# Section 6.4: GRAM-SCHMIDT PROCESS

Question: How to find an orthogonal basis of a nonzero subspace of $\mathbb{R}^n$?

Example: Let $W = \text{Span}\{x_1, x_2\}$, where

$$x_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Construct an orthogonal basis $\{v_1, v_2\}$ of $W$.

Solution:

Decompose $x_2 = p + (x_2 - p)$ where $p$ is the projection of $x_2$ on $x_1$ (or on $V = \text{Span}\{x_1\}$), and $x_2 - p$ is the orthogonal component of $x_2$ to $V$.

Let $v_1 = x_1$ and $v_2 = x_2 - p = x_2 - \dfrac{x_2 \cdot x_1}{x_1 \cdot x_1} x_1$

$$\Rightarrow v_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

Then $\{v_1, v_2\}$ is an orthogonal set of nonzero vectors in $W$. $\Rightarrow$ $\{v_1, v_2\}$ is independent. Since $\dim W = 2$, $\{v_1, v_2\}$ is a basis for $W$.

Example: Let

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Then $\{x_1, x_2, x_3\}$

is linearly independent. $\Rightarrow \{x_1, x_2, x_3\}$ is a basis for a subspace $W$ of $\mathbb{R}^4$ ($\dim W = 3$).

Construct an orthogonal basis for $W$.

Solution:

Step 1: Let $v_1 = x_1$ and $W_1 = \text{Span}\{x_1\} = \text{Span}\{v_1\}$.

Step 2:

$$v_2 := x_2 - \text{proj}_{W_1} x_2$$

$$= x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 \qquad (x_1 = v_1)$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}.$$

We have $v_2$ is the component of $x_2$ orthogonal to $x_1 \Rightarrow \{v_1, v_2\}$ is a basis for $W_2 = \text{Span}\{x_1, x_2\}$ [see the previous example].

Step 2': Clean up (optional).

$$v_2' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \{v_1, v_2'\} \text{ is also an orthogonal}$$

basis for $W_2$.

Step 3: $$\text{proj}_{W_2} x_3 = \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x_3 \cdot v_2'}{v_2' \cdot v_2'} v_2'$$

$$= \frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{2}{12}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\2/3\\2/3\\2/3\end{bmatrix}.$$

Then $v_3 = x_3 - \text{proj}_{W_2} x_3$ is the component of $x_3$ orthogonal to $W_2$.

$$v_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \begin{bmatrix}0\\2/3\\2/3\\2/3\end{bmatrix} = \begin{bmatrix}0\\-2/3\\1/3\\1/3\end{bmatrix},$$

$\Rightarrow \{v_1, v_2', v_3\}$ is an orthogonal basis for $W$.

**Theorem 11:** The Gram-Schmidt Process

Given a basis $\{x_1, \dots, x_p\}$ for a nonzero subspace $W$ of $\mathbb{R}^n$. Define

$$v_1 = w_1$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$\dots$$

$$v_p = x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \dots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1}.$$

Then $\{v_1, \dots, v_p\}$ is an orthogonal basis for $W$.
In addition

$$\text{Span}\{v_1, \dots, v_k\} = \text{Span}\{x_1, \dots, x_k\} \quad 1 \le k \le p$$

<u>Obs</u>: We can construct an orthonormal basis from an orthogonal basis.

$$\{v_1, v_2, \ldots, v_K\} \to \left\{\frac{1}{\|v_1\|} v_1, \frac{1}{\|v_2\|} v_2, \ldots, \frac{1}{\|v_K\|} v_K\right\}.$$

<u>Theorem 12</u>: The QR Factorization.

If $A$ is an $m \times n$ matrix with linearly independent columns, then $A$ can be factored as

$$A = QR,$$

where $Q$ is an $m \times n$ matrix whose columns form an orthonormal basis for $\mathrm{Col}\,A$ an $R$ is an $n \times n$ upper triangular invertible matrix with positive diagonal entries.

<u>Proof</u>: The columns of $A$ form a basis $\{x_1, \ldots, x_n\}$ for $\mathrm{Col}\,A$. Construct an orthonormal basis $\{u_1, \ldots, u_n\}$ for $W = \mathrm{Col}\,A$ (by Gram-Schmidt process).

Let $Q = [u_1\ u_2 \ldots u_n]$.

For $k = 1, 2, \ldots, n$, $x_k \in \mathrm{Span}\{x_1, \ldots, x_k\} = \mathrm{Span}\{u_1, \ldots, u_k\}$

So there are constant $r_{1k}, \ldots, r_{kk}$, s.t.

$$x_k = r_{1k} u_1 + \cdots + r_{kk} u_k + 0 \cdot u_{k+1} + \cdots + 0 \cdot u_n$$

We may assume that $r_{kk} \geq 0$ (if $r_{kk} < 0$, we multiply both $r_{KK}$ and $u_K$ by $-1$).

$\Rightarrow$ $x_K$ is a linear combination of the columns in $Q$ using weights as the entries in the vector

$$r_K = \begin{bmatrix} r_{1K} \\ \vdots \\ r_{KK} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

That is, $x_K = Q r_K$, for $K = 1, \dots, n$.
Let $R = [r_1 \; r_2 \dots r_n]$. Then

$$A = [x_1 \dots x_n] = [Qr_1 \dots Qr_n] = QR.$$

$R$ is invertible because colums of $A$ are linearly independent.
(Otherwise $c_1 r_1 + \dots + c_n r_n = 0$ for not all zero $c_1, \dots, c_n$ $\Rightarrow$ $c_1 Q r_1 + \dots + c_1 Q r_n = 0$
$\Rightarrow$ $c_1 x_1 + \dots + c_1 x_n = 0$, a contradiction)

$\square$.

<u>Example</u>: Find a QR factorization of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

<u>Solution</u>: By previous example

$$\text{Col } A = \text{Span}\{x_1, x_2, x_3\}$$

and

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

form an orthogonal basis for $W = \text{Col}\, A$.

Normalizing $v_1, v_2, v_3$, we have

$$Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}$$

Take $R = Q^T A$.

$(A = QR \Rightarrow Q^T A = Q^T Q R = IR = R)$.

$$R = \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}.$$