Proof of a generalization of Aztec diamond theorem

Tri Lai

Institute for Mathematics and its Applications
University of Minnesota
Minneapolis, MN 55455

December 5, 2014

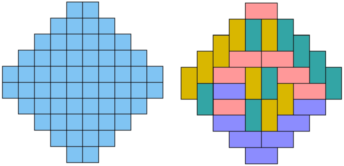
Aztec Diamond Theorem

Definition

The Aztec diamond of order n is the collection of unit squares inside the contour $|x| + |y| = n + 1$.

Theorem (Elkies, Kuperberg, Larsen and Propp 1991)

There are $2^{n(n+1)/2}$ ways to cover the Aztec diamond of order n by dominoes so that there are no gaps or overlaps.



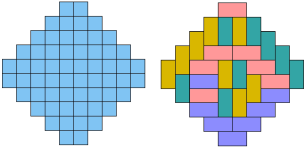
Aztec Diamond Theorem

Theorem (Elkies–Kuperberg–Larsen–Propp, 1991)

There are $2^{n(n+1)/2}$ ways to cover the Aztec diamond of order n by dominoes so that there are no gaps or overlaps.

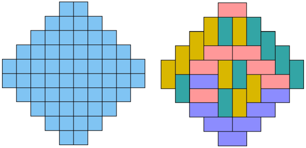
- The first four proofs have been given by Elkies, Kuperberg, Larsen and Propp.
- Many further proofs:
  - Propp 2003
  - Kuo 2004
  - Brualdi and Kirkland 2005
  - Eu and Fu 2005
  - Bosio and Leeuwen 2013
  - Fendler and Grieser 2014

Terminology



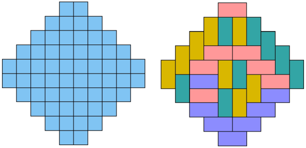
- A lattice divides the plane into fundamental regions called **cells**.

Terminology



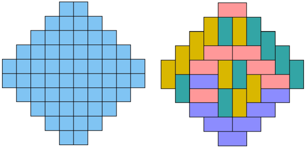
- A lattice divides the plane into fundamental regions called **cells**.
- A **tile** is a union of any two cells sharing an edge.

Terminology



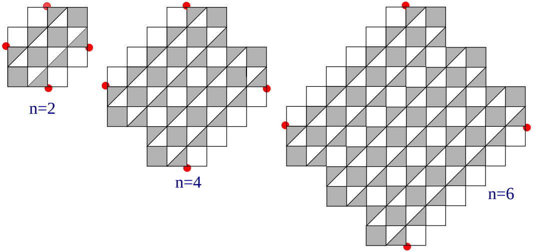
- A lattice divides the plane into fundamental regions called **cells**.
- A **tile** is a union of any two cells sharing an edge.
- A **tiling** of a region is a covering of the region by tiles so that there are no gaps or overlaps.

Terminology



- A lattice divides the plane into fundamental regions called **cells**.
- A **tile** is a union of any two cells sharing an edge.
- A **tiling** of a region is a covering of the region by tiles so that there are no gaps or overlaps.
- Denote by $\mathbf{M}(R)$ the number of tilings of R.

Hybrid Domino-Lozenge Tilings and Douglas' Theorem



Theorem

The number of tilings of the region of order 2k is $2^{2k(k+1)}$.

Hybrid Domino-Lozenge Tilings and Douglas' Theorem

- Are there similar regions on the square lattice with arbitrary diagonals drawn in?

Hybrid Domino-Lozenge Tilings and Douglas' Theorem

- Are there similar regions on the square lattice with arbitrary diagonals drawn in?
- What is the number of tilings of the region?

Hybrid Domino-Lozenge Tilings and Douglas' Theorem

- Are there similar regions on the square lattice with arbitrary diagonals drawn in?
- What is the number of tilings of the region?
- Is it still a power of 2?

Generalized Douglas region $D_a(d_1, d_2, \ldots, d_k)$.

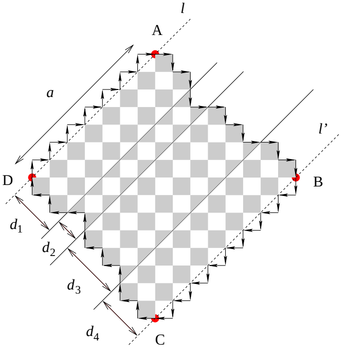


Figure : The region $D_7(4, 2, 5, 4)$.

The width is the numbers of squares running from B to C.

Generalization of Aztec diamond theorem

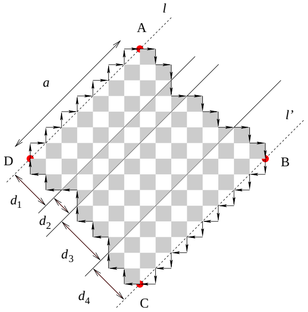
Theorem (L, 2014)

Assume that $D_a(d_1, \ldots, d_k)$ has the width w. Assume that the the vertices B and D are on the same level. Then

$$\mathbf{M}(D_a(d_1, \ldots, d_k)) = 2^{\mathcal{C} - w(w+1)/2} \tag{1}$$

where $\mathcal{C}$ is the number of black squares and black up-pointing triangles (which we called regular cells).

Example



The region $D_7(4,2,5,4)$ has the width $w = 8$ and $\mathcal{C} = 2 \cdot 8 + 1 \cdot 7 + 3 \cdot 8 + 2 \cdot 9 = 65$. Therefore

$$\mathbf{M}(D_7(4,2,5,4)) = 2^{\mathcal{C} - w(w+1)/2} = 2^{65 - \frac{8 \cdot 9}{2}} = 536,870,912.$$

Remark

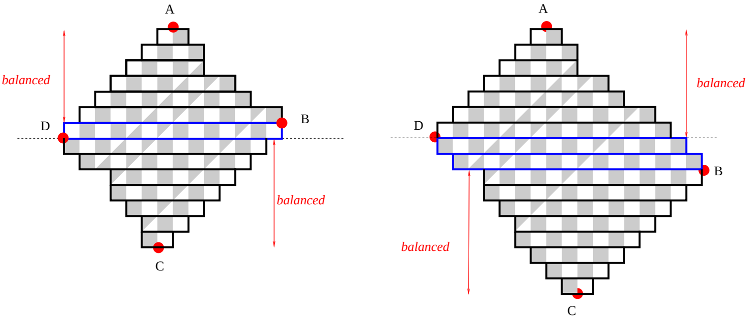
Q: Why did we assume that B and D are on the same level in the main theorem?

Remark

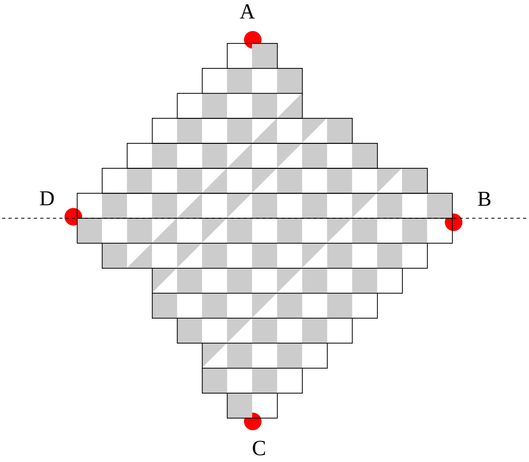
Q: Why did we assume that B and D are on the same level in the main theorem?

A: Otherwise the numbers black and white cells are *not* the same, so the region has no tiling.

A remark



A remark



Bijection between tilings and perfect matchings

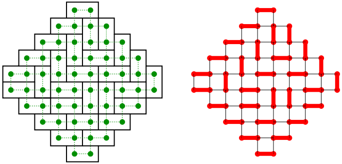
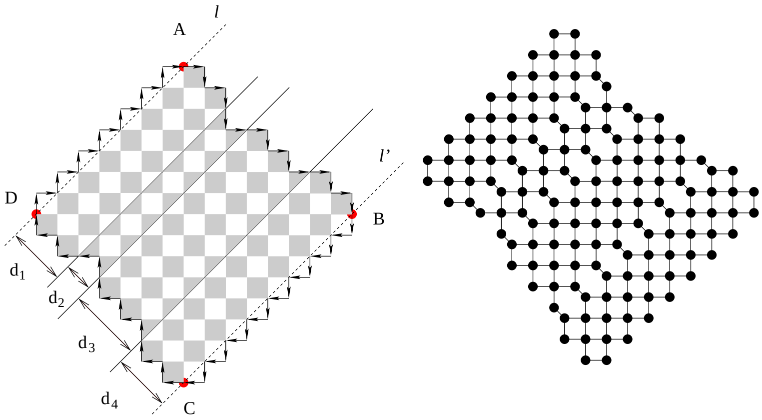


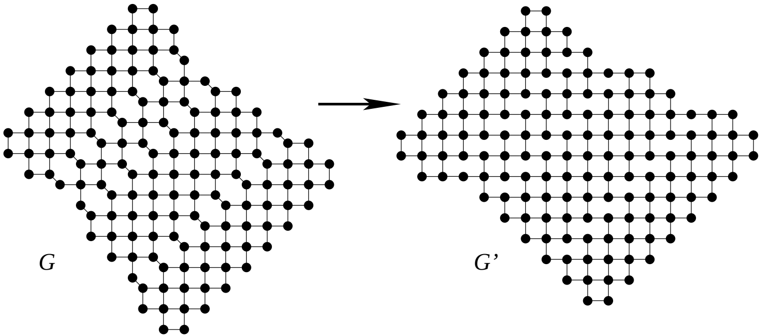
Figure : Bijection between tilings of the Aztec diamond of order 5 and perfect matchings of its dual graph.

- The dual graph of a region R is the graph whose vertices are the cells in R and whose edges connect precisely two adjacent cells.
- A perfect matching of a graph G is a collection of disjoint edges covering all vertices of G.

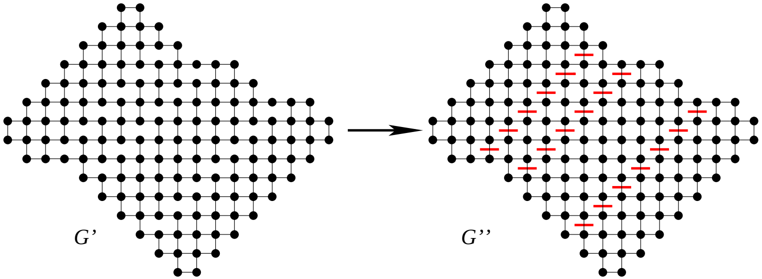
From tilings to families of non-intersecting paths



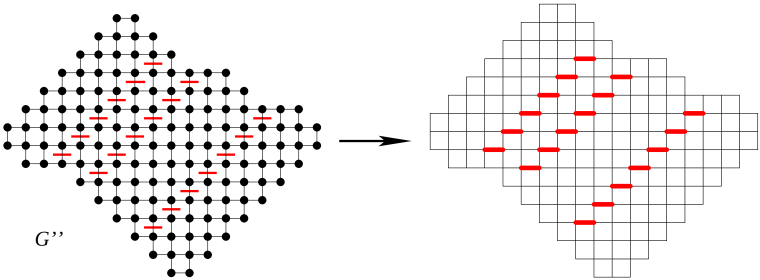
From tilings to families of non-intersecting paths



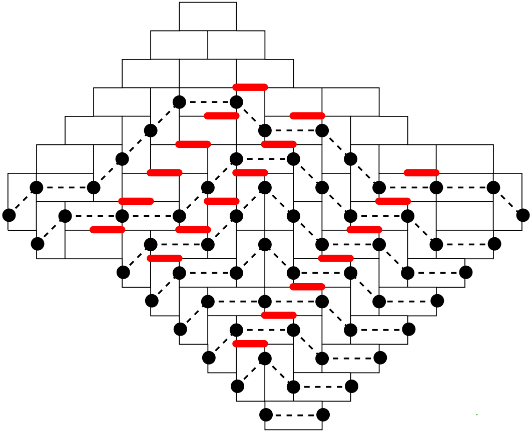
From tilings to families of non-intersecting paths



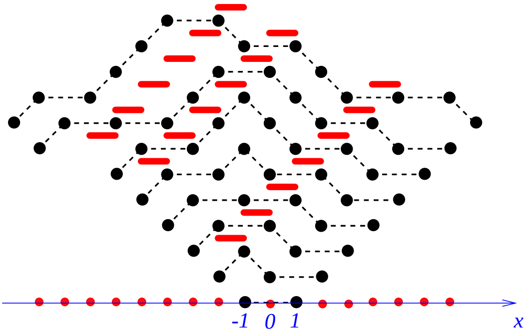
From tilings to families of non-intersecting paths



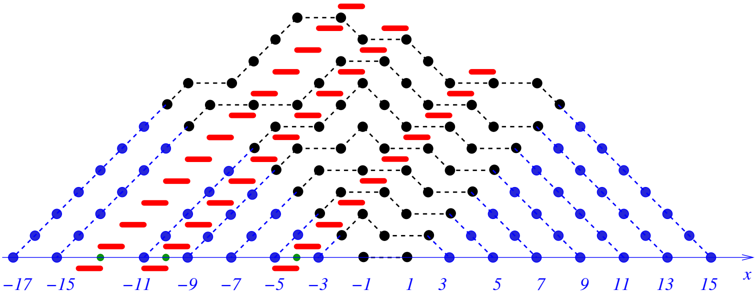
From tilings to families of non-intersecting paths



From tilings to families of non-intersecting paths



From tilings to families of non-intersecting paths

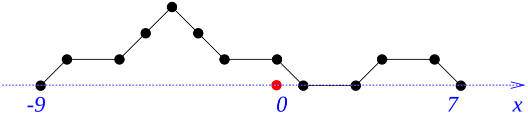


Schröder paths

Definition

A Schröder path is a lattice path:

1. starting and ending on the x-axis;
2. never go below x-axis;
3. using the steps: (1,1) up, (1,-1) down, and (2,0) flat.



Large and small Schröder numbers

http://people.brandeis.edu/ gessel/homepage/slides/schroder.pdf

- $r_n =$ the number of Schröder paths from $(0,0)$ to $(2n,0)$.

Large and small Schröder numbers

http://people.brandeis.edu/ gessel/homepage/slides/schroder.pdf

- r_n = the number of Schröder paths from $(0,0)$ to $(2n,0)$.
- s_n = the number of Schröder paths from $(0,0)$ to $(2n,0)$ with no flat step on the x-axis.

Large and small Schröder numbers

http://people.brandeis.edu/ gessel/homepage/slides/schroder.pdf

- r_n = the number of Schröder paths from $(0, 0)$ to $(2n, 0)$.
- s_n = the number of Schröder paths from $(0, 0)$ to $(2n, 0)$ with no flat step on the x-axis.
- $r_n = 2s_n$

Large and small Schröder numbers

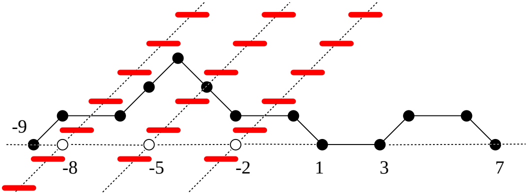
http://people.brandeis.edu/ gessel/homepage/slides/schroder.pdf

- r_n = the number of Schröder paths from $(0,0)$ to $(2n,0)$.
- s_n = the number of Schröder paths from $(0,0)$ to $(2n,0)$ with no flat step on the x-axis.
- $r_n = 2s_n$
- $R(x) = \sum_{n=0}^{\infty} r_n x^n = \dfrac{1 - x - \sqrt{1 - 6x + x^2}}{2x}$

Schröder paths with barriers

Definition

1. For integers $a_m > a_{m-1} > \ldots > a_1 \geq 0$, define $\mathcal{B}(a_1, a_2, \ldots, a_m)$ is the set of length-1 barriers running along the lines $y = x + a_i$ (i.e. connecting $(-a_i + k, k + \frac{1}{2})$ and $(-a_i + k + 1, k + \frac{1}{2})$).
2. If a Schröder path P avoids all barriers in $\mathcal{B} = \mathcal{B}(a_1, a_2, \ldots, a_m)$, we say P is compatible with $\mathcal{B}$.



Two families of non-intersecting Schröder paths with barriers

Definition

1. Let x_i be the i-th largest negative odd number in $\mathbb{Z}\backslash\{-a_1, \ldots, -a_m\}$.
2. $\Pi_n(a_1, \ldots, a_m)$ set of n-tuples of non-intersecting Schröder paths $(\pi_1, \ldots, \pi_n)$, where π_i connects $A_i = (x_i, 0)$ and $B_i = (2i-1, 0)$ and is compatible with $\mathcal{B}(a_1, a_2, \ldots, a_m)$.

Families of non-intersecting paths in $D_a(d_1, d_2, \dots, d_k)$

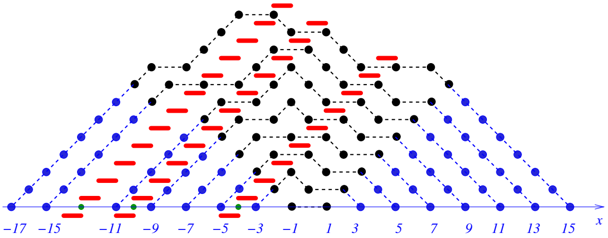
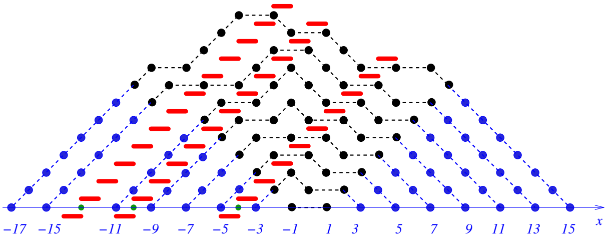


Figure : Families of non-intersecting paths in $D_7(4, 2, 5, 4)$.

$$\Pi_w(a_1, \dots, a_{k-1})$$

where $a_i := d_k + \dots + d_{k-i+1} + i - 1$, for $i = 1, 2, \dots, k-1$.

Families of non-intersecting paths in $D_a(d_1, d_2, \ldots, d_k)$



Lemma

$$\mathbf{M}(D_a(d_1, \ldots, d_k)) = |\Pi_w(a_1, \ldots, a_{k-1})|, \tag{2}$$

where $a_i := d_k + \ldots + d_{k-i+1} + i - 1$, *for* $i = 1, 2, \ldots, k-1$.

Inductive proof on the number layers

Theorem

$$\mathbf{M}(D_a(d_1, d_2, \ldots, d_k)) = 2^{\mathcal{C} - w(w+1)/2} \qquad (*)$$

- If $k = 1$, the theorem follows from the Aztec diamond theorem.
- Assume that (*) is true for any generalized Douglas regions with less than k layers ($k \geq 2$). We need to show for any $D_a(d_1, d_2, \ldots, d_k)$.
- Consider the $d_k = 2x$ (the case $d_k = 2x - 1$ can be obtained similarly).

Reduction using Lindström–Gessel–Viennot Lemma.

Lemma (Linström–Gessel–Viennot)

1. *G is an acyclic digraph.*
2. $U = \{u_1, u_2, \ldots, u_n\}$, $V = \{v_1, v_2, \ldots, v_n\}$
3. *For any $1 \leq i < j \leq n$, $\mathcal{P} : u_i \to v_j$ intersects $\mathcal{Q} : u_j \to v_i$.*

Then

$$\#(\tau_1, \tau_2, \ldots, \tau_n)_{U \to V} = \det \begin{bmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \ldots & a_{n,n} \end{bmatrix},$$

where $a_{i,j}$ is the number of paths from u_i to v_j.

Reduction using Linström–Gessel–Viennot Lemma.

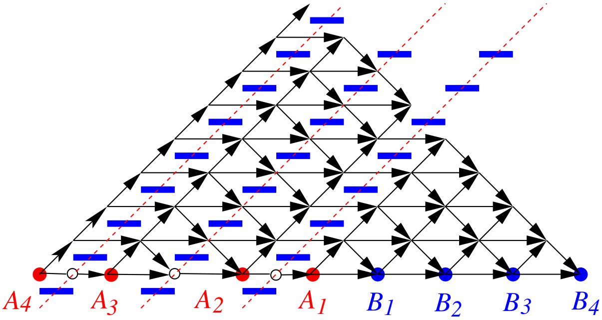
$$\mathbf{M}(D_a(d_1,\dots,d_k)) = |\Pi_w(a_1,\dots,a_m)| = \det(H_w(a_1,\dots,a_m)), \tag{3}$$

where

$$H_w := H_w(a_1,\dots,a_m) := \begin{bmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,w} \\ r_{2,1} & r_{2,2} & \dots & r_{2,w} \\ \vdots & \vdots & & \vdots \\ r_{w,1} & r_{w,2} & \dots & r_{w,w} \end{bmatrix}, \tag{4}$$

and where $r_{i,j}$= the number of Schröder paths connecting A_i to B_j compatible with $\mathcal{B}(a_1,\dots,a_m)$.

Reduction using Linström–Gessel–Viennot Lemma.



Reduction using Linström–Gessel–Viennot Lemma.

Definition

Given $\mathcal{B}(a_1, \ldots, a_m)$

1. A bad level step is a level step on x-axis avoiding all points $(-a_i, 0)$.
2. $\Lambda_n(a_1, \ldots, a_m)$ set of n-tuples of non-intersecting Schröder paths $(\lambda_1, \ldots, \lambda_n)$, where λ_i connects $A_i = (x_i, 0)$ and $B_i = (2i - 1, 0)$, is compatible with $\mathcal{B}(a_1, a_2, \ldots, a_m)$, and has no bad flat step.

Reduction using Linström–Gessel–Viennot Lemma.

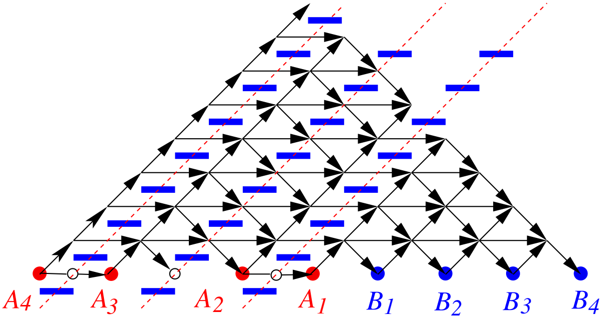
$$|\Lambda_w(a_1, \dots, a_m)| = \det(Q_w(a_1, \dots, a_m)), \tag{5}$$

where

$$Q_w := Q_w(a_1, \dots, a_m) := \begin{bmatrix} s_{1,1} & s_{1,2} & \dots & s_{1,w} \\ s_{2,1} & s_{2,2} & \dots & s_{2,w} \\ \vdots & \vdots & & \vdots \\ s_{w,1} & s_{w,2} & \dots & s_{w,w} \end{bmatrix}, \tag{6}$$

and where $s_{i,j}$= the number of Schröder paths connecting A_i to B_j, is compatible with $\mathcal{B}(a_1, \dots, a_m)$, and has no bad flat step.

Reduction using Linström–Gessel–Viennot Lemma.



Reduction using Linström–Gessel–Viennot Lemma.

Lemma

If $a_1 > 1$, *then* $r_{i,j} = 2s_{i,j}$.

- $\det(H_w(a_1, \ldots, a_m)) = 2^w \det(Q_w(a_1, \ldots, a_m))$

Reduction using Linström–Gessel–Viennot Lemma.

Lemma

If $a_1 > 1$, *then* $r_{i,j} = 2s_{i,j}$.

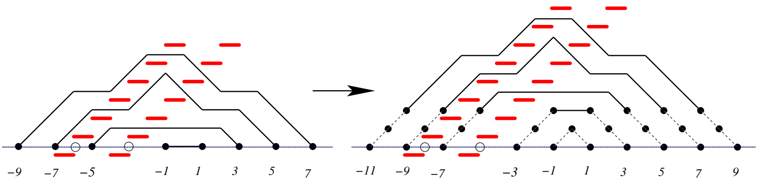
- $\det(H_w(a_1, \dots, a_m)) = 2^w \det(Q_w(a_1, \dots, a_m))$
- $|\Pi_w(a_1, \dots, a_m)| = 2^w |\Lambda_w(a_1, \dots, a_m)|$

Reduction using Linström–Gessel–Viennot Lemma.

$$|\Pi_w(a_1,\dots,a_m)| = 2^w|\Lambda_w(a_1,\dots,a_m)|$$

Lemma

If $a_1 > 1$, *then* $|\Pi_{w-1}(a_1-2,\dots,a_m-2)| = |\Lambda_w(a_1,\dots,a_m)|$.



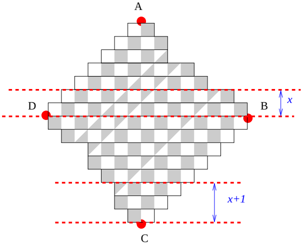
Reduction using Linström–Gessel–Viennot Lemma.

- $|\Pi_w(a_1, \ldots, a_m)| = 2^w |\Pi_{w-1}(a_1 - 2, \ldots, a_m - 2)|$

Reduction using Linström–Gessel–Viennot Lemma.

- $|\Pi_w(a_1, \ldots, a_m)| = 2^w |\Pi_{w-1}(a_1 - 2, \ldots, a_m - 2)|$
- $= 2^{\sum_{i=0}^{x-1}(w-i)} |\Pi_{w-x}(0, a_2 - a_1, \ldots, a_m - a_1)|$

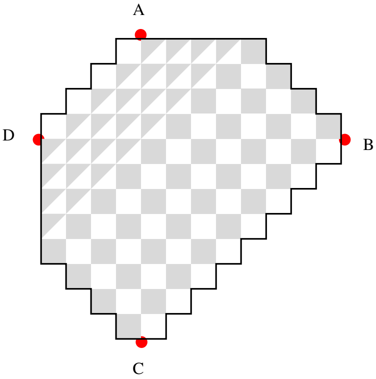
Comparing a and x



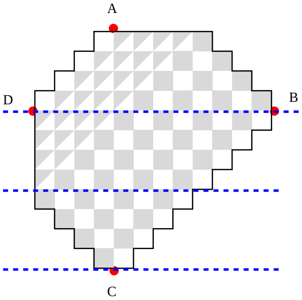
$$a \geq x$$

$$w \geq x + 1$$

Case $a = x$



Case $a = x$

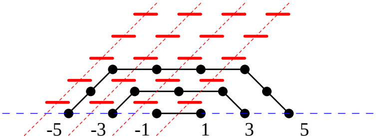


We have $d_1 = d_2 = \ldots = d_{k-1} = 1$ and $w = k + x - 1$ so

$$|\Pi_{w-x}(0, a_2 - a_1 \ldots, a_m - a_1)| = |\Pi_{k-1}(0, 2, 4, \ldots, 2(k-1))| = 1.$$

Case $a = x$

$$|\Pi_{k-1}(0, 2, 4, \ldots, 2(k-1))| = 1$$



Case $a = x$

$$\mathbf{M}(D_a(d_1, \ldots, d_k)) = |\Pi_w(a_1, \ldots, a_m)| = 2^{\sum_{i=0}^{x-1}(w-i)}.$$

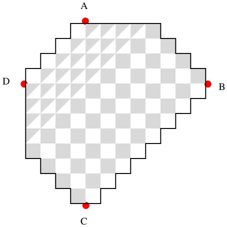
Need to show that

$$\sum_{i=0}^{x-1}(w-i) = \mathcal{C} - w(w+1)/2$$

Case $a = x$

Need to show that

$$\sum_{i=0}^{x-1}(w-i) = \mathcal{C} - w(w+1)/2$$



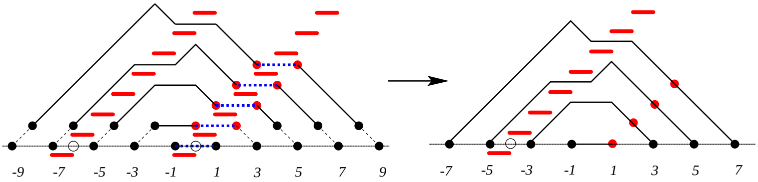
$$\mathcal{C} = (w+1)x + \sum_{i=0}^{w-x-1}(w-i)$$

Case $a > x$

$$\mathbf{M}(D_a(d_1,\dots,d_k)) = 2^{\sum_{i=0}^{x-1}(w-i)}|\Pi_{w-x}(0, a_2 - a_1, \dots, a_m - a_1)|$$

Lemma

$$|\Pi_{w-x}(0, a_2 - a_1, \dots, a_m - a_1)| = |\Pi_{w-x-1}(a_2 - a_1 - 2, \dots, a_m - a_1 - 2)|$$



Case $a > x$

$$\mathbf{M}(D_a(d_1,\ldots,d_k)) = 2^{\sum_{i=0}^{x-1}(w-i)}|\Pi_{w-x-1}(a_2-a_1-2,\ldots,a_m-a_1-2)|$$

- Recall that we have bijection between tilings of $D_a(d_1,\ldots,d_k)$ and elements of $\Pi_w(a_1,a_2,\ldots,a_m)$.

Case $a > x$

$$\mathbf{M}(D_a(d_1, \ldots, d_k)) = 2^{\sum_{i=0}^{x-1}(w-i)} |\Pi_{w-x-1}(a_2 - a_1 - 2, \ldots, a_m - a_1 - 2)|$$

- Recall that we have bijection between tilings of $D_a(d_1, \ldots, d_k)$ and elements of $\Pi_w(a_1, a_2, \ldots, a_m)$.
- Tilings of $D' := D_{a-x}(d_1, d_2, \ldots, d_{k-1} - 1)$ correspond to elements of $\Pi_{w-x-1}(a_2 - a_1 - 2, \ldots, a_m - a_1 - 2)$.

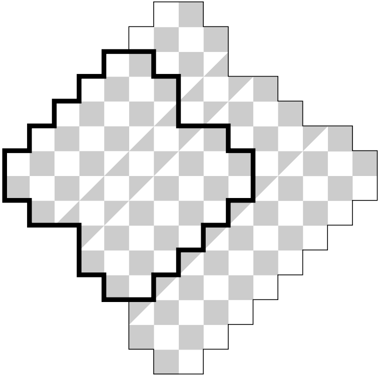
Case $a > x$

$$\mathbf{M}(D_a(d_1,\dots,d_k)) = 2^{\sum_{i=0}^{x-1}(w-i)}|\Pi_{w-x-1}(a_2-a_1-2,\dots,a_m-a_1-2)|$$

- Recall that we have bijection between tilings of $D_a(d_1,\dots,d_k)$ and elements of $\Pi_w(a_1,a_2,\dots,a_m)$.
- Tilings of $D' := D_{a-x}(d_1,d_2,\dots,d_{k-1}-1)$ correspond to elements of $\Pi_{w-x-1}(a_2-a_1-2,\dots,a_m-a_1-2)$.
- $$\mathbf{M}(D_a(d_1,\dots,d_k)) = 2^{\sum_{i=0}^{x-1}(w-i)}\mathbf{M}(D_{a-x}(d_1,d_2,\dots,d_{k-1}-1))$$

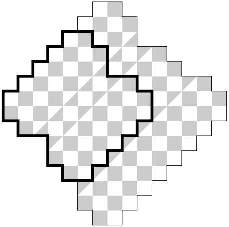
Case $a > x$

$$\mathbf{M}(D_a(d_1, \ldots, d_k)) = 2^{\sum_{i=0}^{x-1}(w-i)}\,\mathbf{M}(D_{a-x}(d_1, d_2, \ldots, d_{k-1} - 1))$$



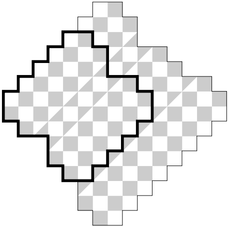
Case $a > x$

- $\mathbf{M}(D_a(d_1,\dots,d_k)) = 2^{\sum_{i=0}^{x-1}(w-i)}2^{\mathcal{C}'-w'(w'+1)/2}$



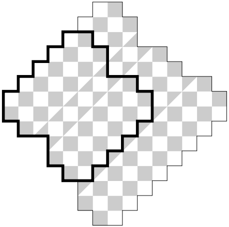
Case $a > x$

- $\mathbf{M}(D_a(d_1, \ldots, d_k)) = 2^{\sum_{i=0}^{x-1}(w-i)} 2^{\mathcal{C}' - w'(w'+1)/2}$
- $w - w' = x + 1$



Case $a > x$

- $\mathbf{M}(D_a(d_1, \dots, d_k)) = 2^{\sum_{i=0}^{x-1}(w-i)} 2^{\mathcal{C}' - w'(w'+1)/2}$
- $w - w' = x + 1$
- $\mathcal{C} - \mathcal{C}' = (w+1)x + w + x(w - x - 1)$



Case $a > x$

$$\mathbf{M}(D_a(d_1, \ldots, d_k)) = 2^{\mathcal{C} - w(w+1)/2}$$

Questions?

Thank you !