Math 958–Topics in Discrete Mathematics
Spring Semester 2018
TR 09:30–10:45 in Avery Hall (AVH) 351

1 Instructor

Dr. Tri Lai
Assistant Professor
Department of Mathematics
University of Nebraska - Lincoln
Lincoln, NE 68588-0130, USA

Email: `tlai3@unl.edu`
Website: `http://www.math.unl.edu/$\sim$tlai3/`
Office: Avery Hall 339
Office Hours: By Appointment

2 Prerequisites

Math 450.

3 Contacting me

The best way to contact with me is by email, `tlai3@unl.edu`. Please put "[MATH 958]" at the beginning of the title and make sure to include your whole name in your email. Using your official UNL email to contact me is strongly recommended. My office is Avery Hall 339. My office hours are by appoinment.

4 Course Description

Bijective Combinatorics is a branch of combinatorics focusing on bijections between mathematical objects. The fact is that there many totally different objects in mathematics that are actually equinumerous, in this case, we always want to seek for a simple bijection between them. For example, the number of ways to divide that a convex $(n+2)$-gon into triangles by non-intersecting diagonals is equal to the number of full binary trees with $n+1$ leaves. Both objects are counted by the well known Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Do you know that there are more than *two hundred* (!!) mathematical objects are counted by the Catalan number? In this course, we will go over a number of such "Catalan objects" and investigate the beautiful bijections between them. One more example is the well-known theorem by Euler about integer partitions stating that *the number of ways to write a positive integer n as the sum of distinct positive integers is equal to the number of ways to write n as the sum of odd positive integers*. We will investigate very nice bijections between the two kinds of partitions by Glaisher and Sylvester. Moreover, you would like the bijection between *planar maps* (connected planar graphs embedded in the sphere considered up to continuous deformation) and decorated trees.

Even though most of the bijections in combinatorics are not hard to understand, but they are very hard to find. The most well-known example is the story about the three combinatorial objects: "*Totally symmetric self-complementary plane partitions*", "*Alternating Sign Matrices*", and "*Descending Plane Partitions*". In the 1980s, Mills–Robbins–Rumsey predicted that these objects are all enumerated by the simple product formula

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!} = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!}.$$

They also conjectured the existence of "natural" bijections between the three objects. Even though the above enumeration has been proved, the bijections between them are still unknown, this makes Mills–Robbins–Rumsey Conjecture one of the hardest most important open problems in combinatorics. We also work on an even stronger conjecture named "*Gog-Magon conjecture*" first posed by Krattenthaler. It is worth to mention that the enumeration of Alternating Sign Matrices was proved independently by Zeilberger and Kuperberg. Especially, Kuperberg's proof was based on bijection between these matrices and the"*square ice model*" (or "*six-vertex model*") in physics.

Another feature of the bijective combinatorics is finding bijective proofs for classical known results. In most cases, combinatorialist do not satisfy with complicated computational proofs of many classical theorems. They always want to find some new elegant "combinatorial proofs" for the theorems. In this course, we will investigate interesting bijective proofs of various classical theorems in linear algebra, hypergeometric series, and graph theory, just to name a few. For example, we will prove bijectively Cayley's formula for labeled trees by using Prüfer code and by using maximal directed pseudoforests, reveal that many classical determinantal identities are actually some correspondences between lattice paths, or simply explain why the number of ways to tile an $m \times n$ chessboard by dominoes is given by

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\left(4\cos^2\left(\frac{i\pi}{m+1}\right)+4\cos^2\left(\frac{j\pi}{n+1}\right)\right)^{1/4}.$$

We will also go over the beautiful proof by Schur for the well-known Rogers-Ramanujan Identities:

$$1+\sum_{k=1}^{\infty}\frac{q^{k^2}}{(1-q)(1-q^2)\dots(1-q^k)} = \prod_{i=1}^{\infty}\frac{1}{(1-q^{5i+1})(1-q^{5i+4})},$$

$$1+\sum_{k=1}^{\infty}\frac{q^{k(k+1)}}{(1-q)(1-q^2)\dots(1-q^k)} = \prod_{i=1}^{\infty}\frac{1}{(1-q^{5i+2})(1-q^{5i+3})}.$$

You would be surprised when knowing that Schur gave the second proof for these identities (after Rogers) and that Ramanujan actually did not provide any proof of the identities.

5 Homework and Grades

The grade is based on 1–2 homework sets and the final presentation.

6 Incompletes

A grade of "Incomplete" may be considered if all but a small portion of the class has been successfully completed, but the student in question is prevented from completing the course by a severe, unexpected, and documented event. Students who are simply behind in their work should consider dropping the course.

7 ADA Statement

Students with disabilities are encouraged to contact the instructor for confidential discussion of their individual needs for academic accommodation. It is the policy of the University of Nebraska-Lincoln to provide flexible and individualized accommodation to students with documented disabilities that may affect their ability to fully participate in course activities or meet course requirements. To receive accommodation services, students must be registered with the Services for Students with Disabilities (SSD) office, 132 Canfield Administration, 472-3787 voice or TTY.

8 Special Dates

January 19, 2018 (Friday):	last day to withdraw from this course and not have it appear on your transcript.
March 2, 2018 (Friday):	last day to change your grade option to or from Pass/No Pass.
April 6, 2018 (Friday):	last day to drop this course and receive a grade of W. (No permission required.) After this date you cannot drop.

9 Departmental Grading Appeals Policy

Students who believe their academic evaluation has been prejudiced or capricious have recourse for appeals to (in order) the instructor, the departmental chair, the departmental appeals committee, and the college appeals committee.

10 Course Evaluation

The Department of Mathematics Course Evaluation Form will be available through your Canvas account during the last two weeks of class. You will get an email when the form becomes available. Evaluations are anonymous and instructors do not see any of the responses until after final grades have been submitted. Evaluations are important—the department uses evaluations to improve instruction. Please complete the evaluation and take the time to do so thoughtfully.

11 Rough schedule:

We will spend the first part of the course to review several basic topics, such as Generating Function and Enumeration. The other topics in the course are pretty independent and can be presented in any order. Here is the a tentative schedule for the course, each topic would take 2–3 weeks.

1. Part 1: Introduction and Review.
2. Part 2: Catalan Objects: Dyck Paths, Triangulations, Full Binary Trees, Permutations etc.
3. Part 3: Bijections on graphs: Cayley's Tree Theorem, Bijections on maps, Temperley's Trick, Matrix-Tree Theorem.
4. Part 4: Bijective Partition Theory.
5. Part 5: Bijective Proofs for Classical Theorems in Linear Algebra.
6. Part 6: Tilings and related objects: Tilings, Perfect matchings, Non-intersecting lattice paths, Fully packed loops, Square Ice, Alternating sign matrices, Plane partitions.

7. Part 7: (If time allows) Symmetric functions, Young tableaux, Robinson–Schensted–Knuth (RSK) correspondence.

12 Recommended books and papers for reading

A textbook is not required for this course. However, here is the list of recommended books and papers:

1. '*Enumerative Combinatorics*' (two volumes) by Richard Stanley.
2. '*Bijective Combinatorics*' by Nicolas Loehr.
3. '*Combinatorial Analysis*' by Percy MacMahon.
4. '*Perfect Matchings and Applications*' (lecture notes) by Mihai Ciucu.
   Link `http://gcoe-mi.jp/temp/publish/b484e6492e396aff6512ae7a2bdf5560.pdf`.
5. '*Catalan Numbers*' by Richard Stanley.
6. '*Proof that Really Count*' by Arthur Benjamin and Jennifer Quinn.
7. Bijective-proof Problems list by Stanley (`http://www-math.mit.edu/~rstan/bij.pdf`).
8. Catalan Webpage by Igor Pak: `http://www.math.ucla.edu/~pak/lectures/Cat/pakcat.htm`.

13 Recommended topics for final presentation

Our final presentation will be held during the last week of the class and the final week. The topic-selecting process will be on a 'first-come-first-served' basis via email. Each presentation should be about 20 minutes, plus about $2--5$ minutes for questions.

Here is the list of recommended topics for the final presentation. *Presenting a topic outside the list is welcome, however, you should email me to get an approval at least 1 week in advance.* In general, bijective proofs of some classical results or bijections between combinatorial objects are acceptable.

1. *Variations of Catalan Numbers (pick one variation), including q-Catalan Number , q,t-Catalan Numbers, two types of super Catalan numbers etc.*
   (a) J. Furlinger and J. Hofbauer, *q-Catalan Numbers*, J. Combin. Theory Ser. A **40** (1985), 248–264.
   `https://www.sciencedirect.com/science/article/pii/0097316585900895`
   (b) A. N. Fan, T. Mansour, and S. Pang, *Elements of the sets enumerated by super-Catalan numbers*
   (`http://math.haifa.ac.il/toufik/enumerative/supercat.pdf`).
   (c) E. Allen and I. Gheorghiciuc, *A weighted interpretation for the super Catalan numbers*, `https://arxiv.org/pdf/1403.5246.pdf`
   (d) K. Lee, L. Li, and N. A. Loehr, *Combinatorics of certain higher q,t-Catalan polynomials: chains, joint symmetry, and the Garsia-Haiman formula*, J. Alg. Comb. **39** (2014), 749–781
   (`https://arxiv.org/abs/1211.2191`).

(e) A.M. Garsia and M. Haiman, *A remarkable q,t-Catalan sequence and q-Lagrange inversion*, J. Algebraic Combin. **5**(3) (1996), pp 191–244. (https://math.berkeley.edu/~mhaiman/ftp/remarkable-qtcat/qtcat.pdf).

(f) Jim Haglund's book "*The q, t-Catalan Numbers and the Space of Diagonal Harmonics*" (https://www.math.upenn.edu/~jhaglund/books/qtcat.pdf).

2. *Other Catalan Objects: 321–avoiding Permutations , Kepler Towers , etc.*

(a) S. Billey, W. Jockusch and R. Stanley, *Some Combinatorial Properties of Schubert Polynomials*, J. Algebraic Combinatorics 2 (1993), Issue 4, 345–374. https://link.springer.com/content/pdf/10.1023/A:1022419800503.pdf

(b) C. Krattenthaler, *Permutations with restricted patterns and Dyck paths*, Adv. Appl Math. 27 (2001), no. 2–3, 510–530. https://arxiv.org/abs/math/0002200

(c) T. Mansour, E. Y. P. Deng and R. R. X. Du, *Dyck paths and restricted permutations*, Discrete Appl. Math. 154 (2006), no. 11, 1593–1605. https://www.sciencedirect.com/science/article/pii/S0166218X06000692

(d) Stanley's list of Catalan Objects (see above link)

(e) X. Viennot, *Kepler towers, Catalan numbers and Strahler distribution*, (http://www.xavierviennot.org/xavier/articles_files/Kepler-Towers2.pdf).

(f) D. Knuth, *Three Catalan bijections*, (https://www-cs-faculty.stanford.edu/~knuth/papers/tcb.ps.gz).

3. *Kuo Condensation*

(a) E. Kuo, *Applications of Graphical Condensation for Enumerating Matchings and Tilings*, Theoretical Computer Science, Vol. 319 (1–3) (2004), pp. 29–57. (https://arxiv.org/abs/math/0304090).

(b) M. Ciucu, *A generalization of Kuo condensation*, Journal of Combinatorial Theory, Series A, Volume 134, August 2015, Pages 221–241. (https://arxiv.org/abs/1404.5003).

(c) T. Lai and G. Musiker, *Beyond Aztec Castles: Toric Cascades in the dP3 Quiver*, Communications in Mathematical Physics, Volume 356, Issue 3 (2017), pp. 823–881. (https://arxiv.org/abs/1512.00507)

4. *Aztec diamond theorem.*

(a) N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, *Alternating–sign matrices and domino tilings*, Journal of Algebraic Combinatorics. 1 (2): 111–132. (https://arxiv.org/abs/math/9201305).

(b) Eu and Fu, *A Simple Proof of the Aztec Diamond Theorem*, Electronic Journal of Combinatorics 12 (2005), #R18. (https://arxiv.org/abs/math/0412041).

(c) Proof using Urban Renewal, see Ciucu's lecture notes.

5. *Dodgson Condensation*

(a) Wikipedia link https://en.wikipedia.org/wiki/Dodgson_condensation.

(b) T. Tao, *Dodgson condensation from Schur complementation*, https://terrytao.wordpress.com/2017/08/28/dodgson-condensation-from-schur-complementation/

(c) Doron Zeilberger, *Dodgson's Determinant-Evaluation Rule proved by Two-Timing Men and Women*,
`https://arxiv.org/abs/math/9808079`.

6. *Shapiro's Catalan convolution*

(a) G. Andrews, *On Shapiro's Catalan convolution*, Advances in Applied Mathematics Volume 46, Issues 1–4, January 2011, Pages 15–24.
`https://www.sciencedirect.com/science/article/pii/S0196885810001028`

(b) A Bijective Proof of Shapiro's Catalan Convolution, Electronic Journal of Combinatorics 21(2) (2014), #P2.4.
(`http://www.combinatorics.org/ojs/index.php/eljc/article/view/v21i2p42/pdf`).

(c) Gabor V. Nagy, *A combinatorial proof of Shapiro's Catalan convolution*, Advances in Applied Mathematics 49 (2012) 391–396.
`https://arxiv.org/abs/1204.5923`.

7. *Generalizations and applications of Temperley's bijection.*

(a) Richard W. Kenyon, James G. Propp, David B. Wilson, *Trees and Matchings*, Electronic Journal of Combinatorics **7** (2000), #R25.
`http://www.combinatorics.org/ojs/index.php/eljc/article/view/v7i1r25`

(b) M. Ciucu, *A visual proof of a result of Knuth on spanning trees of Aztec diamonds in the case of odd order*, Discrete Math. 307 (2007), 1957–1960.
Available on Ciucu's website: `http://pages.iu.edu/~mciucu/list.html`

(c) D. E. Knuth, *Aztec diamonds, checkerboard graphs, and spanning trees*, J. Algebraic Combin. 6 (1997), 253–257.
`https://arxiv.org/abs/math/9501234`.

8. *Hook-length formula.*

(a) J. S. Frame, G. de B. Robinson and R. M. Thrall, *The hook graphs of the symmetric group*, Canad. J. Math. 6 (1954), 316–324.
`https://cms.math.ca/10.4153/CJM-1954-030-1`.

(b) J-C Novelli, I. Pak and A.V. Stoyanovsky, *A direct bijective proof of the hook-length formula*, Discrete Mathematics and Theoretical Computer Science, vol. 1 (1997), 53–67.
Available on Igor Pak, webpage `http://www.math.ucla.edu/~pak/papers/research.htm`.

(c) Alejandro Morales, Igor Pak, and Greta Panova, *Hook formulas for skew shapes, parts I,II, III.*
Available on Igor Pak, webpage `http://www.math.ucla.edu/~pak/papers/research.htm`.