c+1,\dots,a+b\}) V_{(d,c,d)}(B), \qquad (8)$$

where $A \cup B = \{1,\dots,c+d\}$, $|A| = c$ *and* $|B| = d$.
(c). *If* $d < a$, *then*

$$M\left(C^{d,c+d-1}_{a,b,c}\right) = 2^{\frac{d(d-1)}{2}} \sum V_{(b,a,b)}(A \cup \{a+c+1,\dots,a+b\}) V_{(d,c,d)}(B \cup \{c+a+1,\dots,c+d\}), \qquad (9)$$

where $A \cup B = \{1,\dots,c+a\}$, $|A| = c$ *and* $|B| = a$.

Appendix II: Ben Wieland's case

Lemma (Graph-Splitting Lemma)

Let $G = (V_1, V_2, E)$ be a bipartite graph. Let H be an induced subgraph of G.
(a) Assume that

(i) *The separating condition: there are no edges of G connecting a vertex in $V(H) \cap V_1$ and a vertex in $V(G - H)$,*

(ii) *The balancing condition: $|V(H) \cap V_1| = |V(H) \cap V_2|$.*

Then

$$M(G) = M(H) \cdot M(G - H) \tag{10}$$

(b) If H satisfies the separating condition and but $|V(H) \cap V_1| > |V(H) \cap V_2|$, then $M(G) = 0$.

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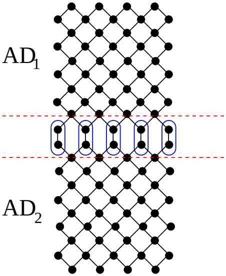

- $T(H) = 2^{C_1 - h_1 w} 2^{C_2 - h_2 w} M(G)$

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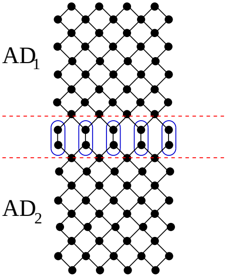

- $T(H) = 2^{C_1 - h_1 w} 2^{C_2 - h_2 w} M(G)$
- $M(G) = M(AD_1) M(G - AD_1)$

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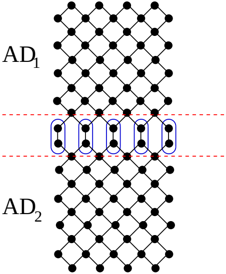

- $T(H) = 2^{C_1 - h_1 w} 2^{C_2 - h_2 w} M(G)$
- $M(G) = M(AD_1) M(G - AD_1)$
- $M(G - AD_1) = M(AD_2) M(G - AD_1 - AD_2)$

Appendix III: Classic local transformations

- Remove leaves: $M(G) = M(G')$.

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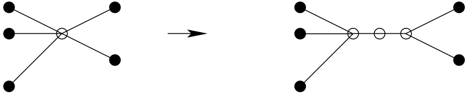

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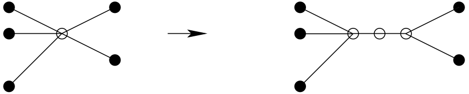
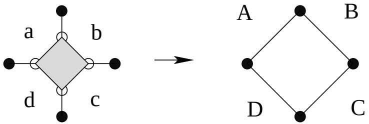


- Spider lemma (Urban renewal): $M(G) = (ac + bd)M(G')$.



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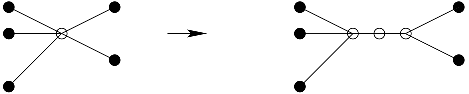
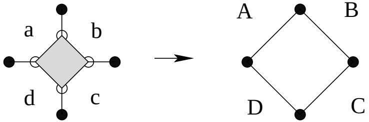
- Vertex-splitting lemma: $M(G) = M(G')$.

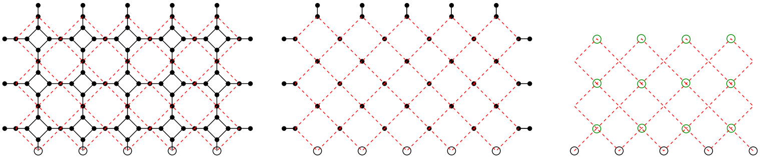


- Spider lemma (Urban renewal): $M(G) = (ac + bd)M(G')$.



- Star lemma: $M(G) = 1/t.M(G')$ for $t > 0$.

Appendix IV: Proof of the transformation T_1



(a) (b) (c)

Appendix V: Proof of the transformation T_2