Section 5.2: THE CHARACTERISTIC EQUATION

Example: Find the eigenvalues of

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

Solution: We need to find all scalar λ s.t.

$$(A - \lambda I)x = 0$$

has a nontrivial solutions.

$\Longleftrightarrow$ Find λ s.t. the matrix

$$A - \lambda I$$

is <u>not</u> invertible.

We have:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 12$$

$$= \lambda^2 - 3\lambda - 10$$

$\Rightarrow$ $A - \lambda I$ is not invertible if and only if

$$\lambda^2 - 3\lambda - 10 = 0$$

$\Delta = 9 + 40 = 49$

$$\lambda_1 = \frac{3 - \sqrt{49}}{2} = -2$$

$$\lambda_2 = \frac{3 + \sqrt{49}}{2} = 5$$

Recall: Let U be an echelon form of A.

$$(*) \quad \det A = \begin{cases} (-1)^r \left\{ \begin{matrix} \text{product} \\ \text{of pivots} \\ \text{in } U \end{matrix} \right\} & , \text{ if } A \text{ is invertible} \\ 0 & , \text{ if } A \text{ is not invertible} \end{cases}$$

Example: Find the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Bring A to an echelon form:

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$r = \#$ interchangings $= 1$

$$\det A = (-1)^1 (1)(-2)(-1) = -2.$$

Remark: $(*)$ only true if we do not use any scaling when transform A into U.

Theorem: The Invertible Matrix Theorem (cont.)

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

(s) The number 0 is <u>not</u> an eigenvalue of A

(t) The determinant of A is <u>not</u> zero.

Proof: A is invertible $\iff$ $\det A \neq 0$.

0 is <u>not</u> an eigenvalue of A $\iff$ $(A - 0I)x = 0$ has only trivial solution

$\iff$ $Ax = 0$ has only trivial solution

$\iff$ The columns of A are linearly independent

$\iff$ A is invertible.

$\square$

Recall: The properties of determinants,

a) $\det AB = \det A \det B$

b) $\det A = \det A^T$

c) det of a triangular matrix equals the product of diagonal entries.

d) A row replacement does not change det
A row interchange change the sign of det
A row scaling also scales the det by the same scalar factor.

e) Part (d) is still true if "row" is replaced by "column".

Obs: A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfy the **characteristic equation**

$$\det(A - \lambda I) = 0$$

Example: Find the characteristic equation of

$$A = \begin{bmatrix} 6 & 2 & 7 & 3 \\ 0 & 1 & 6 & 10 \\ 0 & 0 & -2 & -9 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & 2 & 7 & 3 \\ 0 & 1-\lambda & 6 & 10 \\ 0 & 0 & -2-\lambda & -9 \\ 0 & 0 & 0 & 3-\lambda \end{vmatrix}$$

$$= (6-\lambda)(1-\lambda)(-2-\lambda)(3-\lambda)$$

The characteristic equation is

$$(6-\lambda)(1-\lambda)(-2-\lambda)(3-\lambda) = 0$$

$$\Longleftrightarrow (6-\lambda)(1-\lambda)(2+\lambda)(3-\lambda) = 0$$

$$\Longleftrightarrow (\lambda^2 - 7\lambda + 6)(-\lambda^2 + \lambda + 6) = 0$$

$$\Longleftrightarrow -\lambda^4 + \lambda^3 + 6\lambda^2 + 7\lambda^3 - 7\lambda^2 - 42\lambda - 6\lambda^2 + 6\lambda + 36 = 0$$

$$\Longleftrightarrow -\lambda^4 + 8\lambda^3 - 7\lambda^2 - 36\lambda + 36 = 0$$

$\det(A - \lambda I)$ is always a polynomial of degree n in λ

$\Rightarrow$

$\det(A - \lambda I)$ is the **characteristic polynomial** of A.

<u>Example:</u> The characteristic polynomial of a 6x6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$.

Find the eigenvalues of the matrix.

<u>Solution:</u> Factor $\lambda^6 - 4\lambda^5 - 12\lambda^4 = (\lambda^2 - 4\lambda - 12)\lambda^4$

$$= \lambda^4(\lambda - 6)(\lambda + 2).$$

Thus the eigenvalues are:

0, 6, and -2

multiplicity 4   multiplicity 1

<u>Definition:</u> A is <u>similar</u> to B if there is an invertibe matrix P s.t.

$$A = PBP^{-1}$$

$$\Longleftrightarrow \quad P^{-1}AP = B.$$

Changing A into $P^{-1}AP$ is called a similarity transformation.

<u>Theorem:</u> If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicity)

Proof: If $B = P^{-1} A P$, then

$$
\begin{aligned}
B - \lambda I &= P^{-1} A P - \lambda P^{-1} P \\
&= P^{-1} A P - P^{-1} \lambda P \\
&= P^{-1} (AP - \lambda P) = P^{-1} (A - \lambda I) P.
\end{aligned}
$$

$$
\Rightarrow \det(B - \lambda I) = \det P^{-1} \det(A - \lambda I) \det P
$$
$$
= \det(A - \lambda I).
$$

Remark:

① Having the same eigenvalues $\not\Rightarrow$ similarity

E.g.

$$
\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.
$$

② Similarity $\neq$ row equivalence.

More example:

$$
A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{bmatrix}
$$

has eigenvalues 1, 2, 11.