New Duals of MacMahon's Theorem

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Plane partitions

$$\pi = \begin{array}{ccccc} \pi_{1,1} & \pi_{1,2} & \cdots & \pi_{1,b-1} & \pi_{1,b} \\ \pi_{2,1} & \pi_{2,2} & \cdots & \pi_{2,b-1} & \pi_{2,b} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \pi_{a,1} & \pi_{a,2} & \cdots & \pi_{a,b-1} & \pi_{a,b} \end{array}$$

where $\pi_{i,j} \leq \pi_{i-1,j}$ and $\pi_{i,j} \leq \pi_{i,j-1}$.

Plane partitions

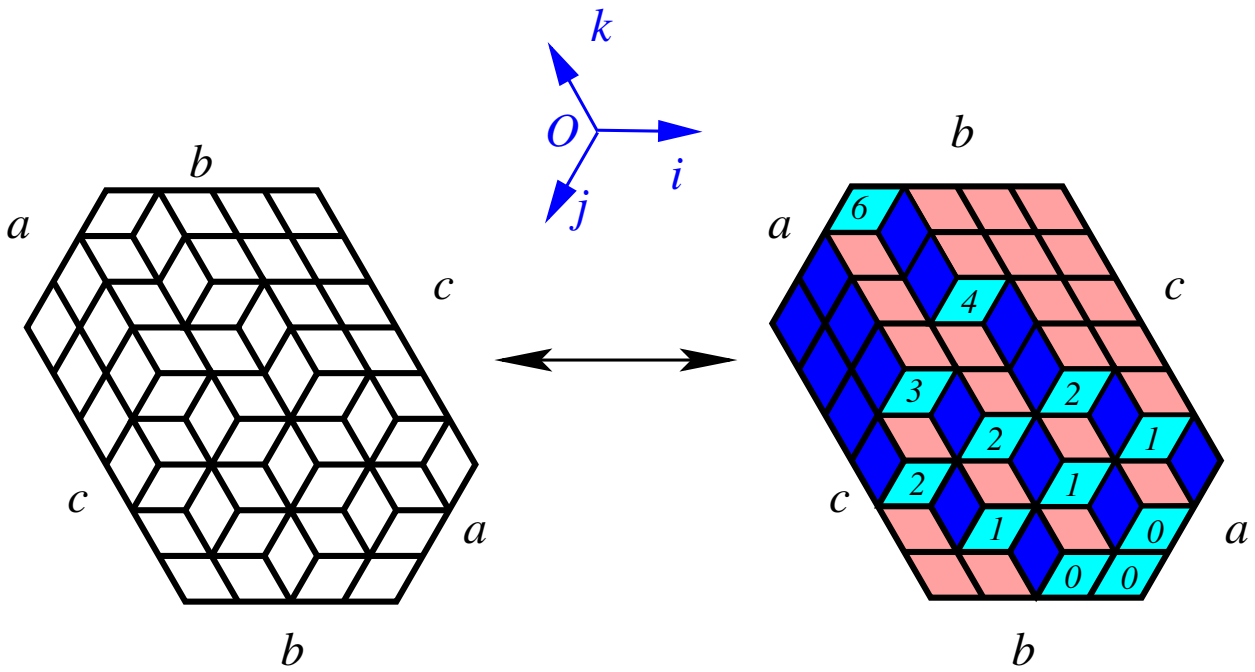
$$\begin{array}{cccc} 6 & 4 & 2 & 1 \\ 3 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{array}$$

Plane partitions as stacks of unit cubes

6 4 2 1

3 2 1 0

2 1 0 0



MacMahon's Theorem

Theorem (MacMahon ∼1900)

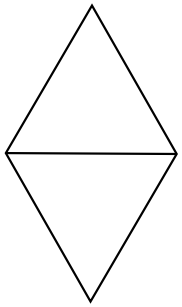
$$\sum_{\pi} q^{|\pi|} = PP_q(a,b,c) = \frac{\mathrm{H}_q(a)\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(a+b+c)}{\mathrm{H}_q(a+b)\,\mathrm{H}_q(b+c)\,\mathrm{H}_q(c+a)},$$

where the sum is taken over all plane partitions π fitting in an $a \times b \times c$ box, and $|\pi|$ is the volume of π.

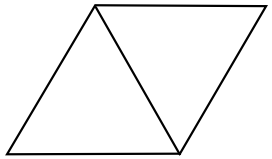
Definition:

- q-integer $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$
- q-factorial $[n]_q! = [1]_q[2]_q \ldots [n]_q$,
- q-hyperfactorial $\mathrm{H}_q(n) = [0]_q![1]_q! \ldots [n-1]_q!$.

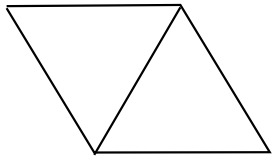
Lozenge tilings



Vertical



Right

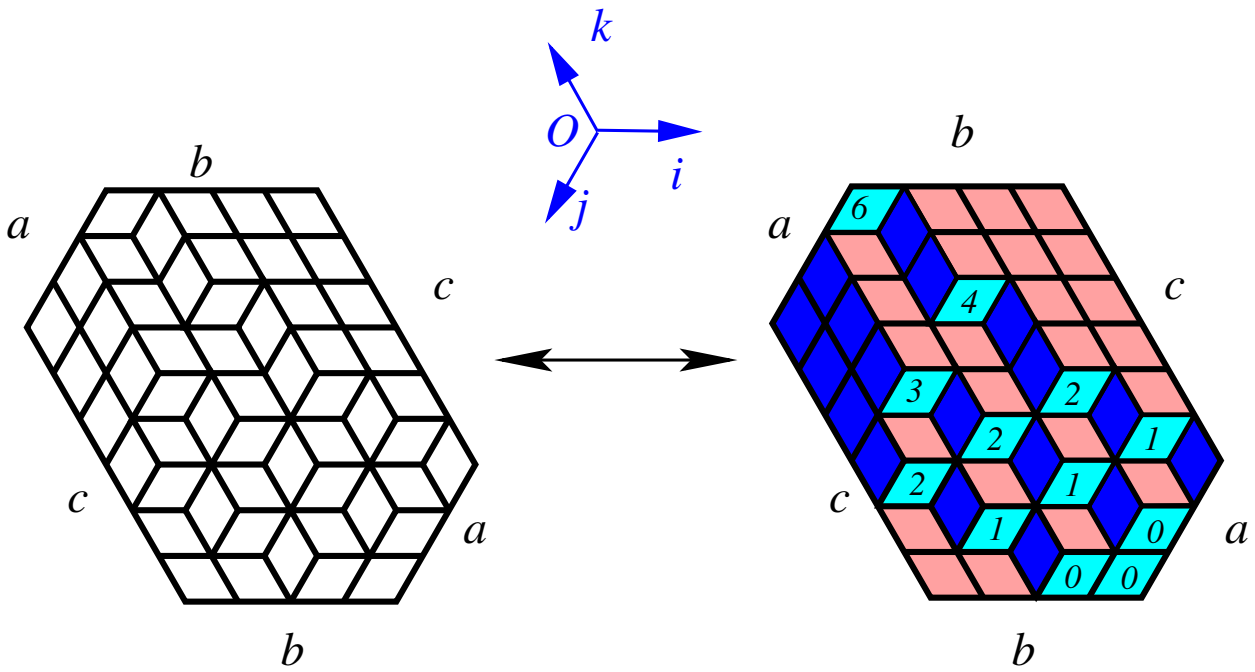


Left

- A lozenge (or unit rhombus) is the union of two adjacent unit equilateral triangles.
- A lozenge tiling of a region R on the triangular lattice is a covering of the region by lozenges, such that there are no gaps or overlaps.
- Denote by $\mathrm{M}(R)$ the number of lozenge tilings of the region R.

Plane partitions as stacks of unit cubes, as lozenge tilings

6	4	2	1
3	2	1	0
2	1	0	0

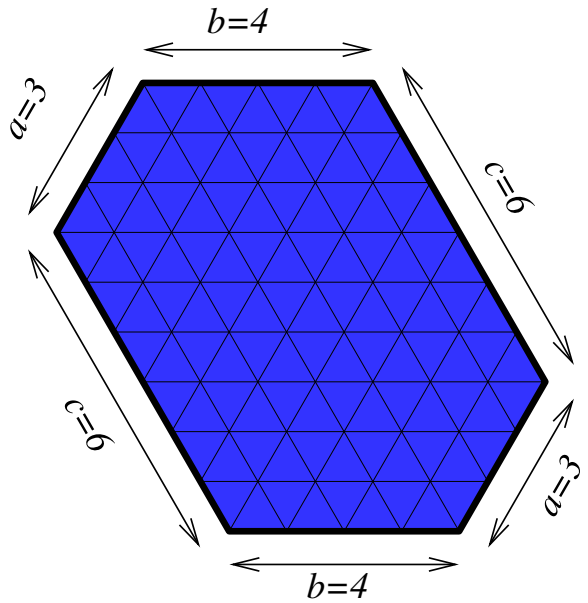


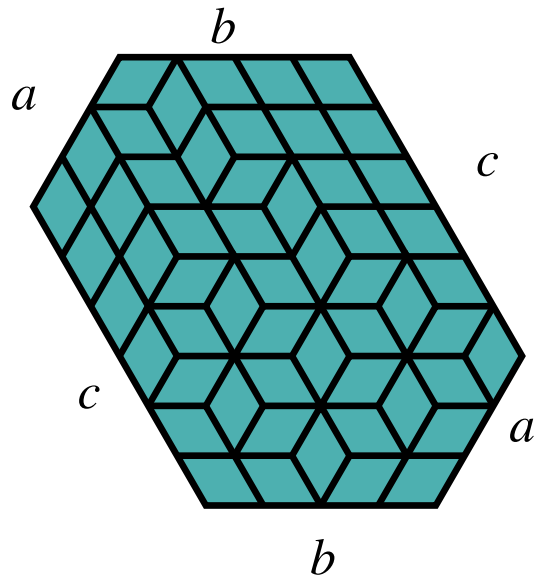
MacMahon's Theorem revisited

Theorem (MacMahon's Theorem for $q = 1$)

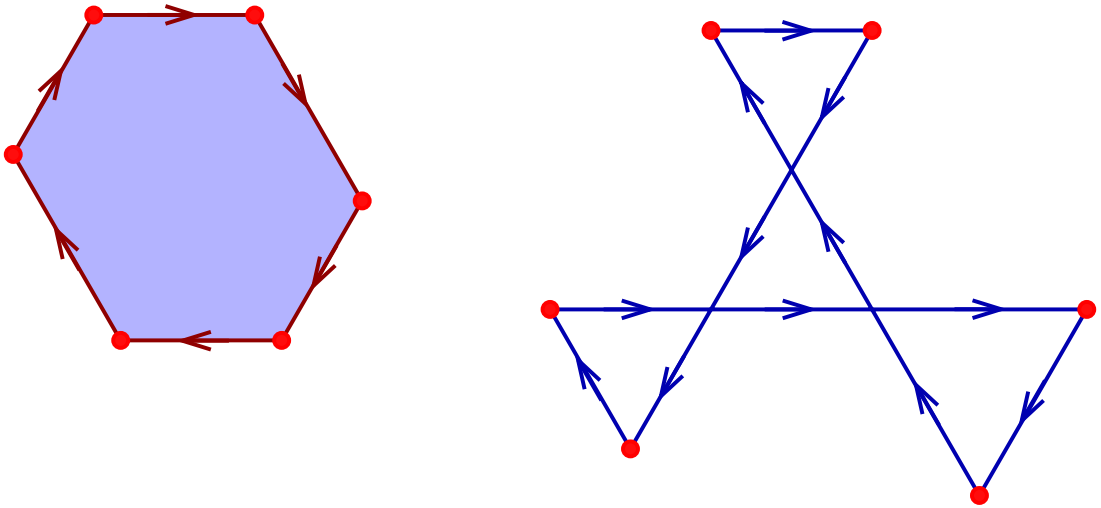
$$\mathrm{M}(Hex(a,b,c)) = PP(a,b,c) = \frac{\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)\,\mathrm{H}(a+b+c)}{\mathrm{H}(a+b)\,\mathrm{H}(b+c)\,\mathrm{H}(c+a)},$$

where the hyperfactorial $\mathrm{H}(n) := 0! \cdot 1! \cdot 2! \ldots (n-1)!$.



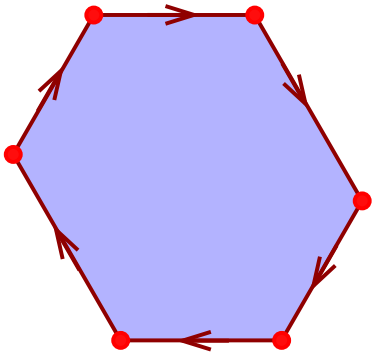


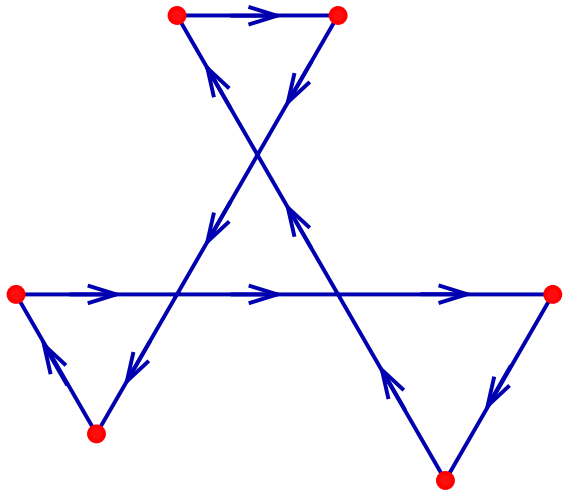
Another hexagonal contour of MacMahon's Theorem



MacMahon's theorem gives the number of lozenge tilings inside the ordinary hexagonal contour.

Another hexagonal contour of MacMahon's Theorem

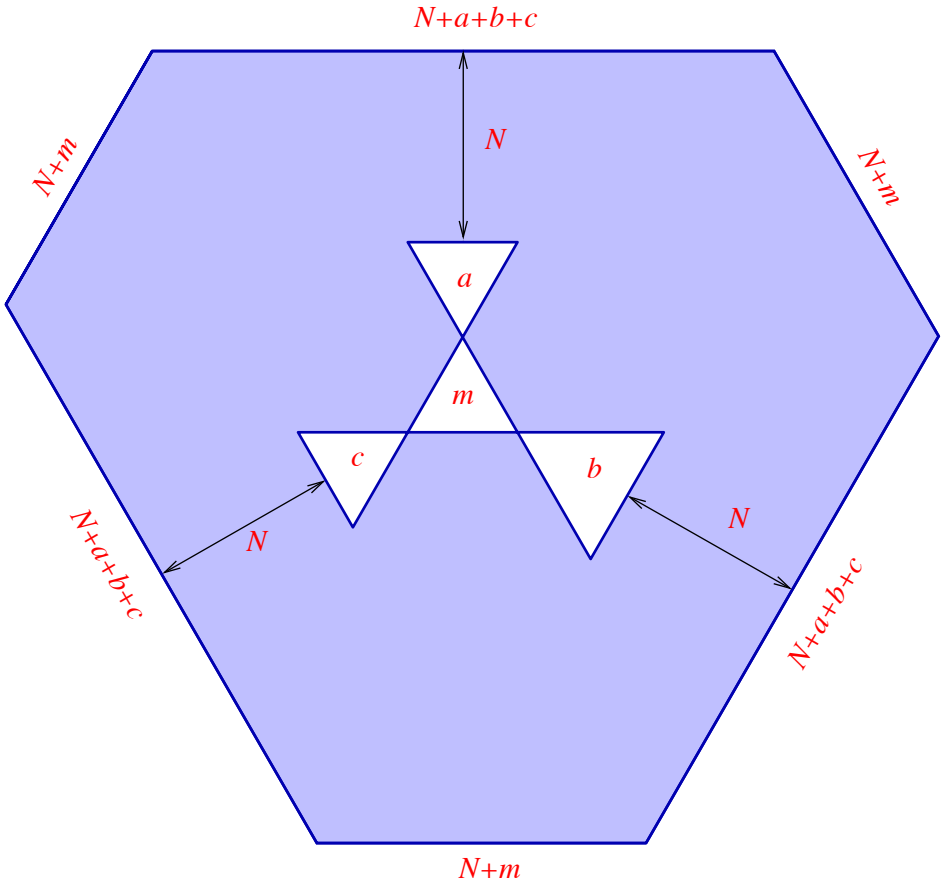




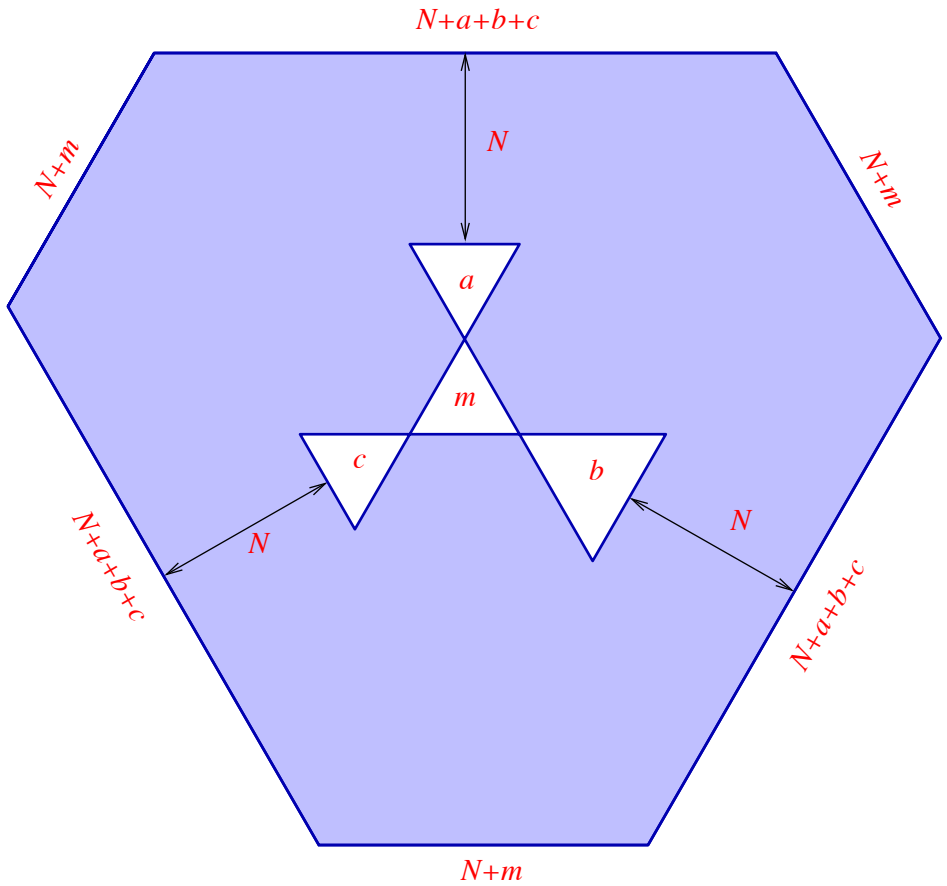
What is the (asymptotic) behavior of the number of lozenge tilings outside the self-intersecting hexagonal contour?

Dual of MacMahon's Theorem

The region $S_{N,N,N}(a, b, c, m)$.

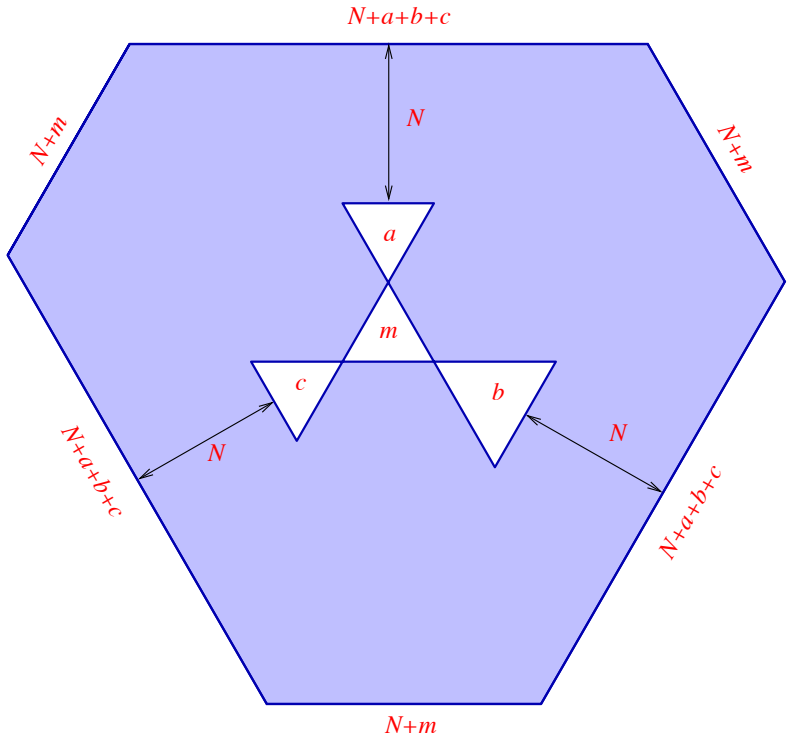


Dual of MacMahon's Theorem



$$\lim_{N\to\infty} \frac{\mathrm{M}(S_{N,N,N}(a,b,c,m))}{\mathrm{M}(S_{N,N,N}(a+b+c,0,0,m))} = ?$$

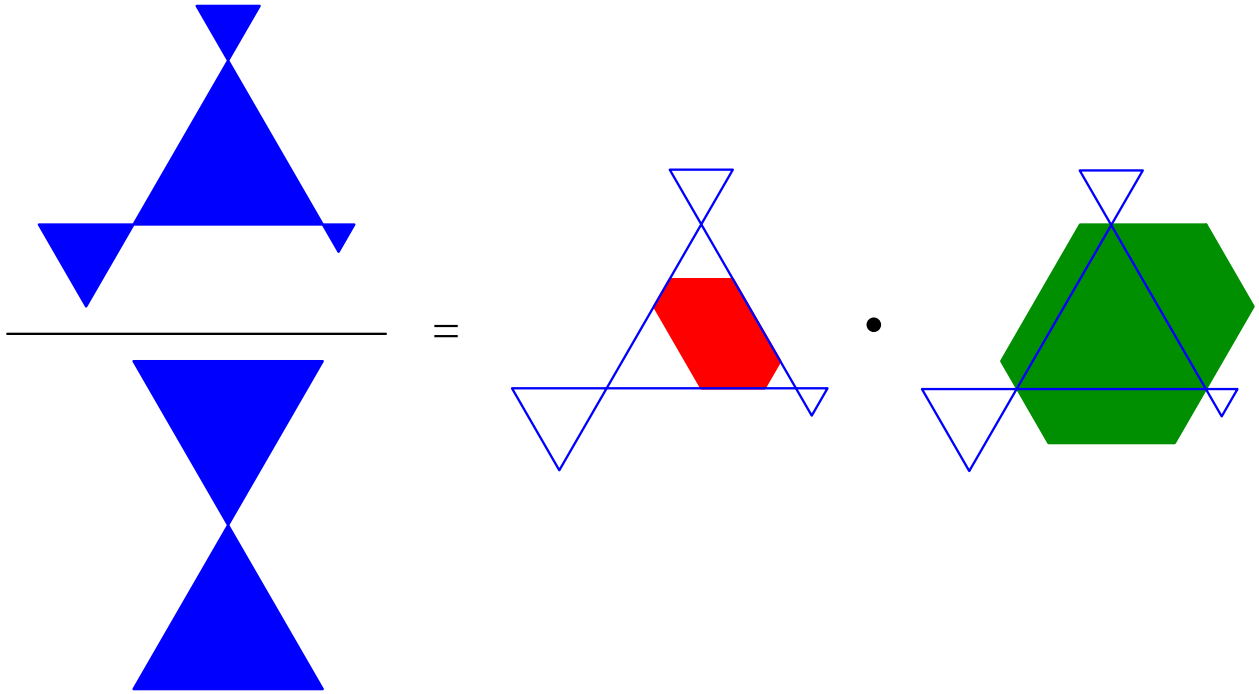
Dual of MacMahon's Theorem



Theorem ('Dual of MacMahon's Theorem' by Ciucu and Krattenthaler)

$$\lim_{N\to\infty} \frac{\mathrm{M}(S_{N,N,N}(a,b,c,m))}{\mathrm{M}(S_{N,N,N}(a+b+c,0,0,m))} = PP(a,b,m)\cdot PP(a+b,c,m)$$

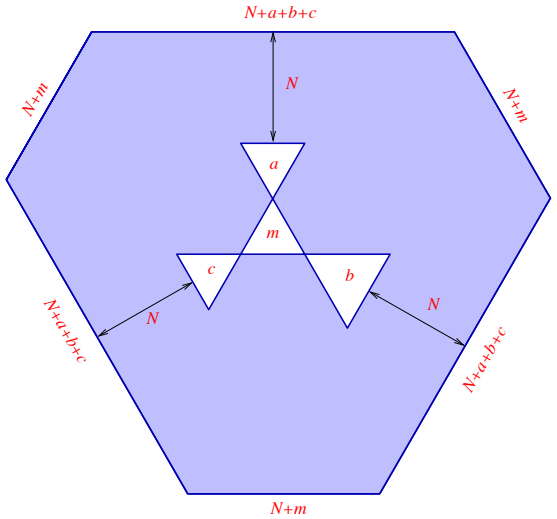
Dual of MacMahon's Theorem



Corollary (Special case $m = a + b + c$)

$$\lim_{N\to\infty} \frac{\mathrm{M}(S_{N,N,N}(a,b,c,a+b+c))}{\mathrm{M}(S_{N,N,N}(a+b+c,0,0,a+b+c))} = PP(a,b,c)\cdot PP(a+b,b+c,c+a)$$

Questions



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- How about the number of tilings when the hole is off-center?

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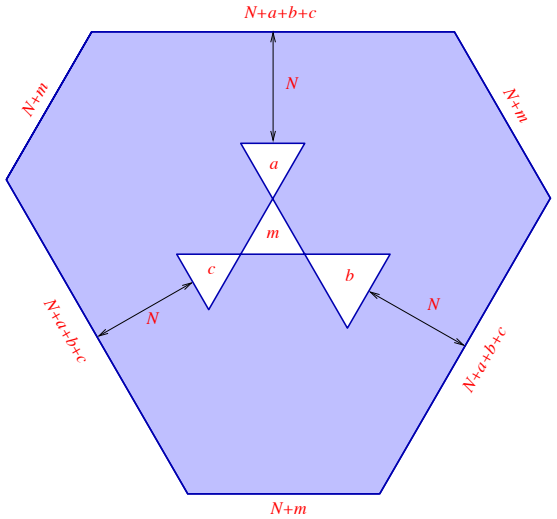

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- How about the number of tilings when the hole is off-center?
- Can we have a q-analog as in the general version of MacMahon's theorem?

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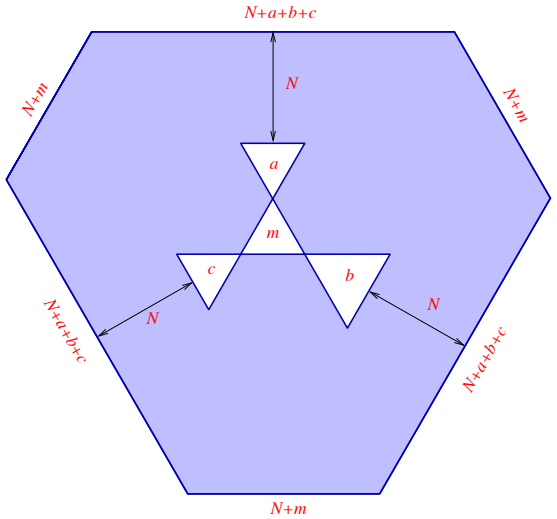

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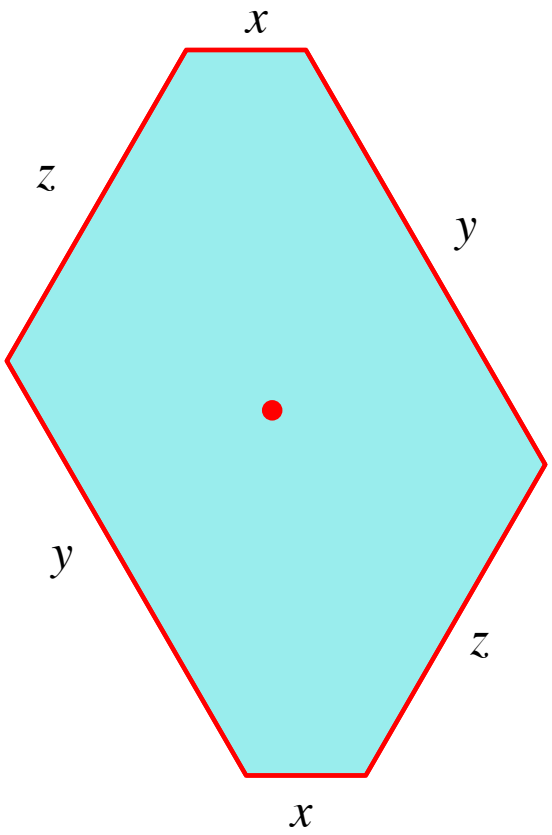
$$\lim_{N\to\infty} \frac{\mathrm{M}(S_{N,N,N}(a,b,c,m))}{\mathrm{M}(S_{N,N,N}(a+b+c,0,0,m))} = PP(a,b,m)\cdot PP(a+b,c,m)$$

- How about the number of tilings when the hole is off-center?
- Can we have a q-analog as in the general version of MacMahon's theorem?
- Can we have more duals?

Definition of S-cored hexagon $S_{x,y,z}(a, b, c, m)$



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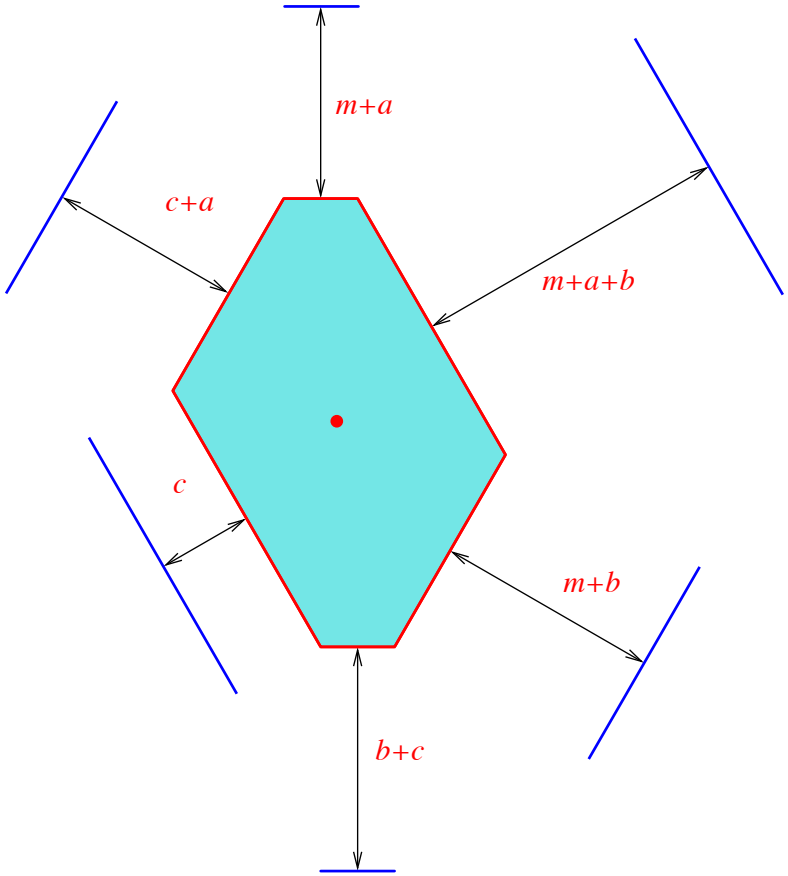

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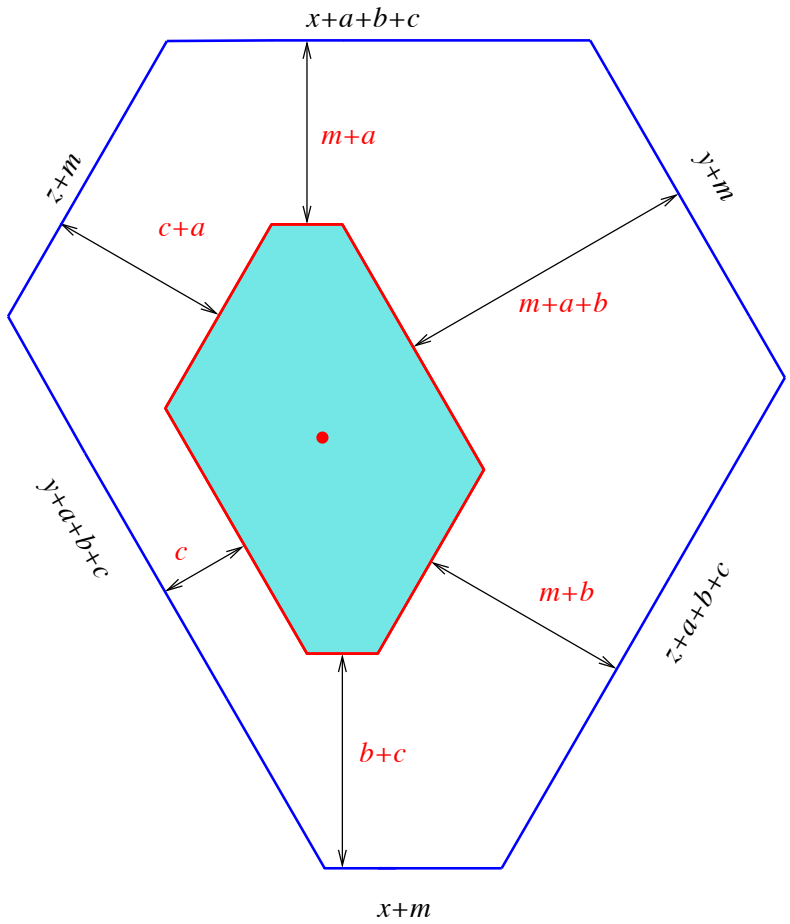

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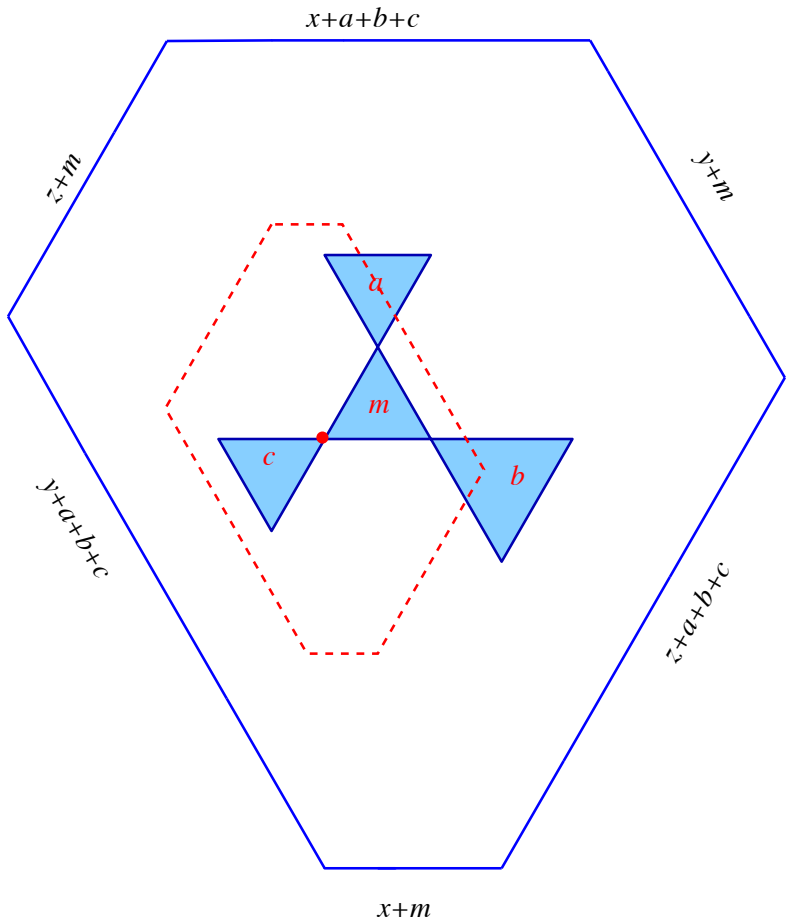

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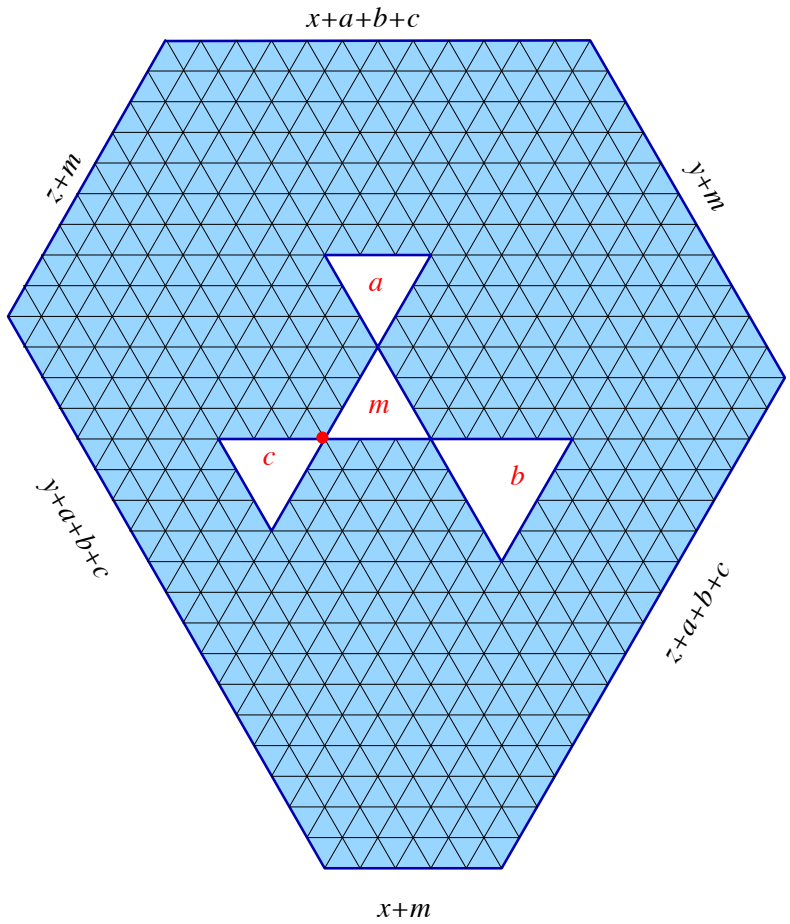
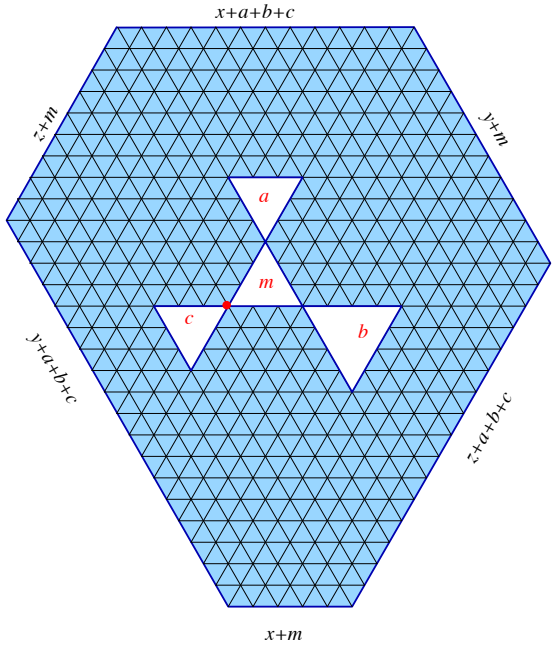
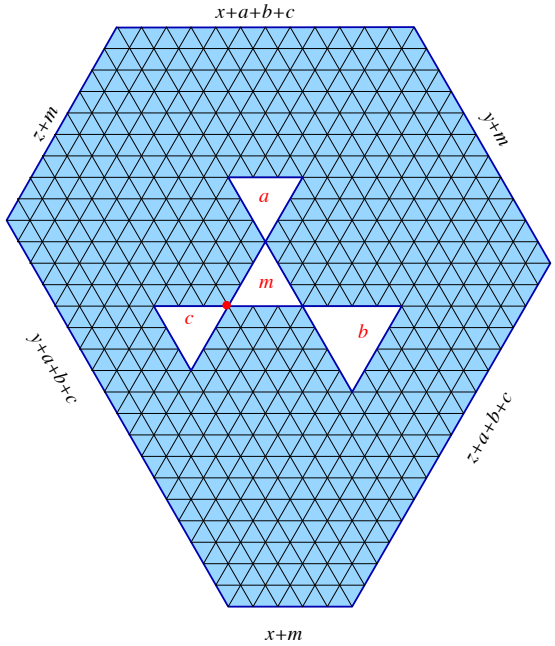


Enumeration of S-cored hexagon



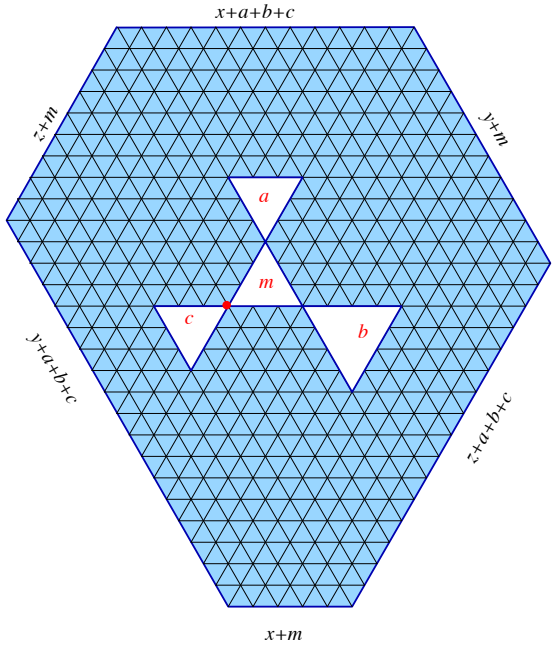
- The 'hyperfactorial' is
  $H(n) := 0! \cdot 1! \cdot 2! \ldots (n-1)! = \prod_{i=0}^{n-1} \Gamma(i+1)$.

Enumeration of S-cored hexagon



- The 'hyperfactorial' is $H(n) := 0! \cdot 1! \cdot 2! \ldots (n-1)! = \prod_{i=0}^{n-1} \Gamma(i+1)$.
- $H(n+\frac{1}{2}) := \prod_{i=0}^{n} \Gamma(i+\frac{1}{2})$.

Enumeration of S-cored hexagon



- The 'hyperfactorial' is $H(n) := 0! \cdot 1! \cdot 2! \ldots (n-1)! = \prod_{i=0}^{n-1} \Gamma(i+1)$.
- $H(n+\frac{1}{2}) := \prod_{i=0}^{n} \Gamma(i+\frac{1}{2})$.
- $\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$

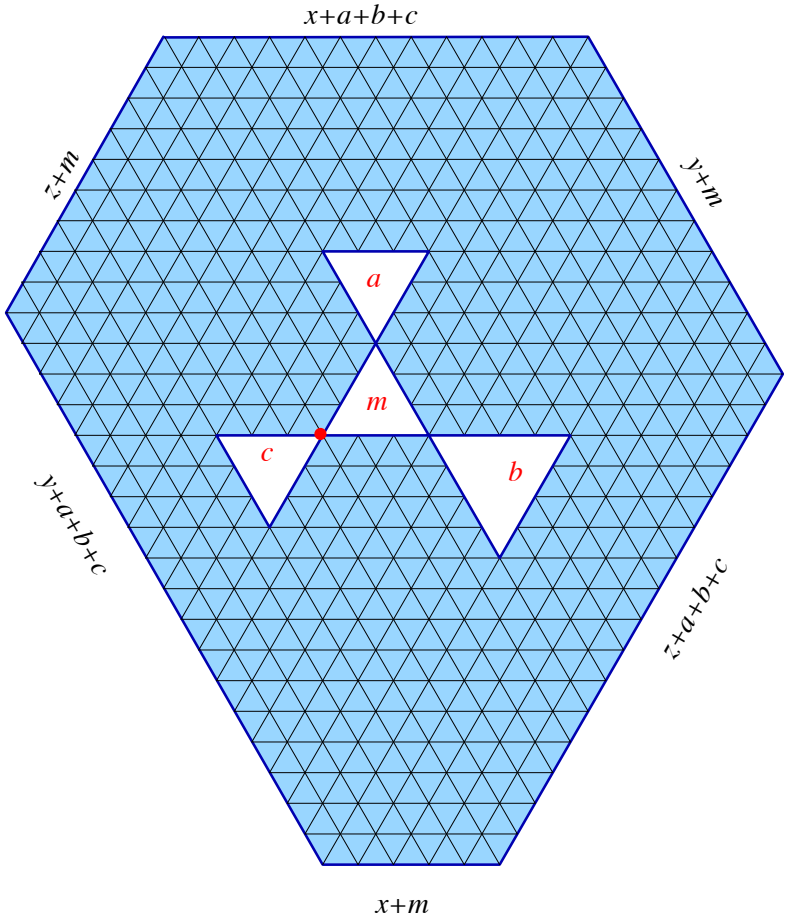
Enumeration of S-cored hexagon

Theorem (Ciucu–Krattenthaler 2013)

$$\mathrm{M}(S_{x,y,z}(a,b,c,m)) = \frac{\mathrm{H}(m)^3\,\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)}{\mathrm{H}(m+a)\,\mathrm{H}(m+b)\,\mathrm{H}(m+c)}$$
$$\frac{\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor+m+a+b)\,\mathrm{H}(\lceil\frac{x+z}{2}\rceil+m+a+c)\,\mathrm{H}(\frac{y+z}{2}+m+b+c)}{\mathrm{H}(\lceil\frac{x+y}{2}\rceil+m+c)\,\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor+m+b)\,\mathrm{H}(\frac{y+z}{2}+m+a)}$$
$$\frac{\mathrm{H}(x+\Delta)\,\mathrm{H}(y+\Delta)\,\mathrm{H}(z+\Delta)\,\mathrm{H}(x+y+z+\Delta)}{\mathrm{H}(x+y+\Delta)\,\mathrm{H}(x+z+\Delta)\,\mathrm{H}(y+z+\Delta)}$$
$$\frac{\mathrm{H}(\lceil\frac{x+y+z}{2}\rceil+\Delta)\,\mathrm{H}(\lfloor\frac{x+y+z}{2}\rfloor+\Delta)}{\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor+\Delta)\,\mathrm{H}(\lceil\frac{x+z}{2}\rceil+\Delta)\,\mathrm{H}(\frac{y+z}{2}+\Delta)}$$
$$\frac{\mathrm{H}(\lceil\frac{x}{2}\rceil)\,\mathrm{H}(\lfloor\frac{x}{2}\rfloor)\,\mathrm{H}(\lceil\frac{y}{2}\rceil)\,\mathrm{H}(\lfloor\frac{y}{2}\rfloor)\,\mathrm{H}(\lceil\frac{z}{2}\rceil)\,\mathrm{H}(\lfloor\frac{z}{2}\rfloor)}{\mathrm{H}(\lceil\frac{x}{2}\rceil+\frac{\Delta}{2})\,\mathrm{H}(\lfloor\frac{x}{2}\rfloor+\frac{\Delta}{2})\,\mathrm{H}(\lceil\frac{y}{2}\rceil+\frac{\Delta}{2})\,\mathrm{H}(\lfloor\frac{y}{2}\rfloor+\frac{\Delta}{2})\,\mathrm{H}(\lceil\frac{z}{2}\rceil+\frac{\Delta}{2})\,\mathrm{H}(\lfloor\frac{z}{2}\rfloor+\frac{\Delta}{2})}$$
$$\frac{\mathrm{H}(\frac{\Delta}{2})^2\,\mathrm{H}(\lceil\frac{x+y}{2}\rceil+\frac{\Delta}{2})\,\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor+\frac{\Delta}{2})}{\mathrm{H}(\lceil\frac{x+y+z}{2}\rceil+\frac{\Delta}{2})\,\mathrm{H}(\lfloor\frac{x+y+z}{2}\rfloor+\frac{\Delta}{2})\,\mathrm{H}(\lceil\frac{x+y}{2}\rceil)}$$
$$\frac{\mathrm{H}(\lceil\frac{x+z}{2}\rceil+\frac{\Delta}{2})\,\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor+\frac{\Delta}{2})\,\mathrm{H}(\frac{y+z}{2}+\frac{\Delta}{2})^2}{\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor)\,\mathrm{H}(\frac{y+z}{2})}.$$

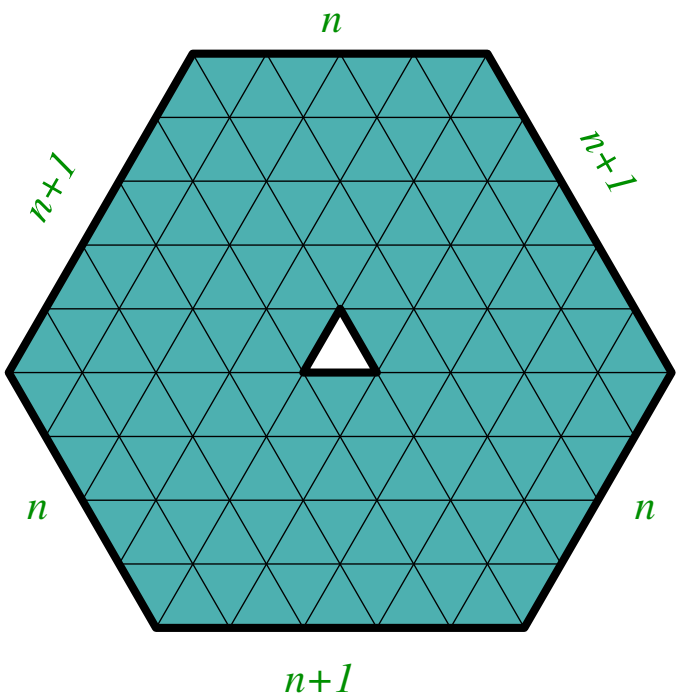
$\Delta = m+a+b+c$

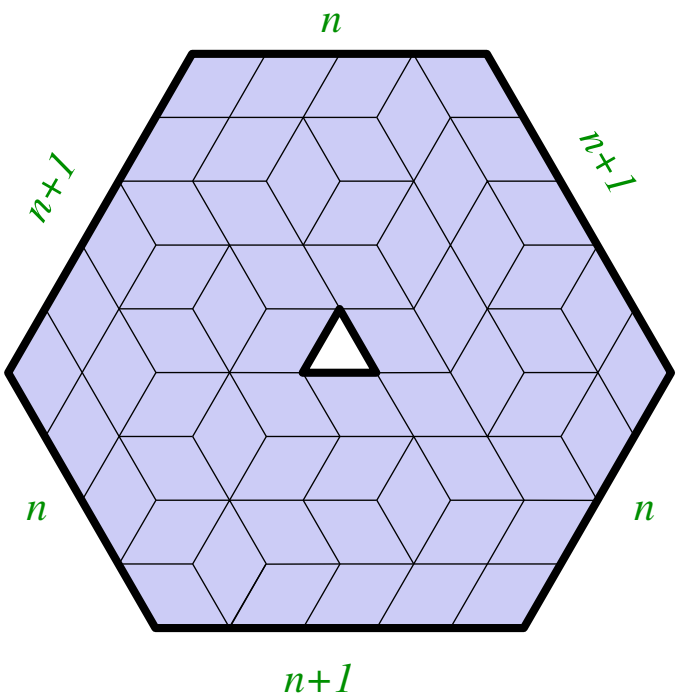
Enumeration of S-cored hexagon



Bad news: The there **no** nice product formula for the off-center case!

History: James Propp's Problem



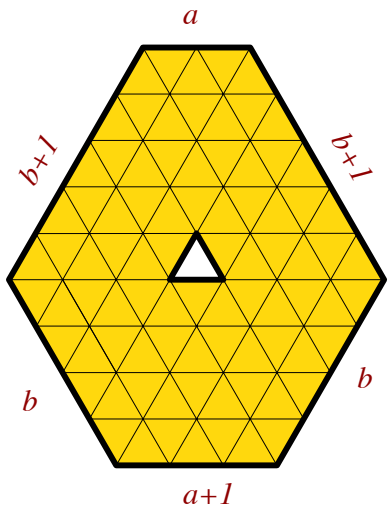


Open Problem (Propp 1997)

Find an explicit formula for the number of tilings of a hexagon of sides $n, n+1, n, n+1, n, n+1$ with the central unit triangle removed.

This is Problem 3 on his list of 20 open problems in the field of enumeration of tilings.

History: Ciucu's Solution



Theorem (Ciucu 1998)

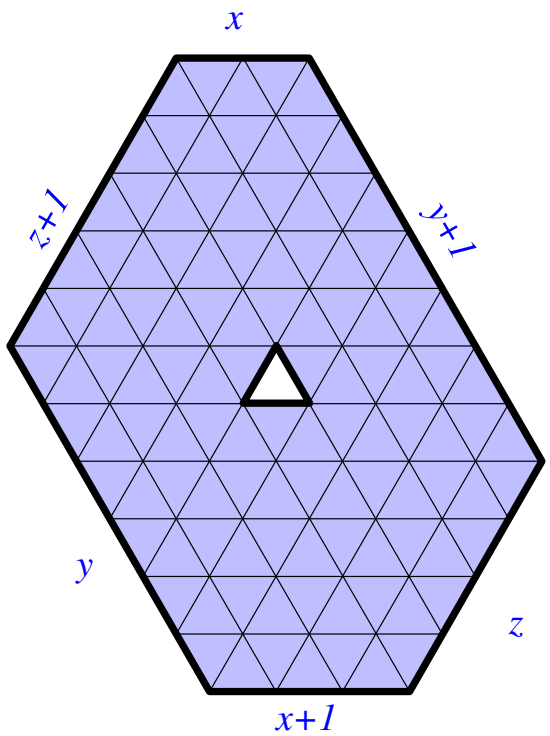
The tiling number of the hexagon of sides $a, b+1, b, a+1, b, b+1$ *with the central unit triangle removed is:*

$$SC(a+1, b, b)SC(a, b+1, b+1),$$

where $SC(a, b, c)$ *is the number of 'self-complementary plane partitions' that fit in an* $a \times b \times c$ *box.*

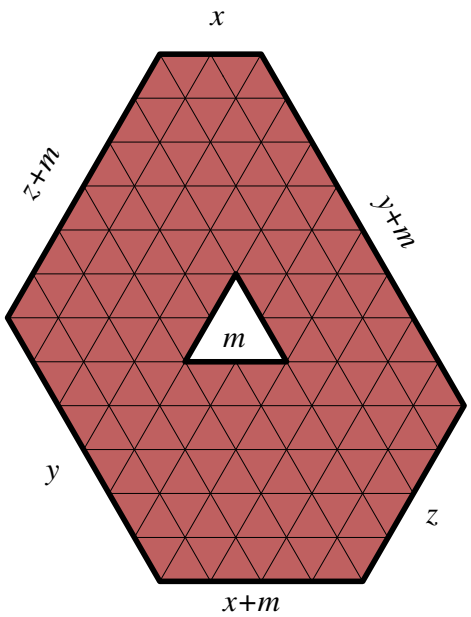
Gessel and Helfgott (1999) re-proved by a different method.

History: Krattenthaler–Okada's generalization



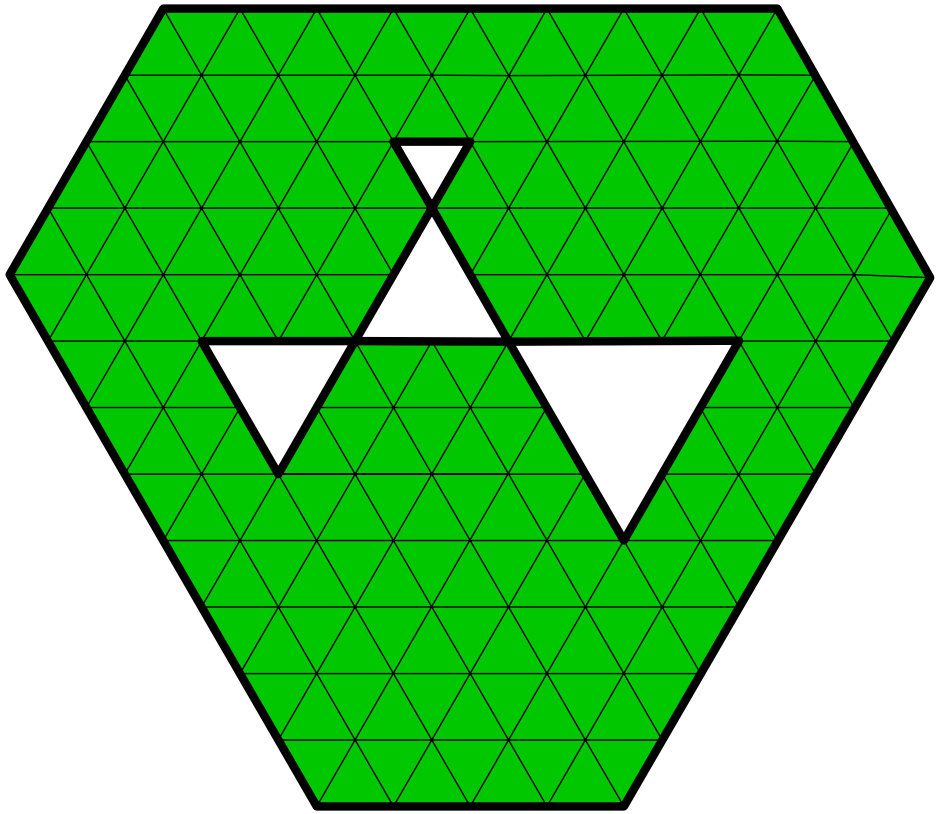
Krattenthaler and Okada (1998) generalized Ciucu's enumeration for a hexagon of sides $x, y+1, z, x+1, y, z+1$ with the central unit triangle removed.

History: Ciucu–Eisenkölbl–Krattenthaler–Zare's cored hexagon



- Unit triangle is replaced by a triangle of any side.
- This is $C_{x,y,z}(m) := S_{x,y,z}(0,0,0,m)$.

History: Ciucu–Krattenthaler's S-cored hexagon



Ciucu–Krattenthaler (2013) generalized to a hexagon with a 'shamrock' removed in the center

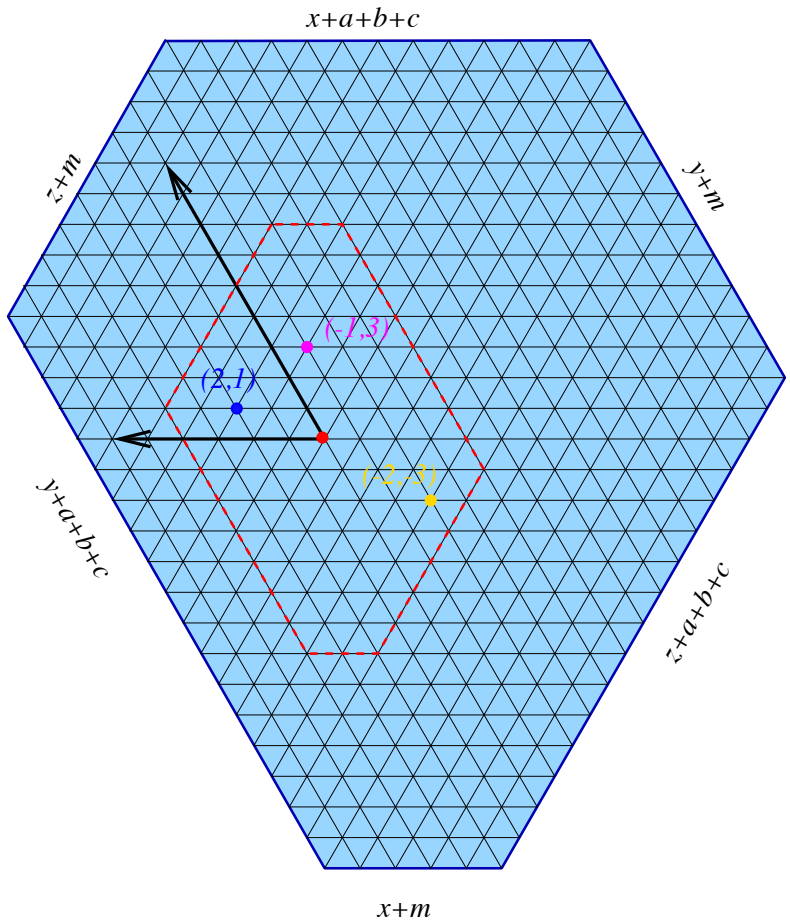
History: Ciucu–Krattenthaler's S-cored hexagon

Ciucu–Krattenthaler Enumeration Reformed

Theorem

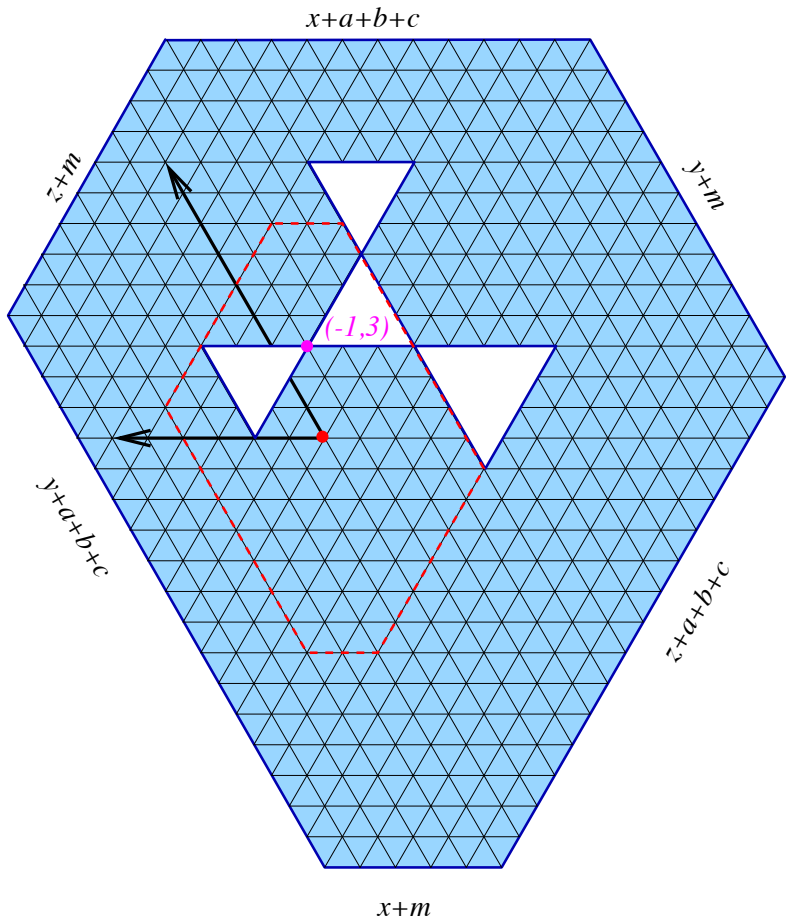
$$\mathrm{M}(S_{x,y,z}(a,b,c,m)) = \mathrm{M}(S_{x,y,z}(0,0,0,m+a+b+c))$$
$$\times \frac{\mathrm{H}(m)^3\,\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)}{\mathrm{H}(m+a)\,\mathrm{H}(m+b)\,\mathrm{H}(m+c)}$$
$$\times \frac{\mathrm{H}(\lceil\frac{x+y}{2}\rceil+c)\,\mathrm{H}(\lceil\frac{x+y}{2}\rceil+\Delta)\,\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor)\,\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor+m+a+b)}{\mathrm{H}(\lceil\frac{x+y}{2}\rceil)\,\mathrm{H}(\lceil\frac{x+y}{2}\rceil+m+c)\,\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor+a+b)\,\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor+\Delta)}$$
$$\times \frac{\mathrm{H}(\lceil\frac{x+z}{2}\rceil+c)\,\mathrm{H}(\lceil\frac{x+z}{2}\rceil+m+a+c)\,\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor+b)\,\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor+\Delta)}{\mathrm{H}(\lceil\frac{x+z}{2}\rceil+a+c)\,\mathrm{H}(\lceil\frac{x+z}{2}\rceil+\Delta)\,\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor)\,\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor+m+b)}$$
$$\times \frac{\mathrm{H}(\frac{y+z}{2}+m+b+c)\,\mathrm{H}(\frac{y+z}{2}+a)}{\mathrm{H}(\frac{y+z}{2}+b+c)\,\mathrm{H}(\frac{y+z}{2}+m+a)}.$$

Moving the ‘shamrock’ hole



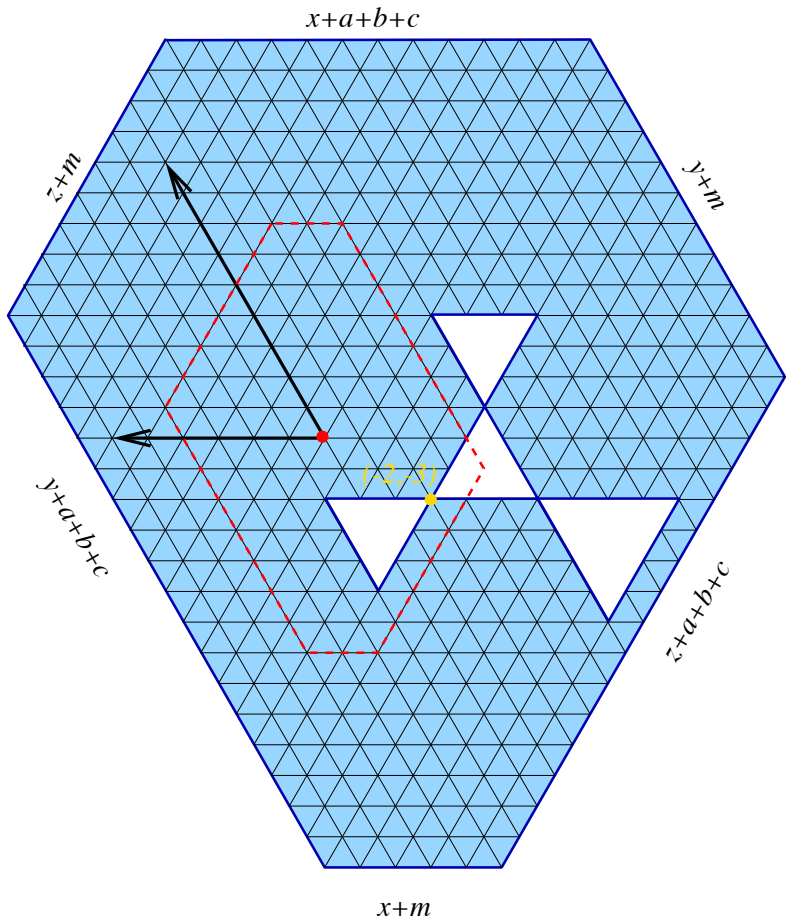
Moving the 'shamrock' hole

The region $S^{-1,3}_{x,y,z}(a,b,c,m)$



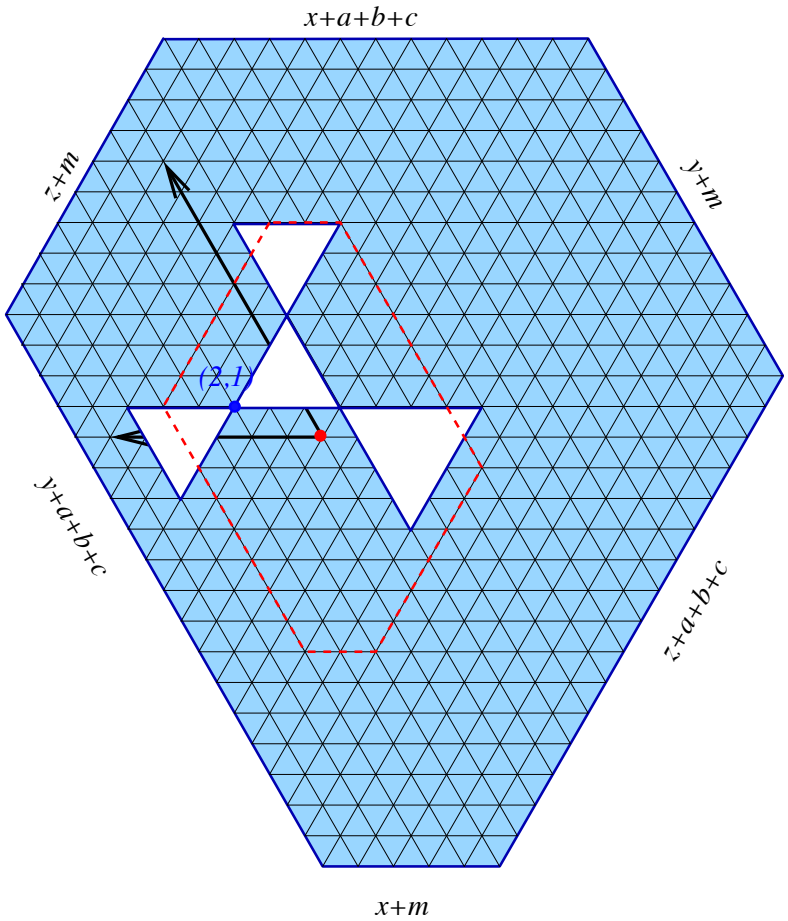
Moving the 'shamrock' hole

The region $S^{-2,-3}_{x,y,z}(a,b,c,m)$



Moving the 'shamrock' hole

The region $S^{2,1}_{x,y,z}(a,b,c,m)$



Factorization Theorem for Generalized S-cored Hexagons

Theorem (Factorization Theorem T.L. 2018)

$$\mathrm{M}(S^{M,N}_{x,y,z}(a,b,c,m)) = \mathrm{M}(S^{M,N}_{x,y,z}(0,0,0,a+b+c+m))$$
$$\times \frac{\mathrm{H}(m)^3\,\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)}{\mathrm{H}(m+a)\,\mathrm{H}(m+b)\,\mathrm{H}(m+c)}$$
$$\times \frac{\mathrm{H}(\lceil\frac{x+y}{2}\rceil + c + M + N)\,\mathrm{H}(\lceil\frac{x+y}{2}\rceil + \Delta + M + N)}{\mathrm{H}(\lceil\frac{x+y}{2}\rceil + M + N)\,\mathrm{H}(\lceil\frac{x+y}{2}\rceil + m + c + M + N)}$$
$$\times \frac{\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor - M - N)\,\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor + m + a + b - M - N)}{\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor + a + b - M - N)\,\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor + \Delta - M - N)}$$
$$\times \frac{\mathrm{H}(\lceil\frac{x+z}{2}\rceil + c + M)\,\mathrm{H}(\lceil\frac{x+z}{2}\rceil + m + a + c + M)}{\mathrm{H}(\lceil\frac{x+z}{2}\rceil + a + c + M)\,\mathrm{H}(\lceil\frac{x+z}{2}\rceil + \Delta + M)}$$
$$\times \frac{\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor + b - M)\,\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor + \Delta - M)}{\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor - M)\,\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor + m + b - M)}$$
$$\times \frac{\mathrm{H}(\frac{y+z}{2} + N)\,\mathrm{H}(\frac{y+z}{2} + m + b + c + N)\,\mathrm{H}(\frac{y+z}{2} + a - N)\,\mathrm{H}(\frac{y+z}{2} + \Delta - N)}{\mathrm{H}(\frac{y+z}{2} + b + c + N)\,\mathrm{H}(\frac{y+z}{2} + \Delta + N)\,\mathrm{H}(\frac{y+z}{2} - N)\,\mathrm{H}(\frac{y+z}{2} + m + a - N)}.$$

Generalized dual

Theorem (Generalized Dual of MacMahon's Theorem, T.L. 2018)

For real number M, N

$$\lim_{k\to\infty} \frac{\mathrm{M}(S_{k,k,k}^{\lfloor Mk\rfloor,\lfloor Nk\rfloor}(a,b,c,m))}{\mathrm{M}(S_{k,k,k}^{\lfloor Mk\rfloor,\lfloor Nk\rfloor}(a+b+c,0,0,m))} = PP(a,b,m)\cdot PP(a+b,c,m)$$

$$\times \frac{(1-(M+N)^2)^{ma+mb}}{(1-M^2)^{mb}(1-N^2)^{ma}}$$

A harder question: q-analog

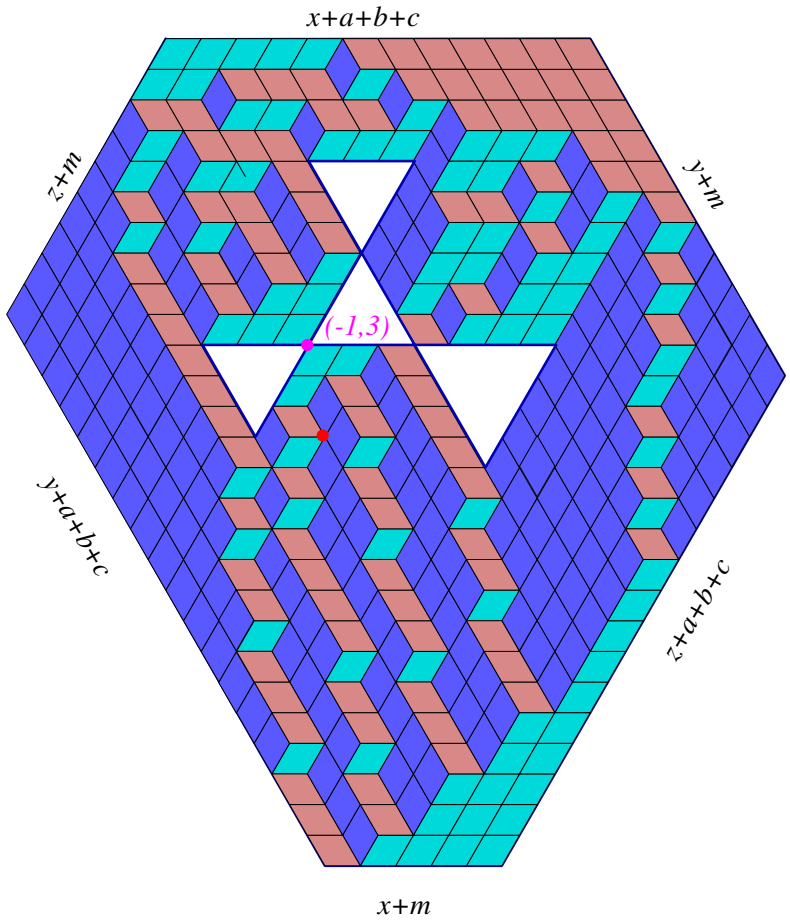
Do we have a dual for MacMahon's theorem for general q?

A harder question: q-analog

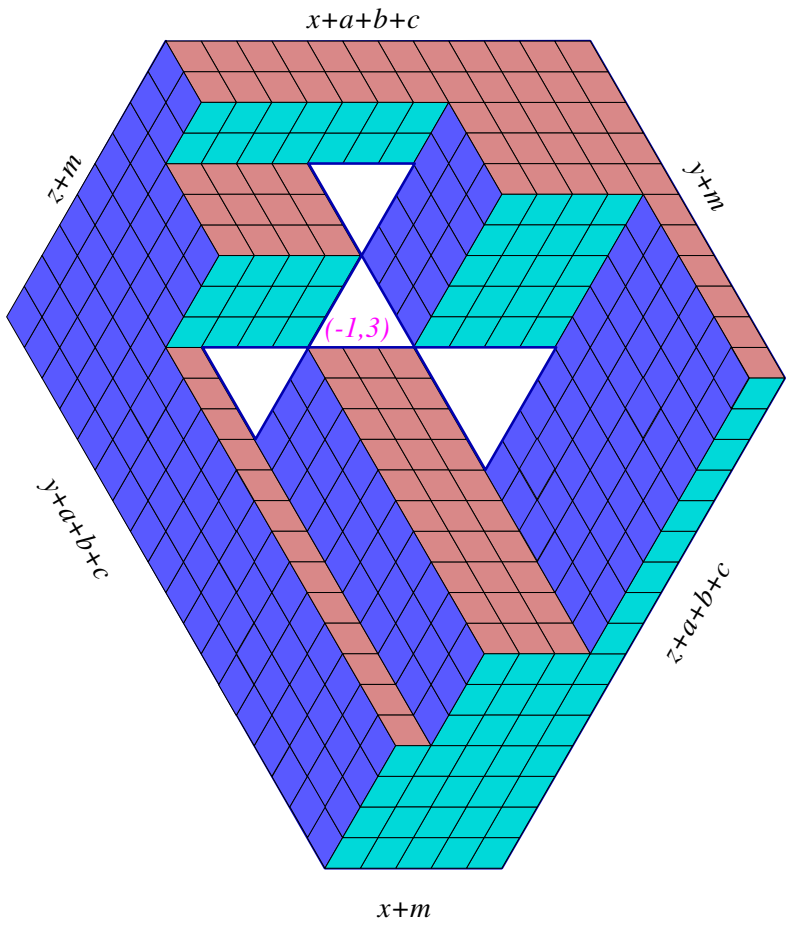
Do we have a dual for MacMahon's theorem for general q?

Facts: There are not many closed-form q-analogs!

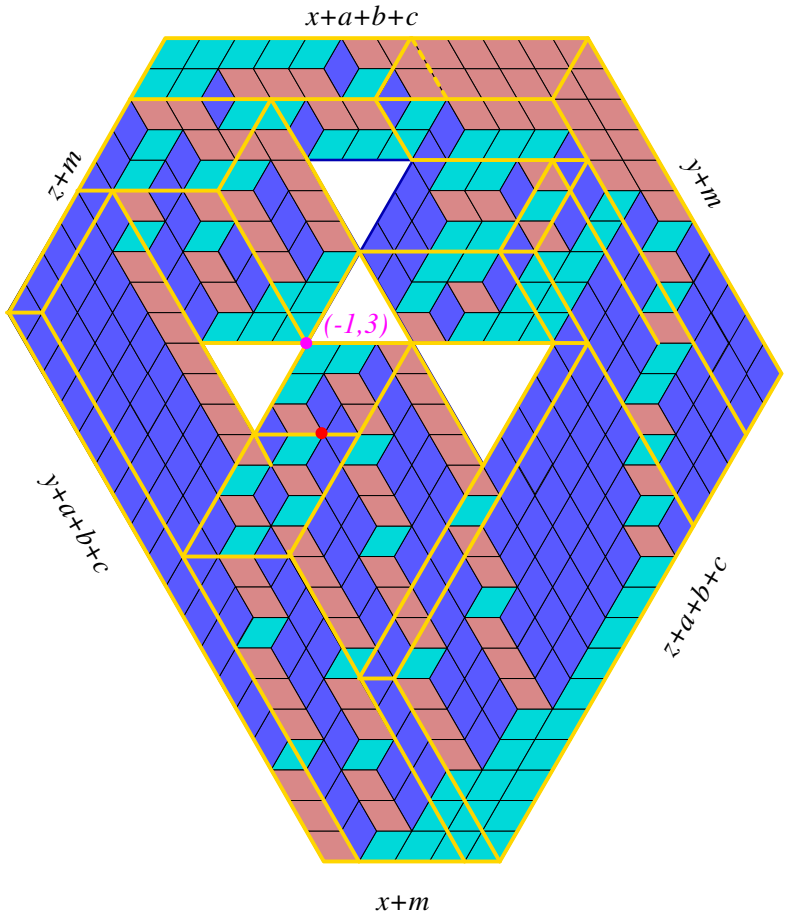
Tilings as Stacks of Cubes



Tilings as Stacks of Cubes

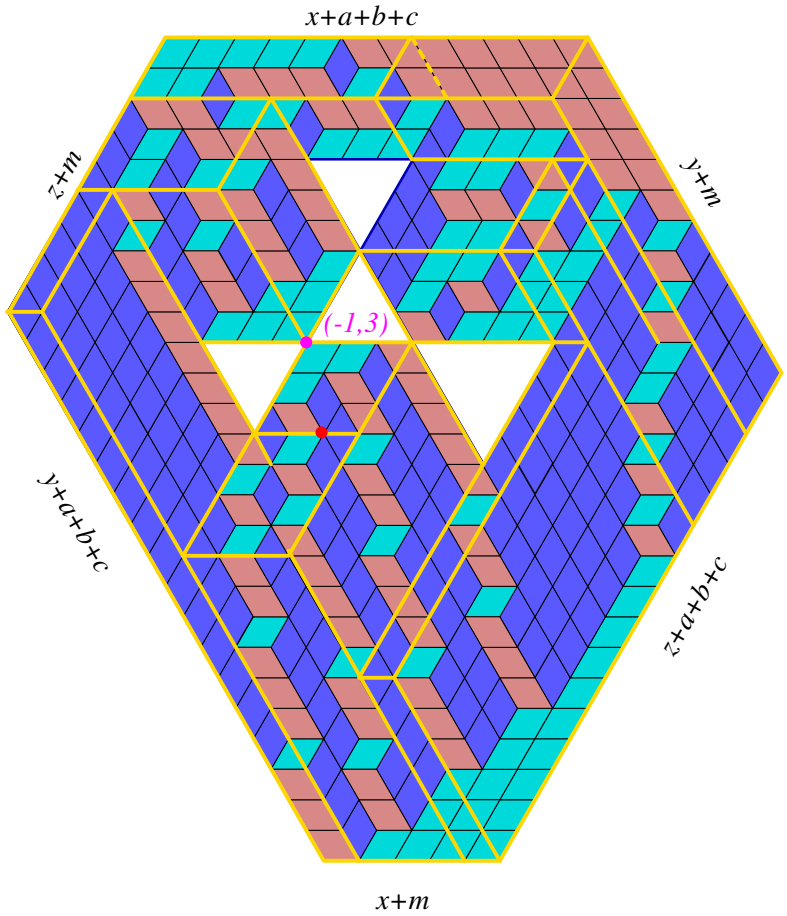


Tilings as Stacks of Cubes



$$\mathrm{M}_q(S^{M,N}_{x,y,z}(a,b,c,m)) = \sum_{\pi} q^{vol(\pi)}$$

Tilings as Stacks of Cubes



$$\mathrm{M}_q(S^{M,N}_{x,y,z}(a,b,c,m)) = \sum_{\pi} q^{vol(\pi)}$$

Bad news: This is not given by a simple product formula, even for the case $M = N = 0$!

Theorem (q-Factorization Theorem T.L. 2018)

$$\mathrm{M}_q(S^{M,N}_{x,y,z}(a,b,c,m)) = \mathrm{M}_q(S^{M,N}_{x,y,z}(0,0,0,a+b+c+m))$$
$$\times \frac{\mathrm{H}_q(m)^3\,\mathrm{H}_q(a)\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)}{\mathrm{H}_q(m+a)\,\mathrm{H}_q(m+b)\,\mathrm{H}_q(m+c)}$$
$$\times \frac{\mathrm{H}_q(\lceil\frac{x+y}{2}\rceil+c+M+N)\,\mathrm{H}_q(\lceil\frac{x+y}{2}\rceil+\Delta+M+N)}{\mathrm{H}_q(\lceil\frac{x+y}{2}\rceil+M+N)\,\mathrm{H}_q(\lceil\frac{x+y}{2}\rceil+m+c+M+N)}$$
$$\times \frac{\mathrm{H}_q(\lfloor\frac{x+y}{2}\rfloor-M-N)\,\mathrm{H}_q(\lfloor\frac{x+y}{2}\rfloor+m+a+b-M-N)}{\mathrm{H}_q(\lfloor\frac{x+y}{2}\rfloor+a+b-M-N)\,\mathrm{H}_q(\lfloor\frac{x+y}{2}\rfloor+\Delta-M-N)}$$
$$\times \frac{\mathrm{H}_q(\lceil\frac{x+z}{2}\rceil+c+M)\,\mathrm{H}_q(\lceil\frac{x+z}{2}\rceil+m+a+c+M)}{\mathrm{H}_q(\lceil\frac{x+z}{2}\rceil+a+c+M)\,\mathrm{H}_q(\lceil\frac{x+z}{2}\rceil+\Delta+M)}$$
$$\times \frac{\mathrm{H}_q(\lfloor\frac{x+z}{2}\rfloor+b-M)\,\mathrm{H}_q(\lfloor\frac{x+z}{2}\rfloor+\Delta-M)}{\mathrm{H}_q(\lfloor\frac{x+z}{2}\rfloor-M)\,\mathrm{H}_q(\lfloor\frac{x+z}{2}\rfloor+m+b-M)}$$
$$\times \frac{\mathrm{H}_q(\frac{y+z}{2}+N)\,\mathrm{H}_q(\frac{y+z}{2}+m+b+c+N)\,\mathrm{H}_q(\frac{y+z}{2}+a-N)\,\mathrm{H}_q(\frac{y+z}{2}+\Delta-N)}{\mathrm{H}_q(\frac{y+z}{2}+b+c+N)\,\mathrm{H}_q(\frac{y+z}{2}+\Delta+N)\,\mathrm{H}_q(\frac{y+z}{2}-N)\,\mathrm{H}_q(\frac{y+z}{2}+m+a-N)}.$$

Corollary (q-Dual of MacMahon's Theorem, T.L. 2018)

For $|q| < 1$

$$\lim_{k\to\infty} \frac{\mathrm{M}_q(S_{k,k,k}^{\lfloor Mk\rfloor,\lfloor Nk\rfloor}(a,b,c,m))}{\mathrm{M}_q(S_{k,k,k}^{\lfloor Mk\rfloor,\lfloor Nk\rfloor}(a+b+c,0,0,m))} = PP_q(a,b,m)\cdot PP_q(a+b,c,m).$$

Theorem (q-Factorization Theorem T.L. 2018)

$$\mathrm{M}_q(S^{M,N}_{x,y,z}(a,b,c,m)) = \mathrm{M}_q(S^{M,N}_{x,y,z}(0,0,0,a+b+c+m))$$
$$\times \frac{\mathrm{H}_q(m)^3\,\mathrm{H}_q(a)\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)}{\mathrm{H}_q(m+a)\,\mathrm{H}_q(m+b)\,\mathrm{H}_q(m+c)}$$
$$\times \frac{\mathrm{H}_q(\lceil\frac{x+y}{2}\rceil+c+M+N)\,\mathrm{H}_q(\lceil\frac{x+y}{2}\rceil+\Delta+M+N)}{\mathrm{H}_q(\lceil\frac{x+y}{2}\rceil+M+N)\,\mathrm{H}_q(\lceil\frac{x+y}{2}\rceil+m+c+M+N)}$$
$$\times \frac{\mathrm{H}_q(\lfloor\frac{x+y}{2}\rfloor-M-N)\,\mathrm{H}_q(\lfloor\frac{x+y}{2}\rfloor+m+a+b-M-N)}{\mathrm{H}_q(\lfloor\frac{x+y}{2}\rfloor+a+b-M-N)\,\mathrm{H}_q(\lfloor\frac{x+y}{2}\rfloor+\Delta-M-N)}$$
$$\times \frac{\mathrm{H}_q(\lceil\frac{x+z}{2}\rceil+c+M)\,\mathrm{H}_q(\lceil\frac{x+z}{2}\rceil+m+a+c+M)}{\mathrm{H}_q(\lceil\frac{x+z}{2}\rceil+a+c+M)\,\mathrm{H}_q(\lceil\frac{x+z}{2}\rceil+\Delta+M)}$$
$$\times \frac{\mathrm{H}_q(\lfloor\frac{x+z}{2}\rfloor+b-M)\,\mathrm{H}_q(\lfloor\frac{x+z}{2}\rfloor+\Delta-M)}{\mathrm{H}_q(\lfloor\frac{x+z}{2}\rfloor-M)\,\mathrm{H}_q(\lfloor\frac{x+z}{2}\rfloor+m+b-M)}$$
$$\times \frac{\mathrm{H}_q(\frac{y+z}{2}+N)\,\mathrm{H}_q(\frac{y+z}{2}+m+b+c+N)\,\mathrm{H}_q(\frac{y+z}{2}+a-N)\,\mathrm{H}_q(\frac{y+z}{2}+\Delta-N)}{\mathrm{H}_q(\frac{y+z}{2}+b+c+N)\,\mathrm{H}_q(\frac{y+z}{2}+\Delta+N)\,\mathrm{H}_q(\frac{y+z}{2}-N)\,\mathrm{H}_q(\frac{y+z}{2}+m+a-N)}.$$

Proof of the main result

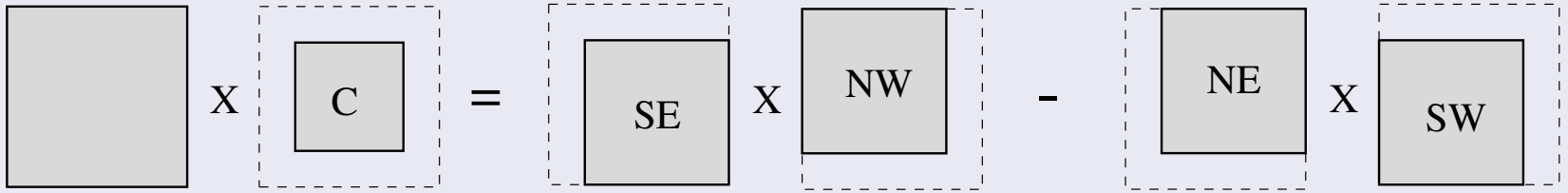
Sketched Proof.

1. Bijection between weighted tilings and weighted perfect matchings.
2. Use a combinatorial version of Dodgson condensation to create the recurrence for both sides.
3. Proof by induction

□

Dodgson Condensation

Lemma (Dodgson condensation)



$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 7 & 8 \end{vmatrix} |5| = \begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} - \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \begin{vmatrix} 4 & 5 \\ 9 & 7 \end{vmatrix}$$

$$-9 \cdot 5 = (-2) \cdot (-3) - (-3) \cdot (-17)$$

The end

Thank you!

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