Section 6.2: ORTHOGONAL SETS

Defn: A set of vector $\{v_1, v_2, \dots, v_n\}$ is orthogonal if each pair of vectors is orthogonal. That is

$$v_i \cdot v_j = 0 \quad \text{for any } 1 \leq i \neq j \leq n.$$

Example:

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -\frac{1}{2} \\ -2 \\ 7/2 \end{bmatrix}$$

$$v_1 \cdot v_2 = 3.(-1) + 1.2 + 1.1 = 0$$
$$v_2 \cdot v_3 = (-1)(-\tfrac{1}{2}) + 2(-2) + 1.\tfrac{7}{2} = 0$$
$$v_3 \cdot v_1 = (-\tfrac{1}{2})3 + (-2).1 + \tfrac{7}{2}.1 = 0$$

$\Rightarrow \{v_1, v_2, v_3\}$ is an orthogonal set.

Theorem 4: If $S = \{u_1, u_2, \dots, u_k\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent.

Proof: If $c_1 u_1 + c_2 u_2 + \dots + c_k u_k = 0$ for some numbers $c_1, c_2, \dots, c_k$, then

$$0 = 0 \cdot u_1 = c_1.(u_1 \cdot u_1) + c_2(u_2 \cdot u_1) + \dots + c_k(u_k \cdot u_1)$$
$$= c_1 \|u_1\|^2$$

$\Rightarrow c_1 = 0$ since $\|u_1\|^2 > 0$ $(u_1 \neq 0)$.

Similar, $c_2 = c_3 = \dots = c_k = 0$.

Thus S is linearly independent $\square$

Defn: An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Theorem 5: Let $\{u_1, u_2, \dots, u_k\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each y in W, the weights in the linear combination

$$y = c_1 u_1 + \cdots + c_k u_k$$

are given by

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j} \qquad (j = 1, 2, \dots, k).$$

Proof:

$$y \cdot u_j = c_1 (u_1 \cdot u_j) + \cdots + c_k (u_k \cdot u_j)$$

$$= c_j (u_j \cdot u_j)$$

$$\Rightarrow \quad c_j = \frac{y \cdot u_j}{u_j \cdot u_j}. \qquad \square$$

Example: The set $S = \{v_1, v_2, v_3\}$ as in the previous example. Express

$$y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$$ as a linear combination of the vectors in S.

Solution:

$$y \cdot v_1 = 6 \cdot 3 + 1 \cdot 1 - 8 \cdot 1 = 11$$

$$y \cdot v_2 = 6 \cdot (-1) + 1 \cdot 2 - 8 \cdot 1 = -12$$

$$y \cdot v_3 = 6 \cdot \left(-\tfrac{1}{2}\right) + 1 \cdot (-2) - 8 \cdot \tfrac{7}{2} = -33$$

$V_1 \cdot V_1 = 3^2 + 1^2 + 1^2 = 11$
$V_2 \cdot V_2 = (-1)^2 + 2^2 + 1^2 = 6$
$V_3 \cdot V_3 = \left(-\frac{1}{2}\right)^2 + (-2)^2 + \left(\frac{7}{2}\right)^2 = \frac{33}{2}$

$$\Rightarrow \quad y = \frac{y \cdot V_1}{V_1 \cdot V_1} V_1 + \frac{y \cdot V_2}{V_2 \cdot V_2} V_2 + \frac{y \cdot V_3}{V_3 \cdot V_3} V_3$$
$$= V_1 - 2V_2 - 2V_3 .$$

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Orthogonal Projection (move to Section 6.3)

Given $u \in \mathbb{R}^n$, we want to decompose a vector y in $\mathbb{R}^n$ as the sum of two vectors, one a multiple of u and the other orthogonal to u.

That is we want to write

$$y = \hat{y} + z, \qquad (*)$$

where $\hat{y} = \alpha u$ and $z \perp u$.

We have $y - \hat{y} \perp u$ iff

$$0 = (y - \hat{y}) \cdot u = y \cdot u - \alpha (u \cdot u)$$
$$\Leftrightarrow \quad \alpha = \frac{y \cdot u}{u \cdot u} .$$

Thus, $\hat{y} = \frac{y \cdot u}{u \cdot u} u$ and $z = y - \frac{y \cdot u}{u \cdot u} u$

Vector $\hat{y}$ is called the **orthogonal projection of y onto u**, and z is called the **component of y**

orthogonal to u.

Let $L = \text{Span}\{u\}$, we denote by

$$\boxed{\hat{y} = \text{proj}_L\, y = \frac{y \cdot u}{u \cdot u}\, u.}$$

Example: Let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of y onto u. Then write y as the sum of two orthogonal vectors, one in $\text{Span}\{u\}$ and one orthogonal to u.

Solution:

$$y \cdot u = 7 \cdot 4 + 6 \cdot 2 = 40$$
$$u \cdot u = 4^2 + 2^2 = 20$$

$\Rightarrow$

$$\hat{y} = \frac{y \cdot u}{u \cdot u} = \frac{40}{20}\, u = \begin{bmatrix} 8 \\ 4 \end{bmatrix}.$$ ← orthogonal projection of y onto u.

$\Rightarrow$ The component of y orthogonal to u is

$$y - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Thus the orthogonal decomposition of y is

$$\underbrace{\begin{bmatrix} 7 \\ 6 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 8 \\ 4 \end{bmatrix}}_{\hat{y}} + \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{y - \hat{y} = z}$$

$\hat{y} \perp z$

Defn: A set $S = \{u_1, u_2, \ldots, u_k\}$ is an **orthonormal set** if it is an orthogonal set of unit vectors.

Let $W = \text{Span}(S)$, then $\{u_1, .., u_k\}$ is an orthogonal basis for W, so it is linearly independent.

Obs, if $\{v_1, v_2, \ldots, v_k\}$ is an orthogonal set of nonzero vectors, then

$$\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \ldots, \frac{v_k}{\|v_k\|} \right\}$$ is an orthonormal set.

Theorem 6: An $m \times n$ matrix U has orthonormal cols iff $U^T U = I$.

Proof:

Let $U = [u_1 \ u_2 \ldots u_n]$

matrix mul

$$\Rightarrow \quad U^T U = [a_{i,j}]_{n \times n}, \text{where } a_{i,j} = u_i^T u_j$$
$$= u_i \cdot u_j$$

dot prod.

If $\{u_1, .., u_n\}$ is orthonormal, then

$$a_{i,j} = 0 = u_i \cdot u_j \quad \text{if } i \neq j$$
$$a_{i,i} = 1 = \|u_i\|^2.$$

$$\Rightarrow \quad U^T U = I.$$

If $U^T U = I \Rightarrow \begin{cases} u_i \cdot u_j = 0 & i \neq j \\ \|u_i\| = 1 \end{cases}$

$\rightarrow \{u_1, .., u_n\}$ is orthonormal

$\square$

Theorem 7: Let U be an $m \times n$ matrix with orthonormal cols, and let x and y be in $\mathbb{R}^n$. Then

(a) $\|Ux\| = \|x\|$

(b) $(Ux)\cdot(Uy) = x \cdot y$

Example: Let $U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$ and $x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$.

$\Rightarrow$ U has orthnormal cols and

$$U^T U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$Ux = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \|Ux\| = \sqrt{9+1+1} = \sqrt{11} = \sqrt{2+9} = \|x\|.$$

Proof: Only need to prove part (b).

$$\begin{aligned} (Ux)\cdot(Uy) &= (Ux)^T(Uy) \\ &= x^T U^T U y \\ &= x^T y = x \cdot y \quad \square. \end{aligned}$$