Section 6.3. ORTHOGONAL PROJECTIONS

Example: Let $\{u_1, \dots, u_5\}$ be an orthogonal basis of $\mathbb{R}^5$ and let

$$y = c_1 u_1 + \cdots + c_5 u_5.$$

Consider the subspace $W = \text{Span}\{u_1, u_2\}$. Write y as the sum of a vector z_1 in W and a vector z_2 in $W^\perp$.

Solution:

Write $$y = \underbrace{(c_1 u_1 + c_2 u_2)}_{z_1} + \underbrace{(c_3 u_3 + c_4 u_4 + c_5 u_5)}_{z_2}$$

We have

$$z_1 \in \text{Span}\{u_1, u_2\} = W$$

and

$$z_2 \in \text{Span}\{u_3, u_4, u_5\}.$$

To show $z_2 \perp W$, it suffices to show $z_2 \perp u_1$ and u_2.

$$\begin{aligned} z_2 \cdot u_1 &= c_3 (u_3 \cdot u_1) + c_4 (u_4 \cdot u_1) + c_5 (u_5 \cdot u_1) \\ &= 0 + 0 + 0 \\ &= 0 \end{aligned}$$

Similar, we have $z_2 \cdot u_2 = 0$.

$$\Rightarrow z_2 \in W^\perp. \qquad \square$$

The decomposition $y = z_1 + z_2$ can be computed without having an orthogonal basis.

Theorem 8: The Orthogonal Decomposition Theorem

Let $W \triangleleft \mathbb{R}^n$. Then each vector y in $\mathbb{R}^n$ can be written uniquely in the form

$$y = \hat{y} + z,$$

where

$$\hat{y} \in W \quad \text{and} \quad z \in W^{\perp}.$$

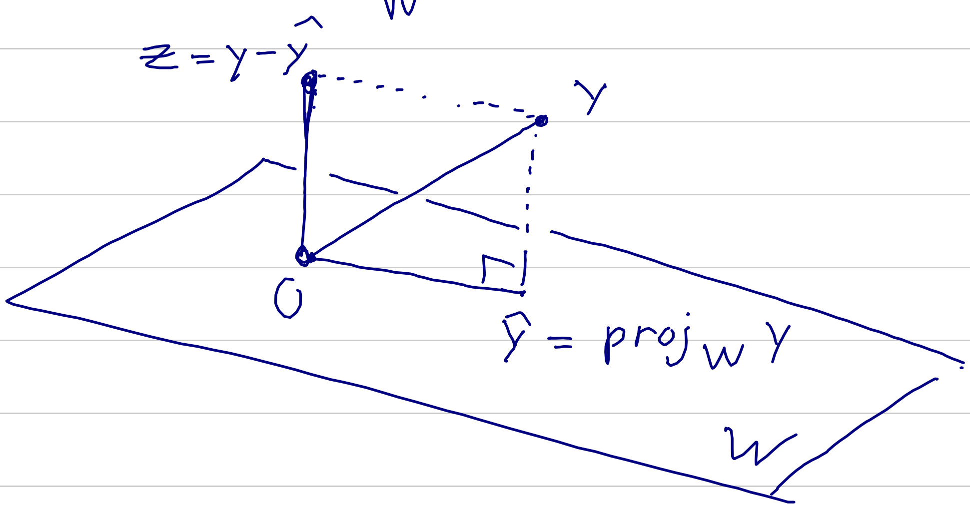
In fact, if $\{u_1, \dots, u_p\}$ is any orthogonal basis of W, then

$$\hat{y} = \frac{y \cdot u_1}{\|u_1\|^2} u_1 + \dots + \frac{y \cdot u_p}{\|u_p\|^2} u_p$$

and $z = y - \hat{y}$.

Vector $\hat{y}$ is called the orthogonal projection of y onto W, and denoted by

$$\text{proj}_W y$$



Proof:

Let $\{u_1, \dots, u_p\}$ be an orthogonal basis for W.

Let $\hat{y} = \frac{y \cdot u_1}{\|u_1\|^2} u_1 + \cdots + \frac{y \cdot u_p}{\|u_p\|^2} u_p$

$\Rightarrow \hat{y} \in W = \text{Span}\{u_1, \ldots, u_p\}$. Let $z = y - \hat{y}$.

Since $u_1 \perp u_2, \ldots, u_p$, we have

$$z \cdot u_1 = (y - \hat{y}) \cdot u_1 = y \cdot u_1 - \frac{y \cdot u_1}{\|u_1\|^2}(u_1 \cdot u_1)$$

$$= y \cdot u_1 - y \cdot u_1 = 0.$$

$\Rightarrow z \perp u_1$.

Similarly, $z \perp u_2, \ldots, u_p$. $\Rightarrow z \in W^{\perp}$.

To show that the decomposition $y = \hat{y} + z$ is unique, we suppose y can also be written as

$$y = \hat{y}_1 + z_1,$$

where $\hat{y}_1 \in W$ and $z_1 \in W^{\perp}$.

$$\Rightarrow \quad \hat{y} + z = \hat{y}_1 + z_1 \iff \underbrace{\hat{y} - \hat{y}_1}_{\in W} = \underbrace{z_1 - z}_{\in W^{\perp}}$$

$\Rightarrow$ $v = \hat{y} - \hat{y}_1$ is in W and $W^{\perp}$

$\Rightarrow$ $v \cdot v = 0 \Rightarrow v = 0$.

$\Rightarrow$ $\hat{y} = \hat{y}_1$ and $z = z_1$. $\square$

Obs

Thus the projection $\text{proj}_W y$ does not depend on the choice of the orthogonal basis $\{u_1, \ldots, u_p\}$ of W.

Example: $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Observe that $\{u_1, u_2\}$ is an orthogonal basis for $W = \operatorname{Span}\{u_1, u_2\}$. Find $\operatorname{proj}_W y$.

Solution:

$$\operatorname{proj}_W y = \hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= \frac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{3}{10} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}.$$

Obs: If $y \in W$, then $\operatorname{proj}_W y = y$.

Thm 9: The Best Approximation Theorem

Let $W \trianglelefteq \mathbb{R}^n$, Let y be any vector in $\mathbb{R}^n$, and $\hat{y} = \operatorname{proj}_W y$. Then $\hat{y}$ is the closest point in W to y. That is

$$\|y - \hat{y}\| \leq \|y - v\|, \text{ for all } v \in W.$$

Vector $\hat{y}$ is called the best approximation to y by elements of W.

Proof: Take $v \in W \Rightarrow \hat{y} - v \in W$.
By Orthogonal Decomposition Theorem, $y - \hat{y} \perp W$
so $y - \hat{y} \perp \hat{y} - v$.

Since, $y - v = (y - \hat{y}) + (\hat{y} - v)$

$$\Rightarrow \quad \|y - v\|^2 = \|y - \hat{y}\|^2 + \|\hat{y} - v\|^2 \geq \|y - \hat{y}\|^2$$

$$\Rightarrow \quad \|y - v\| \geq \|y - \hat{y}\|.$$

The equality holds iff $\|\hat{y} - v\| = 0 \Leftrightarrow \hat{y} = v$. $\square$

Defn: The distance between a vector y in $\mathbb{R}^n$ and a subspace $W \lhd \mathbb{R}^n$ is the distance from y to the nearest point in W.

Example: $y = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}$, $u_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

$W = \text{Span}\{u_1, u_2\}$. Find $\text{dist}(y, W)$.

Solution: $\text{dist}(y, W) = \|y - \hat{y}\|$.

Since $\{u_1, u_2\}$ is an orthogonal basis for W,

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{15}{30} u_1 - \frac{21}{6} u_2$$

$$= \frac{1}{2}\begin{bmatrix}5\\-2\\1\end{bmatrix} - \frac{7}{2}\begin{bmatrix}1\\2\\-1\end{bmatrix} = \begin{bmatrix}-1\\-8\\4\end{bmatrix}.$$

$\Rightarrow$

$$y - \hat{y} = \begin{bmatrix}-1\\-5\\10\end{bmatrix} - \begin{bmatrix}-1\\-8\\4\end{bmatrix} = \begin{bmatrix}0\\3\\6\end{bmatrix}$$

$$\rightarrow \quad \| y - \hat{y} \| = \sqrt{3^2 + 6^2} = \sqrt{45}$$

$\square$

Thm 10: If $\{u_1, \dots, u_p\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then

$$\text{proj}_W y = (y \cdot u_1) u_1 + \cdots + (y \cdot u_p) u_p \quad (*)$$

If $U = [\, u_1 \, u_2 \dots u_p \,]$, then

$$\text{proj}_W y = U U^T y, \text{ for all } y \text{ in } \mathbb{R}^n. \quad (**)$$

Proof: $(*)$ is obtained from the Orthogonal Decomposition Theorem.

$(*) \Rightarrow$ $\text{proj}_W y$ is a linear combin. of col. vectors of U, using weight $y \cdot u_1, \dots, y \cdot u_p$.

These weight can be writen as

$u_1^T y, \; u_2^T y, \dots, u_p^T y$. $\Leftarrow$ entries in $U^T y$

$\rightarrow$

$$\text{proj}_W y = (u_1^T y) u_1 + \cdots + (u_p^T y) u_p$$

$$= U U^T y. \quad \square$$