Section 7.2: QUADRATIC FORMS

Defn: A quadratic form on $\mathbb{R}^n$ is a function Q defined on $\mathbb{R}^n$ whose value at a vector x in $\mathbb{R}^n$ can be computed by an expression of the form

$$Q(x) = x^T A x,$$

where A is an $n \times n$ symmetric matrix.

The matrix A is called the matrix of the quadratic form.

Example: $Q(x) = x^T I x = \|x\|^2.$

Example: Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Comput $x^T A x$

a. $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ b. $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

Solution:

a) $x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4x_1 \\ 3x_2 \end{bmatrix} = 4x_1^2 + 3x_2^2.$

b) $x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{bmatrix} = 3x_1^2 - 4x_1 x_2 + 7x_2^2.$

Obs: $-4 = (-2) + (-2)$ $\quad 3 = 3 \quad 7 = 7$

$\underset{\text{coefficient of } x_1x_2}{-4} = \ldots \underset{2(a_{1,2})}{(-2)}$; $\underset{\text{coefficient of } x_1^2}{3} = \underset{a_{11}}{3}$; $\underset{\text{coefficient of } x_2^2}{7} = \underset{a_{22}}{7}$

Example: For $x \in \mathbb{R}^3$, let

$$Q(x) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3.$$

Write this quadratic form as x^TAx.

Solution: Coefficients of x_1^2, x_2^2, x_3^2 are diagonal entries a_{11}, a_{22}, a_{33}.

Coefficient of x_ix_j $(i \neq j)$ is $a_{i,j} + a_{j,i} = 2a_{i,j}$ (A is symmetric).

$$\Rightarrow \quad a_{1,2} = a_{2,1} = -\frac{1}{2}$$

$$a_{2,3} = a_{3,2} = 4$$

$$a_{3,1} = a_{1,3} = 0$$

$$\Rightarrow Q(x) = [x_1\ x_2\ x_3] \begin{bmatrix} 5 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Example:

Assume that the matrix A of the quadratic form $Q(x)$ is $A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$.

Find the value of $Q(x)$ for

$$x = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \text{ and } \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Solution:

$$Q(x) = x_1^2 - 8x_1x_2 - 5x_2^2.$$

$$\Rightarrow Q\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right) = 9 + 24 - 5 = 28$$

$$Q\left(\begin{bmatrix} 2 \\ -2 \end{bmatrix}\right) = 4 + 32 - 20 = 16$$

$$Q\left(\begin{bmatrix} 4 \\ 5 \end{bmatrix}\right) = 16 - 160 - 125 = 51.$$

Defn: If x presents a variable vector in $\mathbb{R}^n$, then a **change of variable** is an equation:

$$x = Py \quad (\text{i.e. } y = P^{-1}x)$$

for some invertible matrix P, and y is a new variable vector in $\mathbb{R}^n$.

Recall: $y = [x]_{\mathcal{P}}$, where $\mathcal{P}$ is a basis of $\mathbb{R}^n$ formed by cols in P.

Obs: 1) $x^TAx = (Py)^T A (Py)$

$$= y^T(P^TAP)y$$

$\Rightarrow$ $P^T A P$ is a new matrix of the quadratic form $Q(x) = x^T A x$.

2) Since A is symmetric, there is an orthogonal P such that

$$P^T A P = D \text{, a diagonal matrix.}$$

$$\Rightarrow \quad x^T A x = y^T D y.$$

Example: Make a change of variable that transform $Q(x) = x_1^2 - 8x_1x_2 - 5x_2^2$ into a quadratic form $R(y)$ with no term $y_1 y_2$.

Solution: The matrix of Q is

$$A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$$

First, we orthogonally diagonalize A.

Eigenvalues $\lambda = 3, -7$.

$$E_3 = \text{Span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}, \quad E_{-7} = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

Normalize eigenvectors

$$v_1 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\Rightarrow \quad P = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$$

$$A = PDP^{-1} \Rightarrow D = P^{-1}AP = P^TAP$$

$\Rightarrow$ The suitable change of variable is

$$x = Py$$

or

$$y = P^T x = \begin{bmatrix} 2/\sqrt{5} & -\frac{1}{\sqrt{5}} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \quad y_1 = \frac{2x_1 - x_2}{\sqrt{5}}$$

$$y_2 = \frac{x_1 + 2x_2}{\sqrt{5}}.$$

$$Q(x) = x_1^2 - 8x_1x_2 - 5x_2^2 = 3y_1^2 - 7y_2^2.$$

Theorem: The Principal Axes Theorem

Let A be an $n \times n$ symmetric matrix. Then there is an orthogonal change of variable, $x = Py$, that transforms the quadratic x^TAx into a quadratic form y^TDy with no cross-product term

Defn: Columns of P are called the principal axes of the quadratic form $x^T A x$.

A Geometric View of Principal Axes

Suppose $Q(x) = x^T A x$, where A is an invertible 2×2 symmetric matrix, and let c be a constant. It can be shown that the set of all $x \in \mathbb{R}^2$ s.t.

$$x^T A x = c$$

either corresponds to an ellipse, a hyperbola, two intersect lines, or a single point, or constants no point at all.

(Read pages 405 - 407)

<u>Definition:</u> A quadratic form Q is:

a) positive definite if $Q(x) > 0$ for all $x \neq 0$

b) negative definite if $Q(x) < 0$ for all $x \neq 0$

c) indefinite if $Q(x)$ assumes both positive and negative values.

d) positive semidefinite if $Q(x) \geq 0$ for all x.

e) negative semidefinite if $Q(x) \leq 0$ for all x

<u>Theorem 5:</u> Let A be an $n \times n$ symmetric matrix. Then the quadratic form $x^T A x$ is:

a. positive definite iff eigenvalues of A are all positive.

b. negative definite iff eigenvalues of A are all negative, or

c. indefinite iff A has both positive and negative eigenvalues.

<u>Proof</u>: By the Principal Axes Theorem, there exists an orthogonal change of variable $x = Py$ s.t.

$$Q(x) = x^T A x = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A.

Since P is invertibe, there is a one-to-one correspondence between all nonzero