Section 5.2: THE CHARACTERISTIC EQUATION

Example: Find the eigenvalues of

\[ A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \]

Solution: We need to find all scalar \( \lambda \) s.t.

\[(A - \lambda I)x = 0\]

has a nontrivial solution.

\[ \iff \] Find \( \lambda \) s.t. the matrix

\[ A - \lambda I \]

is not invertible.

We have:

\[
\det (A - \lambda I) = \begin{vmatrix} 2 - \lambda & 3 \\ 4 & 1 - \lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 12
\]

\[ = \lambda^2 - 3\lambda - 10 \]

\[ \Rightarrow A - \lambda I \text{ is not invertible if and only if} \]

\[ \lambda^2 - 3\lambda - 10 = 0 \]

\[ \Delta = 9 + 40 = 49 \]

\[ \lambda_1 = \frac{3 - \sqrt{49}}{2} = -2 \]

\[ \lambda_2 = \frac{3 + \sqrt{49}}{2} = 5 \]
Recall: Let \( U \) be an echelon form of \( A \).

\[
\det A = \begin{cases} (-1)^r \text{ product of pivots in } U, & \text{if } A \text{ is invertible} \\ 0, & \text{if } A \text{ is not invertible} \end{cases}
\]

Example: Find the determinant of

\[
A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}
\]

Bring \( A \) to an echelon form:

\[
A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & 1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U
\]

\[r = \# \text{ interchanges} = 1\]

\[\det A = (-1)^1 \times (1) \times (-2) \times (-1) = -2.\]

Remark: \( \odot \) only true if we do not use any scaling when transform \( A \) into \( U \).
Theorem: The Invertible Matrix Theorem (cont.)

Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if:

(i) The number 0 is not an eigenvalue of $A$.
(ii) The determinant of $A$ is not zero.

Proof: $A$ is invertible $\iff$ $\det A \neq 0$.

0 is not an eigenvalue of $A \iff (A - 0 I) x = 0$ has only trivial solution.

$\iff A x = 0$ has only trivial solution.

$\iff$ The columns of $A$ are linearly independent.

$\iff$ $A$ is invertible.

$\blacksquare$

Recall: The properties of determinants.

a) $\det AB = \det A \det B$

b) $\det A = \det A^T$

c) $\det$ of a triangular matrix equals the product of diagonal entries.

d) A row replacement does not change $\det A$.

A row interchange changes the sign of $\det A$.

A row scaling also scales the $\det$ by the same scalar factor.

e) Part (d) is still true if "row" is replaced by "column".
A scalar $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ if and only if $\lambda$ satisfies the characteristic equation
\[
\det(A - \lambda I) = 0
\]

**Example:** Find the characteristic equation of
\[
A = \begin{bmatrix}
  6 & 2 & 7 & 3 \\
  0 & 1 & 6 & 10 \\
  0 & 0 & -2 & -9 \\
  0 & 0 & 0 & 3
\end{bmatrix}
\]
\[
\det(A - \lambda I) = \begin{vmatrix}
  6-\lambda & 2 & 7 & 3 \\
  0 & 1-\lambda & 6 & 10 \\
  0 & 0 & -2-\lambda & -9 \\
  0 & 0 & 0 & 3-\lambda
\end{vmatrix}
\]
\[
= (6-\lambda)(1-\lambda)(-2-\lambda)(3-\lambda)
\]
The characteristic equation is
\[
(6-\lambda)(1-\lambda)(-2-\lambda)(3-\lambda) = 0
\]
\[
\iff (6-\lambda)(1-\lambda)(2+\lambda)(3-\lambda) = 0
\]
\[
\iff (\lambda^2 - 7\lambda + 6)(-\lambda^2 + \lambda + 6) = 0
\]
\[
\iff -\lambda^4 + \lambda^3 + 6\lambda^2 + 7\lambda^3 - 7\lambda^2 - 42\lambda - 6\lambda^2 + 6\lambda + 36 = 0
\]
\[
\iff -\lambda^4 + 8\lambda^3 - 7\lambda^2 - 36\lambda + 36 = 0
\]
\[ \text{Det}(A - \lambda I) \text{ is always a polynomial of degree } n \text{ in } \lambda \]
\[ \Rightarrow \quad \text{det}(A - \lambda I) \text{ is the characteristic polynomial of } A. \]

Example: The characteristic polynomial of a 6x6 matrix is \[ \lambda^6 - 4\lambda^5 - 12\lambda^4. \]
Find the eigenvalues of the matrix.

Solution: Factor \[ \lambda^6 - 4\lambda^5 - 12\lambda^4 = (\lambda^2 - 4\lambda - 12)\lambda^4 \]
\[ = \lambda^4 (\lambda - 6)(\lambda + 2). \]
Thus the eigenvalues are:
0, 6, and -2
up, up, and down

multiplicity 4, multiplicity 1

Definition: \( A \) is similar to \( B \) if there is an invertible matrix \( P \) such that
\[ A = PBP^{-1} \]
\[ \Rightarrow P^{-1}AP = B. \]

Changing \( A \) into \( P^{-1}AP \) is called a similarity transformation.

Theorem: If \( n \times n \) matrices \( A \) and \( B \) are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicity).
Proof: If $B = P^{-1}AP$, then

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P$$

$$= P^{-1}AP - P^{-1}\lambda P$$

$$= P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P.$$ 

$$\Rightarrow \quad \det(B - \lambda I) = \det P^{-1} \det (A - \lambda I) \det P$$

$$= \det (A - \lambda I).$$

Remark:

1. Having the same eigenvalues $\Rightarrow$ similarity

E.g. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

2. Similarity $\neq$ row equivalence.

More example:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{bmatrix}$$

has eigenvalues 1, 2, 11.