Section 7.1: DIAGONALIZATION OF SYMMETRIC MATRICES

A symmetric matrix is a matrix A such that $A^T = A$.

$\iff$ $a_{i,j} = a_{j,i} \quad \forall i,j$.

$\iff$ entries are symmetric over the main diagonal.

Obs: A symmetric matrix must be a square matrix.

Example 1: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 5 & -1 \\ 5 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & -1 & 5 \\ 0 & 1 & 3 & 2 \\ -1 & 3 & 0 & 1 \\ 5 & 2 & 1 & 4 \end{bmatrix}$

Example 2: Diagonalize the matrix A (if possible)

$$A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

Solution:

$$\det(A - \lambda I) = \det \begin{bmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{bmatrix} \begin{matrix} 6-\lambda & -2 \\ -2 & 6-\lambda \\ -1 & -1 \end{matrix}$$

$$= (6-\lambda)^2(5-\lambda) - 2 - 2 - (6-\lambda) - (6-\lambda) - 4(5-\lambda)$$

$$= (6-\lambda)^2(5-\lambda) - 2(6-\lambda) - 4(6-\lambda)$$

$$= (6-\lambda)\left((6-\lambda)(5-\lambda) - 2 - 4\right)$$

$$= (6-\lambda)(\lambda^2 - 11\lambda + 24)$$
$$= (6-\lambda)(\lambda-8)(\lambda-3)$$

$\Rightarrow$ The characteristic eq. is:

$$(6-\lambda)(\lambda-8)(\lambda-3) = 0$$

$\Rightarrow$ The eigenvalues are 3, 6, 8.

$\boxed{\lambda = 3}$ $\quad A - 3I = \begin{bmatrix} 3 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

The augmented matrix is

$$\begin{bmatrix} 3 & -2 & -1 & 0 \\ -2 & 3 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 2 & 0 \\ -2 & 3 & -1 & 0 \\ 3 & -2 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & -1 & 2 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & -5 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_3$
$x_2 = x_3$
x_3 is free

$$\Rightarrow E_3 = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} : v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$\boxed{\lambda=6}$ The augmented matrix is

$$\begin{bmatrix} 0 & -2 & -1 & 0 \\ -2 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = -\frac{1}{2} x_3 \\ x_2 = -\frac{1}{2} x_3 \\ x_3 \text{ is free} \end{array}$$

$$\Rightarrow E_6 = \text{Span}\left\{ \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}, \text{ take } v_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

$\boxed{\lambda=8}$ The augmented matrix is

$$\begin{bmatrix} -2 & -2 & -1 & 0 \\ -2 & -2 & -1 & 0 \\ -1 & -1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = -x_2 \\ x_2 \text{ is free} \\ x_3 = 0 \end{array}$$

$\Rightarrow E_8 = \text{Span}\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$, take $v_3 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$.

— * —

$\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$

$\Rightarrow$ A is digonalizable, and

Moreover, $\{v_1, v_2, v_3\}$ is orthogonal. Since an orthonormal basis is useful, $\Rightarrow$ normalize v_1, v_2, v_3 as

$$u_1 = \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}, \quad u_2 = \begin{bmatrix}-1/\sqrt{6}\\-1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix}, \text{ and } u_3 = \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}.$$

Take

$$P = \begin{bmatrix}1/\sqrt{3} & -1/\sqrt{6} & -\frac{1}{\sqrt{2}}\\1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2}\\1/\sqrt{3} & 2/\sqrt{6} & 0\end{bmatrix} \text{ and } D = \begin{bmatrix}3 & 0 & 0\\0 & 6 & 0\\0 & 0 & 8\end{bmatrix}.$$

<u>Theorem 1</u>: If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

<u>Proof</u>: Let v_1 and v_2 are eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2.

$\boxed{\text{NTS} \quad v_1 \cdot v_2 = 0}$

We have $\lambda_1 v_1 \cdot v_2 = (\lambda_1 v_1)^T v_2 = (A v_1)^T v_2$

$$= (v_1^T A^T) v_2 = v_1^T (A v_2) \quad (\text{Since } A^T = A)$$
$$= v_1^T (\lambda_2 v_2)$$
$$= \lambda_2 \, v_1 \circ v_2 .$$

Since $\lambda_1 \neq \lambda_2$, $v_1 \circ v_2$ must be 0. $\square$

<u>Defn</u>: An $n \times n$ matrix A is orthogonally diagonalizable if there is an orthonormal matrix P and a diagonal matrix A s.t.
$$A = PDP^{-1} \; (= PDP^T).$$

<u>Theorem 2</u>: An $n \times n$ matrix A is orthogonally diagonalizable iff A is symmetric.

<u>Example</u>: Orthogonally diagonalize $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

<u>Solution</u>:
$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix} \begin{matrix} 3-\lambda & -2 \\ -2 & 6-\lambda \\ 4 & 2 \end{matrix}$$

$$= (3-\lambda)^2 (6-\lambda) - 16 - 16 - 16(6-\lambda) - 4(3-\lambda)$$

$$= -\lambda^3 + 12\lambda^2 - 21\lambda - 98 = -(\lambda - 7)^2 (\lambda + 2)$$

$\Rightarrow$ Eigenvalues are -2 and 7.

$\boxed{\lambda=-2}$ The augmented matrix is

$$\begin{bmatrix} 5 & -2 & 4 & 0 \\ -2 & 8 & 2 & 0 \\ 4 & 2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & -2 & 4 & 0 \\ -1 & 4 & 1 & 0 \\ 4 & 2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & 1 & 0 \\ 4 & 2 & 5 & 0 \\ 5 & -2 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 4 & 1 & 0 \\ 0 & 18 & 9 & 0 \\ 0 & 18 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = -\tfrac{1}{2}x_3 \\ x_3 \text{ is free} \end{cases}$$

$$\Rightarrow E_{-2} = \text{Span}\left\{\begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}\right\}, \text{ take } v_1 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}.$$

$\boxed{\lambda=7}$ The augmented matrix is

$$\begin{bmatrix} -4 & -2 & 4 & 0 \\ -2 & -1 & 2 & 0 \\ 4 & 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} -4 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad x_1 = -\tfrac{1}{2}x_2 + x_3$$

x_2, x_3 are free

$$\Rightarrow E_7 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Take $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$.

We have $v_1 \perp v_2, v_3$, But $v_2 \not\perp v_3$.

Replace v_3 by $v_3' = v_2 - \dfrac{v_3 \cdot v_2}{v_2 \cdot v_2} v_2$. (Gram-Schmidt)

$$v_3' = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1/2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}.$$

Next, normalize $\{v_1, v_2, v_3'\}$:

$$u_1 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad u_3 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}.$$

Thus

$$P = \begin{bmatrix} -2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -1/3 & 0 & 4/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & 1/\sqrt{18} \end{bmatrix} \text{ and } D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

*

The set of eigenvalues of a matrix A are sometimes called the <u>spectrum</u> of A

Theorem 3: The Spectral Theorem for Symmetric Matrices

A $n \times n$ symmetric A has the following properties:

a) A has n eigenvalues (counting multiplicities)

b) $\dim E_\lambda$ = multiplicity of λ

c) Eigenspaces are mutually orthogonal.

d) A is orthogonally diagonalizable.

— * —

Assume A has