Section 6.1 Inner product, lenght, and orthogonality.

If $u, v \in \mathbb{R}^n$, then u and v are $n \times 1$ matrices.
The transpose u^T of u is an $n \times 1$ matrix.
The **inner product** (or the **dot product**) of u and v is $u^T v$. Denoted by $u \cdot v$

$$\begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Example:

$$u = \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$$

then $u \cdot v = 2 \cdot 0 + 7 \cdot 1 + 0 \cdot 6 = 7$.

Theorem 1: Let $u, v, w \in \mathbb{R}^n$, c be a scalar. Then

a) $u \cdot v = v \cdot u$

b) $(u + v) \cdot w = u \cdot w + v \cdot w$

c) $(cu) \cdot v = u \cdot (cv) = c\,(u \cdot v)$

d) $u \cdot u \geq 0$ and $u \cdot u = 0$ iff $u = 0$.

e) $(c_1 u_1 + \dots + c_n u_n) \cdot w = c_1 (u_1 \cdot w) + \cdots + c_n (u_n \cdot w)$.

Definition: The length (or norm) of v is the nonnegative scalar $\|v\|$ defined by

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + \dots + v_n^2}$$

$(\Rightarrow \ \|v\|^2 = v \cdot v)$

Suppose $v \in \mathbb{R}^2$, v is interpreted as a point (a,b) $\Rightarrow$ $\|v\|$ is the distance from the origin to (a,b).

Similarly, $w \in \mathbb{R}^3$ interpreted as a point (a,b,c) $\Rightarrow$ $\|v\|$ is also the distance from the origin to (a,b,c).

Obs: $\|cv\| = |c|\,\|v\|$.

Defn: A vector of length 1 is called a **unit vector**. The unit vector $\frac{v}{\|v\|}$ is the **norm vector** of v, or the unit vector in the same direction as v.

Example: Let $v = \begin{bmatrix} 2 \\ 3 \\ -1 \\ 1 \end{bmatrix}$, Find a unit vector u in the same direction as v.

$$\|v\|^2 = 2^2 + 3^2 + (-1)^2 + 1^2 = 15$$

$$\Rightarrow \quad \|v\| = \sqrt{15}$$

$$\Rightarrow \quad u = \frac{v}{\|v\|} = \begin{bmatrix} 2/\sqrt{15} \\ 3/\sqrt{15} \\ -1/\sqrt{15} \\ 1/\sqrt{15} \end{bmatrix}.$$

Example: $W = \text{Span}\left\{ \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \right\} \triangleleft \mathbb{R}^2$. Find a unit vector z that is a basis for W.

Solution:

W consists of all scalar multiples of $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$.

Any nonzero in W is a basis for W. To simplify the calculation, we can scale $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$ to eliminate fractions. Multiply x by 3, we get

$$y = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

$$\|y\| = \sqrt{2^2 + 3^2} = \sqrt{13}.$$

$$\Rightarrow \quad z = \frac{y}{\|y\|} = \begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}.$$

Defn **Distance in $\mathbb{R}^n$.**

For u and v in $\mathbb{R}^n$, the **distance between u and v** written as $\text{dist}(u, v)$, is the length of the vector $u - v$. That is

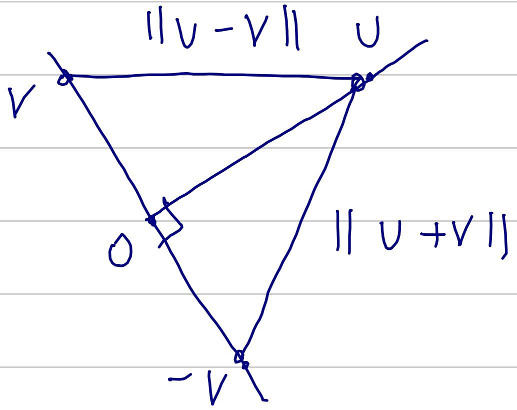
$$\text{dist}(u, v) = \|u - v\|.$$

Example: $u = (3, 4, 5)$ and $v = (4, 2, -1)$

Compute the distance between u and v.

$$u - v = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$$

$$\Rightarrow \quad \text{dist}(u, v) = \|u - v\| = \sqrt{(-1)^2 + 2^2 + 6^2} = \sqrt{41}.$$



We have
$$\operatorname{dist}(u,-v)^2 = \|u-(-v)\|^2 = \|u+v\|^2$$
$$= (u+v)\cdot(u+v)$$
$$= u\cdot(u+v) + v\cdot(u+v)$$
$$= u\cdot u + 2\,u\cdot v + v\cdot v$$
$$= \|u\|^2 + \|v\|^2 + 2\,u\cdot v$$

Similarly,
$$\operatorname{dist}(u,v)^2 = \|u-v\|^2 = \|u+(-v)\|^2$$
$$= \|u\|^2 + \|-v\|^2 + 2\,u\cdot(-v)$$
$$= \|u\|^2 + \|v\|^2 - 2\,u\cdot v.$$

Thus, $\operatorname{dist}(u,-v) = \operatorname{dist}(u,v)$
$$\Updownarrow$$
$$2\,u\cdot v = -2\,u\cdot v$$
$$\Updownarrow$$
$$u\cdot v = 0.$$

Geometrically, $\operatorname{dist}(u,-v) = \operatorname{dist}(u,v)$
$$\Updownarrow$$
The line passing 0 and u and the line passing 0 and v are orthogonal.

Defn: Two vectors u and v in $\mathbb{R}^n$ are orthogonal if $u \cdot v = 0$

Theorem 2: The Pythagorean Theorem.

Two vectors u and v are orthogonal iff
$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

Theorem 2'. The parallelogram Theorem

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

$\ast$.

Defn: If a vector z is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then we say z is orthogonal to W. Denoted by $z \perp W$.

The set of all vectors z that are orthogonal to W is called the **orthogonal complement** of W. Denoted by $W^\perp$.

Obs: (1) $x \perp W$ iff x is orthogonal to every vector in a spanning set of W

(2) $W^\perp \trianglelefteq \mathbb{R}^n$.

Theorem 3: Let A be an $m \times n$ matrix

$$(\text{Row } A)^\perp = \text{Nul } A \text{ and } (\text{Col } A)^\perp = \text{Nul } A^T$$

Proof:

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathrm{Nul}\, A$

$$\Rightarrow \quad Ax = 0 \quad \Leftrightarrow \quad \underbrace{\begin{bmatrix} r_1 x \\ r_2 x \\ \vdots \\ r_n x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\text{by row-column rule}}$$

$\Rightarrow$ x is orthogonal to each row of A

$\Rightarrow$

$$x \perp \mathrm{Row}\, A.$$

$\Rightarrow$ $\mathrm{Nul}\, A \subset (\mathrm{Row}\, A)^{\perp}$.

Revereresely, for every $z \in (\mathrm{Row}\, A)^{\perp}$

$\Rightarrow$ $z \perp r_1, r_2, \ldots, r_n \in \mathrm{Row}\, A$

$\Rightarrow$ $Az = 0 \Rightarrow z \in \mathrm{Nul}\, A$

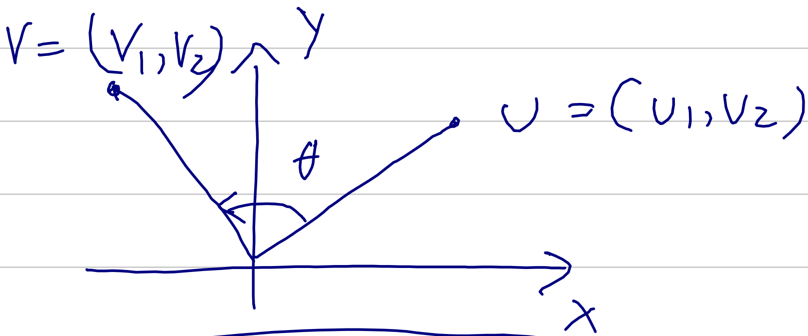
$\Rightarrow$ $(\mathrm{Row}\, A)^{\perp} \subset \mathrm{Nul}\, A$.

Thus, $(\mathrm{Row}\, A)^{\perp} = \mathrm{Nul}\, A$.

$$\mathrm{Col}\, A = \mathrm{Row}\, A^T \Rightarrow (\mathrm{Col}\, A)^{\perp} = (\mathrm{Row}\, A^T)^{\perp}$$
$$= \mathrm{Nul}\, A^T.$$

$\square$

Angles:



$$\boxed{u \cdot v = \|u\| \, \|v\| \cos\theta.}$$

By the law of cosines

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta$$

$$\Rightarrow \quad \|u\|\|v\|\cos\theta = \frac{1}{2}\left[\|u\|^2 + \|v\|^2 - \|u-v\|^2\right]$$

$$= \frac{1}{2}\left[u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1-v_1)^2 - (u_2-v_2)^2\right]$$

$$= u_1v_1 + u_2v_2 = u \cdot v$$

$\square$.