Section 5.4: Eigenvectors and Linear Transformations

Consider a linear transformation $T: V \to W$.
Assume that $\dim V = n$, and $\mathcal{B} = \{b_1, \dots, b_n\}$ a basis; $\dim W = m$ and $\mathcal{C} = \{c_1, c_2, \dots, c_m\}$ is a basis.

Given $x \in V$, x has coordinate vector $[x]_{\mathcal{B}} \in \mathbb{R}^n$ and the coordinate vector of the image is $[T(x)]_{\mathcal{C}} \in \mathbb{R}^m$

Assume that $[x]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$ (i.e $x = r_1 b_1 + \cdots + r_n b_n$)

$$\Rightarrow T(x) = r_1 T(b_1) + \cdots + r_n T(b_n)$$

$$\Rightarrow [T(x)]_{\mathcal{C}} = r_1 [T(b_1)]_{\mathcal{C}} + \cdots + r_n [T(b_n)]_{\mathcal{C}}$$

$$= \left[[T(b_1)]_{\mathcal{C}} \cdots [T(b_n)]_{\mathcal{C}} \right] \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

$$= M [x]_{\mathcal{B}}.$$

The matrix $M = \left[[T(b_1)]_{\mathcal{C}} \cdots [T(b_n)]_{\mathcal{C}} \right]$ is called the **matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$**.

$$\begin{tikzcd} x \arrow[r, "T"] \arrow[d] & T(x) \\ {[x]_{\mathcal{B}}} \arrow[r, "\times M"'] & {[T(x)]_{\mathcal{C}}} \end{tikzcd}$$

Example: Suppose $\mathcal{B} = \{b_1, b_2\}$ is a basis for V

$\mathcal{C} = \{c_1, c_2, c_3\}$ is a basis for W

$T: V \to W$ a linear transformation s.t.

$T(b_1) = 6c_1 - 2c_2 + 4c_3$ and

$T(b_2) = 4c_1 + 8c_3$.

$\Rightarrow$ The matrix for T relative to $\mathcal{B}$ and $\mathcal{C}$ is

$$M = \left[[T(b_1)]_{\mathcal{C}} \quad [T(b_2)]_{\mathcal{C}} \right]$$

$$= \begin{bmatrix} 6 & 4 \\ -2 & 0 \\ 4 & 8 \end{bmatrix}.$$

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Next, we consider the case $V = W$ and $\mathcal{B} = \mathcal{C}$. The matrix M is now called the **matrix for T relative to $\mathcal{B}$**, or simply the **$\mathcal{B}$-matrix for T**, and is denoted by $[T]_{\mathcal{B}}$.

Thus we have: $[T(x)]_{\mathcal{B}} = [T]_{\mathcal{B}} [x]_{\mathcal{B}}$.

Example: The mapping $T: \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(a_0 + a_1 t + a_2 t^2) = 2a_1 + 3a_0 t^2$$

is a linear transformation.

a) Find the $\mathcal{B}$-matrix for T, where

$$\mathcal{B} = \{1, t, t^2\}$$

b) Verify that: $[T(x)]_{\mathcal{B}} = [T]_{\mathcal{B}} [x]_{\mathcal{B}}$.

a) Compute the images of basis vectors

$$T(1) = 3t^2$$
$$T(t) = 2$$
$$T(t^2) = 0$$

$\Rightarrow$ $[T(1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$, $[T(t)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $[T(t^2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

b) For general $x(t) = a_0 + a_1 t + a_2 t^2$

$\Rightarrow$ $[T(x)]_{\mathcal{B}} = [\, 2a_1 + 3a_0 t^2 \,] = \begin{bmatrix} 2a_1 \\ 0 \\ 3a_0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = [T]_{\mathcal{B}} [x]_{\mathcal{B}} .$$

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Linear Transformation on $\mathbb{R}^n$

Theorem 8: Diagonal Matrix Representation

Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is a basis for $\mathbb{R}^n$ formed from columns of P, then D is the $\mathcal{B}$-matrix for the transformation $x \mapsto Ax$.

Proof: Denote the columns of P by $b_1, \ldots, b_n$. So $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ and $P = [b_1 \ldots b_n]$. In this case P is the change-of-coordinate matrix $P_{\mathcal{B}} := \underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$:

Defn $\Longrightarrow$
$$x = P[x]_{\mathcal{B}} \quad \text{and} \quad [x]_{\mathcal{B}} = P^{-1}x.$$

If $T(x) = Ax$, then

$$
\begin{aligned}
[T]_{\mathcal{B}} &= \left[\, [T(b_1)]_{\mathcal{B}} \cdots [T(b_n)]_{\mathcal{B}} \,\right] \\
&= \left[\, [Ab_1]_{\mathcal{B}} \cdots [Ab_n]_{\mathcal{B}} \,\right] && \leftarrow T(b_i) = Ab_i \\
&= \left[\, P^{-1}Ab_1 \ \cdots \ P^{-1}Ab_n \,\right] && \leftarrow [x]_{\mathcal{B}} = P^{-1}x \\
&= P^{-1}A[b_1 \cdots b_n] \\
&= P^{-1}AP
\end{aligned}
$$

Since

$$A = PDP^{-1} \Rightarrow [T]_{\mathcal{B}} = P^{-1}(PDP^{-1})P$$
$$= D$$

Example 3:

Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$, where

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}.$$

Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that the $\mathcal{B}$-matrix for T is a diagonal matrix.

Sol: We can verify that: $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}.$$

Obs:

The proof of Thm 8 did not use the property that D was diagonal. Thus, if A is similar to C, i.e $A = PCP^{-1}$, then C is the $\mathcal{B}$-matrix for the transf. $x \mapsto Ax$.

$$\begin{tikzcd}
x \arrow[r, "A"] \arrow[d, "P^{-1}"'] & Ax \\
{[x]_{\mathcal{B}}} \arrow[r, "C"'] & {[Ax]_{\mathcal{B}}} \arrow[u, "P"']
\end{tikzcd}$$

Conversely, if $T: \mathbb{R}^n \to \mathbb{R}^n$ is defined by $T(x) = Ax$, and if $\mathcal{B}$ is any basis for $\mathbb{R}^n$, then $\mathcal{B}$-matrix of T is similar to A:

$$[T]_{\mathcal{B}} = P^{-1} A P$$

where P is the matrix with columns in $\mathcal{B}$.

Example: $A = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}$, $b_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $b_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

The characteristic polynomial of A is $(\lambda+2)^2$, but the eigenspace for $\lambda = -2$ is only 1-dimensional.

$\Rightarrow$ A is not diagonalizable.

However, the basis $\mathcal{B} = \{b_1, b_2\}$ has the property that $\mathcal{B}$-matrix for the transf. $x \mapsto Ax$ is triangular matrix called the **Jordan form** of A.

Find this $\mathcal{B}$-matrix.

$P = [b_1, b_2]$ then the $\mathcal{B}$-matrix is $P^{-1}AP$

$$AP = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -1 \\ -4 & 0 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -6 & -1 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}.$$