Section 5.1: Eigenvectors and Eigenvalues.

Example 1:

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have $A\mathbf{v} = 2\mathbf{v}$

Although a transformation $\mathbf{x} \longmapsto A\mathbf{x}$ may move vectors in a variety directions, it often happens that there are special vectors on which the action of A is simple.

Definition: An **eigenvector** of an $n \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ. A scalar λ is call an **eigenvalue** of A if there is a nontrivial solution of $A\mathbf{x} = \lambda\mathbf{x}$; such an x is called an "eigenvector corresponding to λ."

Example 2: Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

$$A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\mathbf{u}$$

$$A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\mathbf{v}$$

So $\mathbf{u}$ is an eigenvector of A, but $\mathbf{v}$ is not.

Example 3: Show that 7 is an eigenvalue of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, and find the corresponding eigenvector.

Solution: Consider $A\mathbf{x} = 7\mathbf{x}$. ①

$\Leftrightarrow \quad (A - 7I_2)\mathbf{x} = 0$ ②

$$A - 7I_2 = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}.$$

The matrix $A - 7I_2$ is invertible. (The two columns are linearly independent.) $\Rightarrow$ The equation ② has nontrivial solutions. $\Rightarrow$ ① has nontrivial solutions, and 7 is an eigenvalue of A.

To find the corresponding eigenvector, we use row reduction:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} -6 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 is free

row 1 $\Rightarrow x_1 - x_2 = 0$

$\Rightarrow$ The general solution has form

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftarrow \text{eigenvector.}$$

<u>Observation:</u> λ is an eigenvalue of A if and only if the equation

$$(A-\lambda I)\mathbf{x} = 0 \qquad (3)$$

has a non-trivial solution.

The null space of $A-\lambda I$ is called the **eigenspace** of A corresponding to λ.

<u>Example 4:</u>

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

Check that 2 is an eigenvalue of A. Find the corresponding eigenspace.

<u>Solution:</u>

$$A-2I = \begin{bmatrix} 4-2 & -1 & 6 \\ 2 & 1-2 & 6 \\ 2 & -1 & 8-2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

Row reduce the augmented matrix of eq. $(A-2I)x = 0$.

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_2 and x_3 are free

$\Rightarrow$ $(A-2I)x = 0$ has nontrivial solutions.

$\Rightarrow$ 2 is an eigenvalue of A.

The general solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

So the general solution is:

$$x = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Thus: $\mathrm{Nul}\,(A-2I) = \mathrm{Span}\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ ← eigenspace.

<u>Theorem</u>: The eigenvalues of a triangular matrix are the entries on the main diagonal.

<u>Proof</u>: W.l.o.g., we prove for 4×4 upper triangular matrix A:

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & a_{14} \\ 0 & a_{22}-\lambda & a_{23} & a_{24} \\ 0 & 0 & a_{33}-\lambda & a_{34} \\ 0 & 0 & 0 & a_{44}-\lambda \end{bmatrix}$$

λ is an eigenvalue iff $(A-\lambda I)x = 0$ has nontrivial solutions. $\Leftrightarrow$ the equation has free variables $\Leftrightarrow$ a column does not have a pivot $\Leftrightarrow$ one of $a_{ii}-\lambda$ is zero $(i=1,2,3,4)$
$\Leftrightarrow$ $\lambda = a_{ii}$ for some i. $\square$

<u>Example</u>: $A = \begin{bmatrix} 4 & 4 & 1 \\ 0 & 6 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ has eigenvalues $4, 6, -1$.

$B = \begin{bmatrix} 0 & 0 & 0 \\ 8 & 1 & 0 \\ 2 & 5 & 7 \end{bmatrix}$ has eigenvalues $0, 1, 7$.

Obs: 0 is an eigenvalue of A iff
$Ax = 0$ has nontrivial solution,
iff A is not invertible.

Theorem 2: If $v_1, v_2, \dots, v_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$ of an $n \times n$ matrix A, then $\{v_1, v_2, \dots, v_r\}$ is linearly independent.

Proof: Assume otherwise that $\{v_1, \dots, v_r\}$ is linearly dependent. Since $v_1 \neq 0$, we can find v_{p+1} that can be written as a linear combination of $v_1, \dots, v_p$. W.l.o.g. we can assume that q is the least index with the property.

$$c_1 v_1 + \dots + c_p v_p = v_{p+1} \qquad (5)$$

$$\Rightarrow \quad c_1 A v_1 + \dots + c_p A v_p = A v_{p+1}$$

$$\Rightarrow \quad c_1 \lambda_1 v_1 + \dots + c_p \lambda_p v_p = \lambda_{p+1} v_{p+1} \qquad (6)$$

Multiplying (5) by λ_{p+1}, then subtracting the result from (6):

$$(c_1 \lambda_1 - c_1 \lambda_{p+1}) v_1 + \dots + (c_p \lambda_p - c_p \lambda_{p+1}) v_p = 0$$

Since $\{v_1, \dots, v_p\}$ is linearly independent (by the choice of the index p),

$$c_1 \lambda_1 = c_1 \lambda_{p+1}$$
$$c_2 \lambda_2 = c_2 \lambda_{p+1}$$
$$\dots\dots$$
$$c_p \lambda_p = c_p \lambda_{p+1}$$

$\Rightarrow c_1 = c_2 = \dots = c_p = 0$ (since $\lambda_1, \dots, \lambda_r$ are distinct)

By (5). $v_{p+1} = 0$