Section 5.5 COMPLEX EIGENVALUES

Example: Find all eigenvalues of

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

$$\Rightarrow \quad \det(A - \lambda I) = 0$$

$$\Leftrightarrow \quad \lambda^2 + 1 = 0$$

However, this equation does not have real solution. It only has complex solutions:

$$\left.\begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \end{aligned}\right\} \quad i = \sqrt{-1}, \text{ or } i^2 = -1.$$

These are complex eigenvalues of A.

Fact: Each polynomial of degree n has exactly n roots, counting multiplicities, provided that possibly complex roots are included.

$\Rightarrow$ Each $n \times n$ matrix has exactly n complex eigenvalues, counting multiplicities.

Example: Find eigenvalues of $A = \begin{bmatrix} -2 & 5 \\ -1 & -3 \end{bmatrix}$.

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 5 \\ -1 & -3-\lambda \end{vmatrix}$$

$$= (2+\lambda)(3+\lambda) + 5$$
$$= \lambda^2 + 5\lambda + 11$$

The characteristic eq.

$$\lambda^2 + 5\lambda + 11 = 0$$

has two complex solutions

$$\lambda_{1,2} = \frac{-5 \pm \sqrt{25-44}}{2}$$

$$= \frac{-5 \pm \sqrt{-19}}{2}$$

$$= \frac{-5 \pm i\sqrt{19}}{2}.$$

Example: Let $A = \begin{bmatrix} \frac{1}{2} & -\frac{3}{5} \\ \frac{3}{4} & \frac{11}{10} \end{bmatrix}$. Find eigenvalues of A, and find a basis for each eigenspace.

$$\det(A - \lambda) = \begin{vmatrix} \frac{1}{2}-\lambda & -\frac{3}{5} \\ \frac{3}{4} & \frac{11}{10}-\lambda \end{vmatrix}$$

$$= \left(\frac{1}{2}-\lambda\right)\left(\frac{11}{10}-\lambda\right) + \frac{3}{5}\cdot\frac{3}{4}$$

$$= \lambda^2 - \frac{16}{10}\lambda + 1$$

By quadratic formular, A has two complex eigenvalues

$$\lambda_{1,2} = \left(\frac{16}{10} \pm \sqrt{\left(\frac{-16}{10}\right)^2 - 4}\right)/2$$

$$= \frac{4}{5} \pm \frac{3}{5} i$$

For the eigenvalue $\lambda_1 = \frac{4}{5} - \frac{3}{5} i$,

$$A - \lambda_1 I = \begin{bmatrix} \frac{1}{2} - \frac{4}{5} + \frac{3}{5} i & -\frac{3}{5} \\ \frac{3}{4} & \frac{11}{10} - \frac{4}{5} + \frac{3}{5} i \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-3}{10} + \frac{3}{5} i & -\frac{3}{5} \\ \frac{3}{4} & \frac{3}{10} + \frac{3}{5} i \end{bmatrix}$$

Since λ_1 is an eigenvalue, the system

$$\left(\frac{-3}{10} + \frac{3}{5} i\right) x_1 - \frac{3}{5} x_2 = 0$$

$$\frac{3}{4} x_1 + \left(\frac{3}{10} + \frac{3}{5} i\right) x_2 = 0$$

has a nontrivial solution. Two eqs must be equivalent.

The second eq. leads to

$$x_1 = \left(-\frac{2}{5} - \frac{4}{5} i\right) x_2$$

Choose $x_2 = -5$, $x_1 = 2 + 4i$

A basis for the eigenspace corresponding to $\lambda_1 = \frac{4}{5} - \frac{3}{5}i$ is

$$v_1 = \begin{bmatrix} 2+4i \\ -5 \end{bmatrix}.$$

Similarly $\lambda_2 = \frac{4}{5} + \frac{3}{5}i$ yields a basis for the corresponding eigenspace

$$v_2 = \begin{bmatrix} 2-4i \\ -5 \end{bmatrix}.$$

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<u>Example:</u>

$$x = \begin{bmatrix} 3-i \\ i \\ 2+5i \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 6 \end{bmatrix} + i \begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix}$$

Real part of x is $\operatorname{Re} x = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 6 \end{bmatrix}$

Imaginary part of x is $\operatorname{Im} x = \begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix}.$

The complex conjugate of x is

$$\bar{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 6 \end{bmatrix} - i \begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix}.$$

If $B = [b_{ij}]$, then $\overline{B} := [\overline{b_{i,j}}]$.

Simple properties:

(1) $\overline{rx} = \overline{r}\,\overline{x}$

(2) $\overline{Bx} = \overline{B}\,\overline{x}$

(3) $\overline{BC} = \overline{B}\,\overline{C}$

(4) $\overline{rB} = \overline{r}\,\overline{B}$.

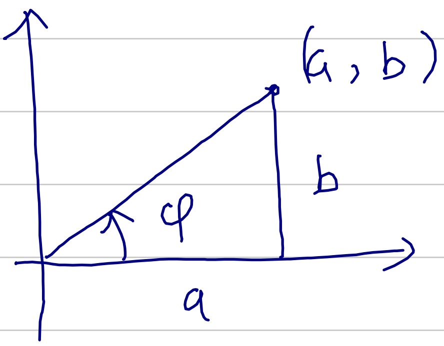
Example: Let A be a 2×2 real matrix. If A does not have real eigenvalues, then A has two conjugate complex eigenvalues.

Example: If $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where a, b are real, then $\lambda = a \pm bi$ are complex eigenvalues of C.

Also, if $r = |\lambda| = \sqrt{a^2+b^2}$, then

$$C = r\begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

where



φ is called the argument of $\lambda = a+bi$

$\Rightarrow$ the transformation $x \longmapsto Cx$ can be viewed as the composition of a rotation through the angle φ and a scaling by $|\lambda| = r$.

Example 7: Let $A = \begin{bmatrix} \frac{1}{2} & -\frac{3}{5} \\ \frac{3}{4} & \frac{11}{10} \end{bmatrix}$

$\lambda = \frac{4}{5} - \frac{3}{5} i$, and $v_1 = \begin{bmatrix} 2+4i \\ -5 \end{bmatrix}$ as in previous example. Also, let P be the 2×2 matrix

$$P = [\operatorname{Re} v_1 \quad \operatorname{Im} v_1] = \begin{bmatrix} 2 & 4 \\ -5 & 0 \end{bmatrix}$$

and let

$$C = P^{-1} A P$$

$$= \frac{1}{20} \begin{bmatrix} 0 & -4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{3}{5} \\ \frac{3}{4} & \frac{11}{10} \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

C is purely a rotation, since $|\lambda| = r = 1$,

Thm 9: Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector v in $\mathbb{C}^2$. Then

$$A = PCP^{-1}, \text{ where } P = [\operatorname{Re} v \quad \operatorname{Im} v]$$

and

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$