Enumeration of Tilings and Related Problems

Tri Lai

Institute for Mathematics and its Applications
Minneapolis, MN 55455

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University of British Columbia – Vancouver
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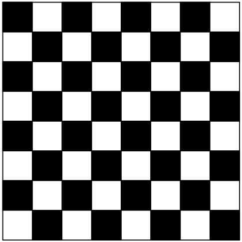
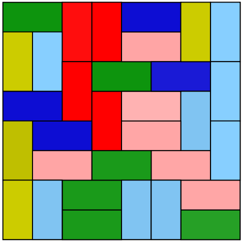
Outline

1. Introduction to the Enumeration of Tilings.
2. Electrical networks
3. Tiling expression of minors
4. Future work.

Definition of Tilings





- A lattice divides the plane into disjoint pieces, called **fundamental regions**.

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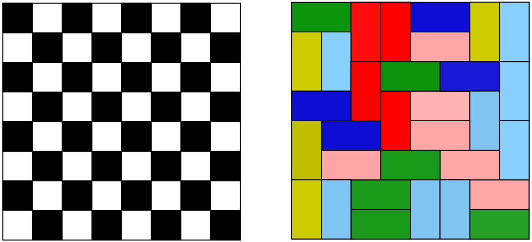

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- A **tile** is a union of any two fundamental regions sharing an edge.

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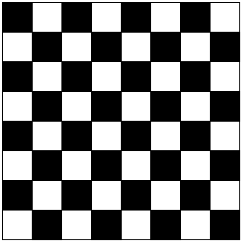

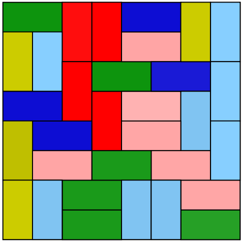

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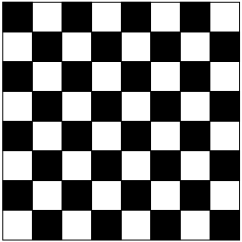

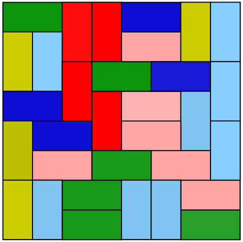

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- A **tile** is a union of any two fundamental regions sharing an edge.
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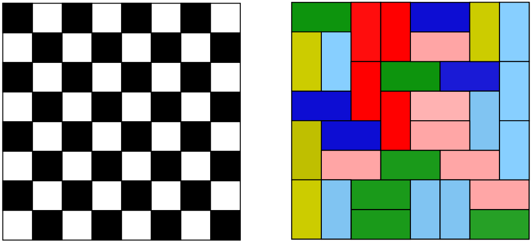
- The number of tilings of a 8×8 chessboard is 12,988,816.

Tilings



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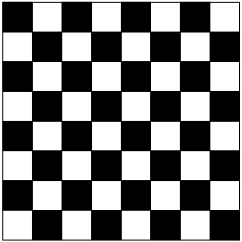
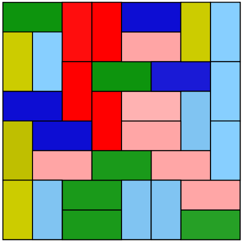
Tilings





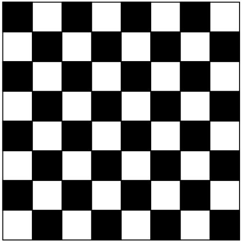
- We would like to find the number of tilings of certain regions.
- Tilings can carry weights, we also care about the weighted sum of tilings: $\sum_T wt(T)$, called the tiling generating function.

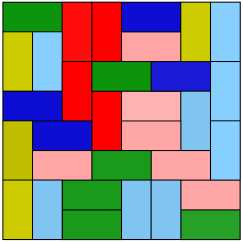
Kasteleyn–Temperley–Fisher Theorem

Theorem (Kasteleyn, Temperley and Fisher 1961)

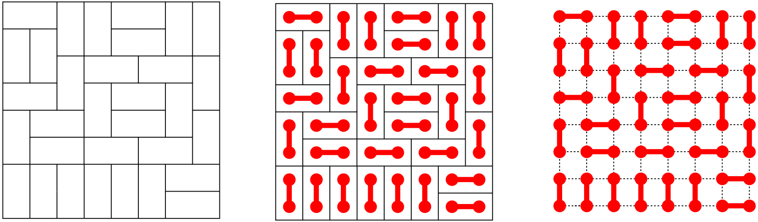
The number of tilings of a $2m \times 2n$ chessboard equals

$$\prod_{j=1}^{m} \prod_{k=1}^{n} \left(4\cos^2\left(\frac{j\pi}{2m+1}\right) + 4\cos^2\left(\frac{k\pi}{2n+1}\right) \right).$$





Tilings and dimer coverings



Pfaffian

Let $A = (a_{i,j})$ be a $2N \times 2N$ skew-symmetric matrix. The Pfaffian of A is defined by the equation

$$\mathrm{Pf}(A) = \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} \mathrm{sign}(\sigma) \prod_{i=1}^{N} a_{\sigma(2i-1),\sigma(2i)}$$

where S_{2N} is the symmetric group and $\mathrm{sign}(\sigma)$ is the signature of σ.

Pfaffian

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- The adjacent matrix of $\overrightarrow{G}_{m,n}$ is $A = (a_{i,j})_{2N \times 2N}$ $(N = mn)$ defined as:

$$a_{i,j} = \begin{cases} 1 & \text{if } i \to j, \\ -1 & \text{if } j \to i, \\ 0 & \text{if } \{i,j\} \notin E. \end{cases}$$

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- There exists an orientation so that $|\,\mathrm{Pf}(A)| = \mathrm{M}(G_{m,n})$.

Pfaffian and Determinant

Theorem (Muir)

$$\det A = (\mathrm{Pf}(A))^2.$$

$$(\det(A))^2 = \prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\left(\frac{j\pi}{2m+1}\right) + 4\cos^2\left(\frac{k\pi}{2n+1}\right)\right)^4$$

Connections and Applications to Other Fields

- **Statistical Mechanics:** Dimer model, double-dimer model, square ice model, 6-vertex model, fully packed loops configuration...
- **Probability:** Arctic Curves.
- **Graph Theory:** Bijection between tilings and perfect matchings, Temperley's correspondence.
- **Cluster Algebras:** Combinatorial interpretation of cluster variables.
- **Electrical networks:**
- ...
- **Other topics in combinatorics:** Alternating sign matrices, monotone triangles, plane partitions, lattice paths, symmetric functions...

Aztec Diamond

Theorem (Elkies, Kuperberg, Larsen and Propp 1991)

The Aztec diamond of order n has $2^{n(n+1)/2}$ *domino tilings.*

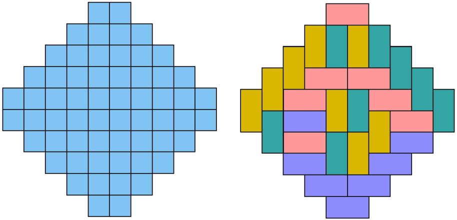


Figure: The Aztec diamond of order 5 and one of its tilings.

An Aztec Temple

“Urban renewal”

Definition

$$\mathrm{M}(G) = \sum_{\pi \in \mathcal{M}(G)} wt(\pi)$$

Lemma (Spider Lemma)

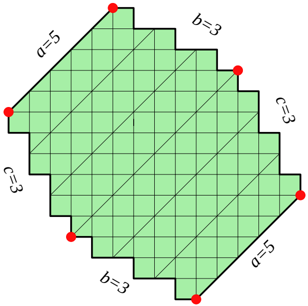
$$\mathrm{M}(G) = (xz + yt)\,\mathrm{M}(G')$$

Lemma (Vertex-spitting Lemma)

$$\mathrm{M}(G) = \mathrm{M}(G')$$

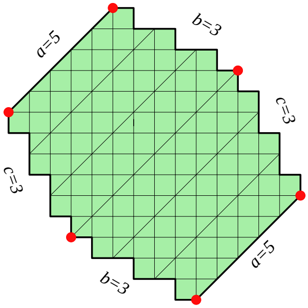
Proof using “Urban renewal”

Quasi-hexagon



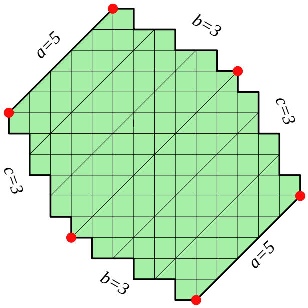
- In 1999, James Propp collected 32 open problems in enumeration of tilings.

Quasi-hexagon



- In 1999, James Propp collected 32 open problems in enumeration of tilings.
- Problem 16 on the list asks for the number of tilings of a quasi-hexagon.

Quasi-hexagon



Theorem (L. 2014)

The number of tilings of a quasi-hexagon is a power of 2 times the number of tilings of a semi-regular hexagon.

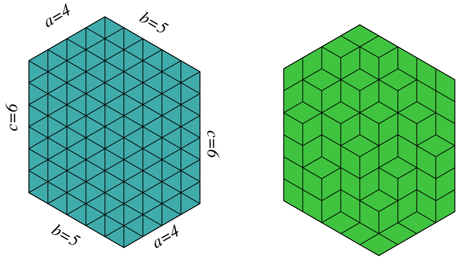
MacMahon's Formula

Theorem (MacMahon 1900)

The number of (lozenge) tilings of a semi-regular hexagon of sides a, b, c, a, b, c on the triangular lattice is

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{t=1}^{c}\frac{i+j+t-1}{i+j+t-2} = \frac{\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)\,\mathrm{H}(a+b+c)}{\mathrm{H}(a+b)\,\mathrm{H}(b+c)\,\mathrm{H}(c+a)},$$

where the hyperfactorial $\mathrm{H}(n) = 0!\cdot 1!\cdot 2!\ldots(n-1)!$.



Tilings and non-intersecting lattice paths

Bijection between lozenge tilings and non-intersecting lattice paths

Lindström-Gessel-Viennot Lemma

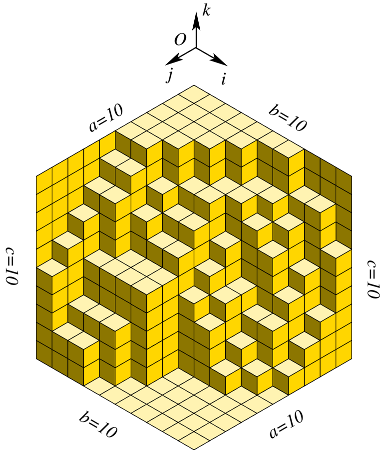
Lemma

$G = (E, V)$ is an acyclic directed graph, $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$ are disjoint subsets of V. Assume that any pair of paths connecting a_i and b_j, and a_j and b_i are intersected. The number of families of non-intersecting lattice paths connecting A and B is given by

$$|\mathcal{P}(A \to B)| = \det(p_{i,j})_{k \times k}$$

where $p_{i,j}$ is the number of paths from a_i to b_j.

Connection to Plane Partitions



MacMahon's q-formula

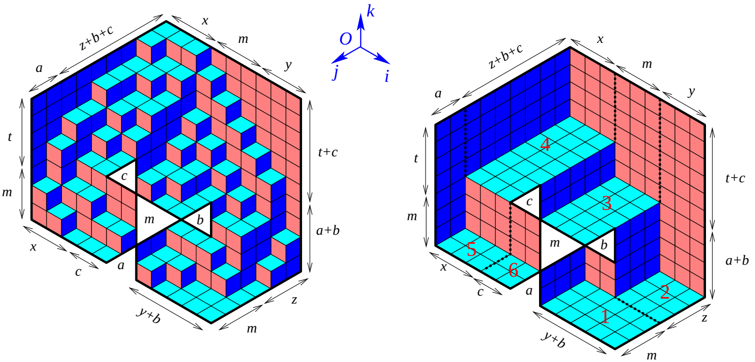
Theorem (MacMahon)

$$\sum_{\pi} q^{vol(\pi)} = \frac{\mathrm{H}_q(a)\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(a+b+c)}{\mathrm{H}_q(a+b)\,\mathrm{H}_q(b+c)\,\mathrm{H}_q(c+a)},$$

where the sum is taken over all stacks π fitting in an $a \times b \times c$ box.

- q-integer $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$
- q-factorial $[n]_q! := [1]_q \cdot [2]_q \cdot [3]_q \ldots [n]_q$
- q-hyperfactorial $\mathrm{H}_q(n) := [0]_q! \cdot [1]_q! \cdot [2]_q! \ldots [n-1]_q!$.

Generalizing MacMahon's Formula



$$\sum_{\text{stacks}} q^{\text{volume of the stack}} = ?$$

Generalizing MacMahon's Formula

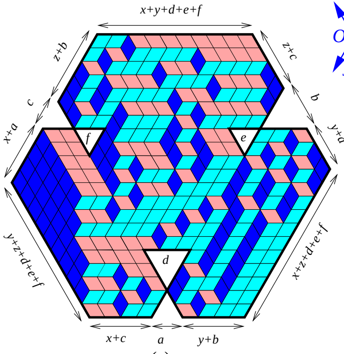
Theorem (L. 2015+)

For non-negative integers x, y, z, t, m, a, b, c

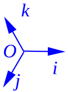
$$\sum_{\pi} q^{vol(\pi)} = \frac{\mathrm{H}_q(\Delta+x+y+z+t)}{\mathrm{H}_q(\Delta+x+y+t)\,\mathrm{H}_q(\Delta+x+y+z)}$$
$$\times \frac{\mathrm{H}_q(\Delta+x+t)\,\mathrm{H}_q(\Delta+x+y)\,\mathrm{H}_q(\Delta+y+z)\,\mathrm{H}_q(\Delta)}{\mathrm{H}_q(\Delta+z+t)\,\mathrm{H}_q(\Delta+x)\,\mathrm{H}_q(\Delta+y)}$$
$$\times \frac{\mathrm{H}_q(m+b+c+z+t)\,\mathrm{H}_q(m+a+c+x)\,\mathrm{H}_q(m+a+b+y)}{\mathrm{H}_q(m+b+y+z)\,\mathrm{H}_q(m+c+x+t)}$$
$$\times \frac{\mathrm{H}_q(c+x+t)\,\mathrm{H}_q(b+y+z)}{\mathrm{H}_q(a+c+x)\,\mathrm{H}_q(a+b+y)\,\mathrm{H}_q(b+c+z+t)}$$
$$\times \frac{\mathrm{H}_q(m)^3\,\mathrm{H}_q(a)^2\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(x)\,\mathrm{H}_q(y)\,\mathrm{H}_q(z)\,\mathrm{H}_q(t)}{\mathrm{H}_q(m+a)^2\,\mathrm{H}_q(m+b)\,\mathrm{H}_q(m+c)\,\mathrm{H}_q(x+t)\,\mathrm{H}_q(y+z)},$$

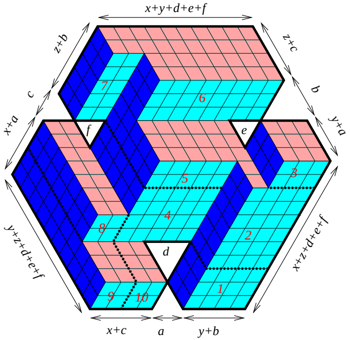
where $\Delta = m+a+b+c$.

Generalizing MacMahon's Formula



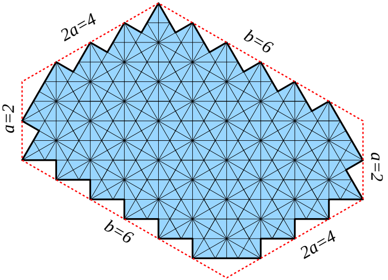
(a)

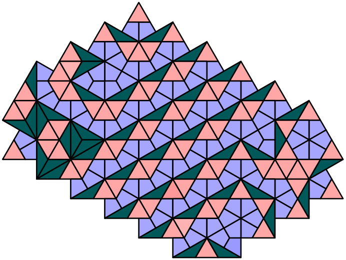




(b)

Blum's Conjecture and Hexagonal Dungeon





Theorem (Blum's (ex-)conjecture)

The number of tilings of the hexagonal dungeon of side-lengths $a, 2a, b, a, 2a, b$ $(b \geq 2a)$ is $13^{2a^2} 14^{\lfloor \frac{a^2}{2} \rfloor}$.

The conjecture was proven by Ciucu and L. (2014).

Kuo condensation

Theorem (Kuo)