Enumeration of Tilings and Related Problems

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My research area is Algebraic and Enumerative Combinatorics, specially the Enumeration of Tilings and related problems. The field of enumeration of tilings dates back to the early 1900s when MacMahon proved his classical theorem about the enumeration of plane partitions that fit in a given box [McM]. By a natural bijection, this theorem gives an elegant product formula for the number of lozenge tilings of a hexagonal region on the triangular lattice (see e.g. [Sta99, Kra15]). In the early 1960s, the work of Kasteleyn [Kas] and Temperley and Fisher [TF] opened up a new research direction in the field related to various topics in statistical mechanics, such as dimer and double-dimer models, limit shape and fluctuations, dynamics, integrability, and gauge theory. The connections to algebra were later revealed in the work of Elkies, Kuperberg, Larsen, and Propp [EKLP1, EKLP2] about the well-known Aztec diamonds. This work set off the last two-and-a-half flourishing decades of the enumeration of tilings (see e.g. the excellent surveys by James Propp [Pro99, Pro15]).

Nowadays, enumeration of tilings is a vibrant subfield of enumerative and algebraic combinatorics with connections and applications to diverse areas of mathematics, including representation theory, linear algebra, cluster algebra, group theory, mathematical physics, graph theory, probability, and dynamical systems.

In 1999, James Propp published his well-known article [Pro99] tracking the progress on a list of 32 open problems in the field of enumeration of tilings, which he presented in a 1996 lecture as part of the special program on algebraic combinatorics organized at MSRI. The paper, "*Enumeration of Matchings: Problems and Progress,*" is world-renowned, and many mathematicians have worked to solve the problems. In a review on AMS MathSciNet, Christian Krattenthaler (University Professor at the University of Vienna) wrote about this list of problems: "*This list of problems was very influential; it called forth tremendous activity, resulting in the solution of several of these problems (but by no means all), in the development of interesting new techniques, and, very often, in results that move beyond the problems.*" Many of the problems had been solved prior to 2014 and the remaining few were extremely challenging. I made a contribution to the field by solving two of the remaining problems (Problems 16 and 25 on the list). For Problem 16, I found the tiling formula for a quasi-hexagon, a natural hybrid between a hexagon and the Aztec diamond [Lai14a]. None of the methods previously used in the study of the hexagon or the Aztec diamond work for their hybrid, making the quasi-hexagon a particularly challenging topic in the field. I was able to develop a new tool, "local transformations," to transform a quasi-hexagon into a normal hexagon, which solved the problem. In fact, my method allowed me to solve a harder problem that has Propp's problem as only a special case. I worked with Mihai Ciucu to solve Problem 25 about Blum's conjecture on hexagonal dungeons [CL14]. Matt Blum investigated the tiling number of a hexagonal dungeon, a hexagonal counterpart of the '*Aztec dungeon*' introduced by James Propp in the 1990s. Blum conjectured a striking pattern for the number of tilings of a hexagonal dungeon, which is a certain product of a power of 13 and a power of 14. This conjecture was considered as one of the most challenging open problems in the field for decades. We approached the problem with innovative thinking by using "Kuo condensation," which paid off as our novel idea became the solution.

Besides solving these two challenging and longstanding open problems, I obtained a number of interesting results about the *Aztec diamond*, a central object of the field. Three highlights are my extension of a well-known 1990s result of Bo-Yin Yang about the diabolo tilings of the '*fortresses*' (or '*Penta Aztec diamond*') [Lai13a], my four different proofs for a generalization of the classical Aztec diamond theorem by Noam Elkies, Greg Kuperberg, Michael Larsen, and James Propp [Lai14b, Lai16c], and especially my series of papers about the quartered Aztec diamonds originally introduced by Jockusch and Propp [Lai14c, Lai14d, Lai16d, Lai15].

Besides continuing my Ph.D. research about different aspects of the enumeration of tilings, I have been broadening my research to related areas during my assistant professor appointment at the University of Nebraska–Lincoln. I am currently working on the following four main projects.

In the first project, I have been working on special weighted tiling problems that relate to three significant generalizations of Percy MacMahon's 100-year-old theorem [McM], one of the most important results in algebraic and enumerative combinatorics. As a plane partition can be pictured as a stack of unit cubes fitting in a rectangular box, MacMahon's theorem gives an elegant product formula for the generating function of the volume of such stacks. However, for more than 100 years, there have *not* been many generalizations of MacMahon's theorem in terms of volume generating functions of unit-cube stacks. My work led to three such generalizations [Lai17c, Lai17d, CL17]. My method also opened a new way to investigate the so-called weighted lozenge tilings, that, due to their complexity, have been understudied over the past century.

In the second project, I am working on a problem at the boundary between enumeration of tilings and the theory of electrical networks. R. Kenyon and D. Wilson [KW14] discovered a connection between enumeration of tilings and electrical networks by showing that any contiguous minor of a matrix can be expressed as the generating function of domino tilings of a weighted Aztec diamond. They also conjectured a similar expression for a larger class of minors, the semicontiguous minors. I recently proved this conjecture and continue to generalize it to the case of arbitrary minors [Lai19]. I also aim to various open problems in the same topic.

The third project focuses on connections to cluster algebras. In 2012, S. Zhang [Zha] proved that the generating functions of perfect matchings of a certain class of graphs, the Aztec Dragons, are equal to cluster variables arising from periodic sequences in the dP_3 (del Pezzo 3) quiver. This work has been extended by M. Leoni, G. Musiker, S. Neel, and P. Turner [LMNT] for a new family of graphs, the Aztec Castles. G. Musiker and I generalized further to four different families of graphs restricted by six-sided non-self-intersecting contours [LM17, LM18]. We also conjectured a connection between the case of self-intersecting contours to a new model, that is a natural hybrid between the dimer and the double-dimer models. Besides seeking a proof of the conjecture, we would also like to find different quivers that give the generating functions of tilings of several related regions.

Finally, in the most recent project, I continue the fruitful work of a number of authors on the tiling enumeration of regions with central holes (see e.g. [Ciu98, HG, OK, CEKZ, CK13, Ciu17]). I proved a simple product formula for eight families of hexagons in which three arrays of an arbitrary number of triangles have been removed [Lai18c]. This generalizes Ciucu's previous work [Ciu17] on the so-called '*F-cored hexagons*'. In the same paper, I also provided an extensive list of *thirty* tiling enumerations. Two of these enumerations give multi-parameter generalizations for two longstanding conjectures by M. Ciucu, T. Eisenkölbl, C. Krattenthaler, and D. Zare [CEKZ]. I am further investigating several elegant factorizations for these regions. These factorizations demonstrate a new phenomenon in the field: the tiling generating functions of regions in a certain family are the same, up to a simple multiplicative factor. Interestingly, this fact still holds in the case when the tiling generating functions are not given by a simple product formula. I have also observed that these factorizations can be used to obtain a number of nice results in the asymptotic enumeration of tilings, including the so-called '*duals of MacMahon's theorem*' by Ciucu and Krattenthaler [CK13, Ciu17].

In the remaining part of this statement, I will discuss in detail my four projects above.

1 Generalizing MacMahon's Classical Theorem

One can view a *plane partition*[1] as a monotonic stack of unit cubes fitting in a given rectangular box, and the latter in turn are in bijection with lozenge tilings of a hexagonal region in the triangular lattice. For example, we can write the entries of the plane partition π in the right picture of Figure 1 on a rectangular board of the same side (in this case a 3×4 board) embedded on the plane Oij, and we place the corresponding number of unit cubes on each entry of the board. This way one can interpret the plane partition π as a monotonic stack of unit cubes in the middle picture of Figure 1. This stack in turn can be projected on the plane $i+j+k=0$ to obtain the lozenge tiling of a hexagon shown in the left picture. From this point of view, MacMahon's classical theorem [McM] on plane partitions can be stated as follows.

Let q be an indeterminate. The *q-factorial* is defined as $[n]_q! := \prod_{i=1}^{n} \frac{1-q^n}{1-q}$; and the *q-hyperfactorial* is $\mathrm{H}_q(n) := [0]_q![1]_q! \dots [n-1]_q!$.

Theorem 1 (MacMahon [McM]). *For non-negative integers a, b, c*

$$\sum_{\pi} q^{|\pi|} = \frac{\mathrm{H}_q(a)\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(a+b+c)}{\mathrm{H}_q(a+b)\,\mathrm{H}_q(b+c)\,\mathrm{H}_q(c+a)}, \tag{1}$$

where the sum is taken over all monotonic stacks π fitting in an $a \times b \times c$ box, and where $|\pi|$ denotes the volume *of π.*

The $q = 1$ specialization of MacMahon's theorem gives an elegant tiling formula of a hexagon. This formula inspired a large body of work, focusing on enumeration of lozenge tilings of hexagons with defects. Put differently, MacMahon's theorem gives a q-enumeration of lozenge tilings of a hexagon. However, such q-enumerations are *very rare* in the domain of enumeration of lozenge tilings. Besides my three q-enumerations below, only a few are known (see e.g. [Sta99], [Sta86], [Kra15] and the list of references therein).

In [Lai17c], [Lai17d] and [CL17], I have extended MacMahon's classical theorem 1 by investigating q-enumerations of several large families of hexagons with defects.

The first generalization was motivated by the work of Ciucu and Krattenthaler in [CK13]. I considered the lozenge tilings of a hexagon with four adjacent triangles removed from its boundary, denoted by $Q\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$ (see Figure 2(a)). We view the lozenge tilings of a Q-type region as stacks of unit cubes fitting in a *compound box* $\mathcal{B}$, which is the union of 6 non-overlapping

[1] A *plane partition* can be defined as a rectangular array of non-negative integers with weakly decreasing rows and columns.

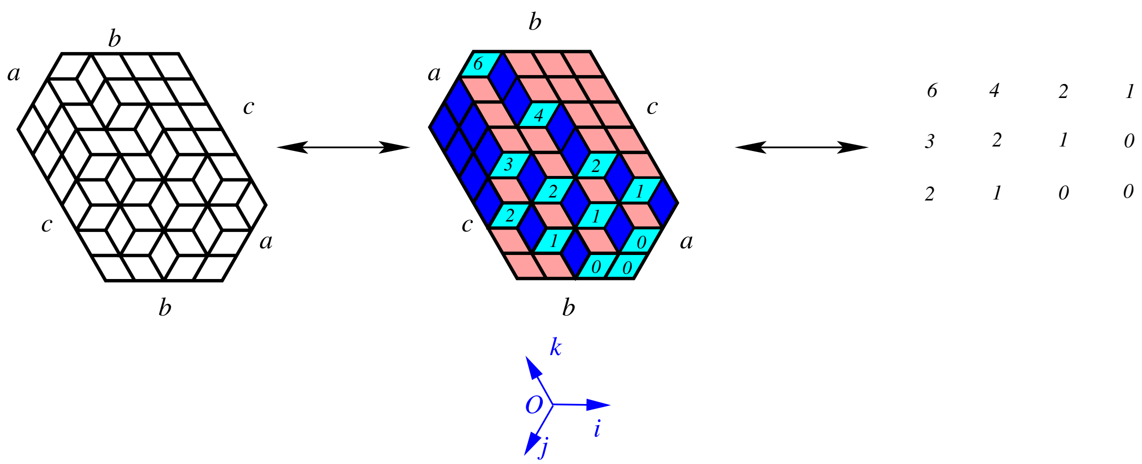

Figure 1: Correspondence between lozenge tilings of a hexagon, stacks of unit cubes fitting in a rectangular box, and plane partitions.

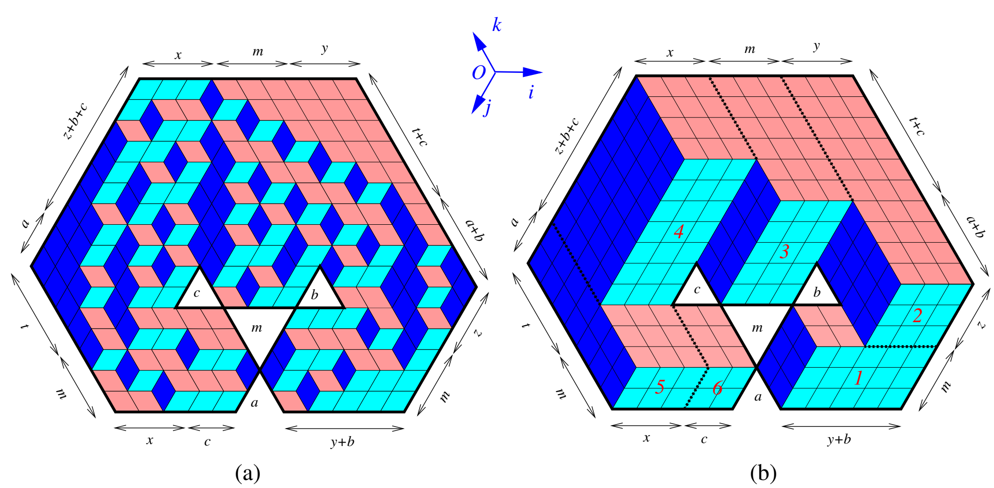

Figure 2: (a) Viewing a lozenge tiling of a Q-type region as a stack of unit cubes fitting in a compound box. (b) The empty stack— a 3-D picture of the compound box.

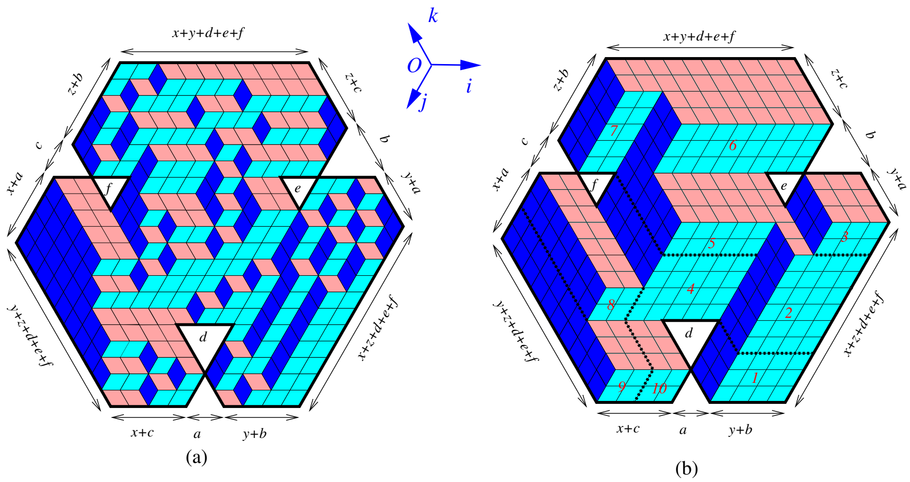

Figure 3: (a) Viewing a lozenge tiling of a hexagon with three dents as a stack of unit cubes fitting in a compound box. (b) The empty compound box with the bases of the component boxes labelled by $1, 2, \dots, 10$.

component (rectangular) boxes (see Figure 2(a)). In Figure 2(b), we have a picture of the lozenge tiling corresponding to the empty stack, and this also gives a 3-D picture of the compound box $\mathcal{B}$. The bases of the component boxes of $\mathcal{B}$ are labelled by $1, 2, \dots, 6$ in Figure 2(b). One readily sees that the stacks of unit cubes here have the same monotonicity as the ordinary plane partitions (i.e., the tops of the columns are weakly deceasing along the directions of the $i-$ and $j-$axes).

Theorem 2 (Theorem 1.2 in [Lai17c]). *Let m, a, b, c, x, y, z, t be 8 non-negative integers. Then*

$$\begin{aligned}\sum_{\pi} q^{|\pi|} &= \frac{\mathrm{H}_q(m+a+b+c+x+y+z+t)}{\mathrm{H}_q(m+a+b+c+x+y+t)\,\mathrm{H}_q(m+a+b+c+x+y+z)}\\ &\times \frac{\mathrm{H}_q(m+a+b+c+x+t)\,\mathrm{H}_q(m+a+b+c+x+y)\,\mathrm{H}_q(m+a+b+c+y+z)}{\mathrm{H}_q(m+a+b+c+z+t)\,\mathrm{H}_q(m+a+b+c+x)\,\mathrm{H}_q(m+a+b+c+y)}\\ &\times \frac{\mathrm{H}_q(x)\,\mathrm{H}_q(y)\,\mathrm{H}_q(z)\,\mathrm{H}_q(t)\,\mathrm{H}_q(m)^3\,\mathrm{H}_q(a)^2\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(m+a+b+c)}{\mathrm{H}_q(x+t)\,\mathrm{H}_q(y+z)\,\mathrm{H}_q(m+a)^2\,\mathrm{H}_q(m+b)\,\mathrm{H}_q(m+c)}\\ &\times \frac{\mathrm{H}_q(m+b+c+z+t)\,\mathrm{H}_q(m+a+c+x)\,\mathrm{H}_q(m+a+b+y)\,\mathrm{H}_q(c+x+t)\,\mathrm{H}_q(b+y+z)}{\mathrm{H}_q(m+b+y+z)\,\mathrm{H}_q(m+c+x+t)\,\mathrm{H}_q(a+c+x)\,\mathrm{H}_q(a+b+y)\,\mathrm{H}_q(b+c+z+t)},\end{aligned} \tag{2}$$

where the sum is taken over all monotonic stacks π corresponding to the region $Q\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$ and $|\pi|$ denotes the volume of π.

In the second generalization [Lai17d], I generalized a family of dented hexagons in a problem posed by James Propp (Problem 3 in [Pro99][2]) by considering a certain hexagon with three bowtie-shaped dents on three non-consecutive sides. Similar to [Lai17c], we can view lozenge tilings of the region as unit-cube stacks (see Figure 3(a)) fitting in a compound box $\mathcal{C}$ (see Figure 3(b)). By the same method as in [Lai17c], I proved that the volume generating function of these stacks is always given by a simple product formula in terms of q-hyperfactorials (see Theorem 1.2 in [Lai17d]).

[2]This problem was first solved by T. Eisenkölbl [Eis].

The third generalization is the main result of my recent joint work with M. Ciucu [CL17]. We proved a simple product formula for the q-enumeration of a hexagon in which two chains of alternating triangles have been removed from the boundary. Moreover, we found out that our result has interesting connections to statistical mechanics and symmetric functions.

I believe that there are many more elegant q-enumerations of lozenge tilings waiting for us to explore. I hope that after collecting enough such q-enumerations, I can solve the following open problem:

Open Problem 1. *Characterize the unit-cube stacks whose volume generating function is given by a simple product formula.*

2 Electrical Networks

Study of *electrical networks* comes from classical physics with the work of Ohm and Kirchhoff more than 100 years ago. A *circular planar electrical network* (or simply *electrical network* in this statement) is a graph $G = (E, V)$ embedded in a disk with a set of distinguished vertices $N \subseteq V$ on the circle, called *nodes*, and a *conductance* function $wt : E \to \mathbb{R}^+$. The electrical networks were first studied systematically by Colin de Verdière [Col] and Curtis, Ingerman, Moores, and Morrow [CIM, CMM]. Recently, a number of new properties of electrical networks have been discovered (see e.g. [ALT, HK, KW11, KW14, Lam15, Lam18, LP11, Yi14]).

We arrange the indices $1, 2, \ldots, n$ of a $n \times n$ matrix M in counter-clockwise order around the circle. Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$ be two sets of indices so that a_1, a_2, $\ldots$, a_k, b_k, b_{k-1}, $\ldots$, b_1 appear in counter-clockwise order around the circle. We call the pair (A, B) a *circular pair* of M. For any circular pair (A, B), the *circular minor* $\det M_A^B$ is defined to be the minor of M obtained from the rows $a_1, a_2, \ldots, a_k$ and the columns $b_k, b_{k-1}, \ldots, b_1$.

Associated with an electrical network is a *response matrix* that measures the response of the network to potential applied at the nodes. It has been shown that a matrix M is the response matrix of an electrical network if and only if it is symmetric with row and column sums equal zero and each circular minor $\det M_A^B$ is non-negative (see Theorem 4 in [CIM]).

An electrical network is called *well-connected* if for any two non-interlaced sets of nodes A and B, there are k pairwise vertex-disjoint paths in G connecting the nodes in A to the nodes in B, where $|A| = |B| = k$. Colin de Verdière showed that an electrical network is well-connected if and only if the response matrix has all circular minors $\det M_A^B$ positive.

A *contiguous minor* of a matrix M is a circular minor whose row indices and whose column indices are contiguous on the circle. The *(small) central minor* is a non-interlaced contiguous minor whose row indices and column indices are opposite (or almost opposite depending on the parity of n) around the circle.

There are $\binom{n}{2}$ central minors, whether n is even or odd. Kenyon and Wilson [KW14] showed how to test the well-connectivity of an electrical network by checking the positivity of the $\binom{n}{2}$ central minors of the response matrix. This is a significant improvement as the previous test by Colin de Verdière [Col] relies on exponentially many circular minors. The test of Kenyon and Wilson is based on their interesting theorem: *Any contiguous minor can be written as a Laurent polynomial in central minors. Moreover, this Laurent polynomial is the generating function of tilings of a (weighted) Aztec diamond.*

The *Aztec diamond* of order h with the center located at the lattice point (x_0, y_0) is the region consisting of all unit squares inside the contour $|x - x_0| + |y - y_0| \leqslant h + 1$. Elkies, Kuperberg, Larsen, and Propp [EKLP1, EKLP2] have shown that the number of *domino tilings* (coverings

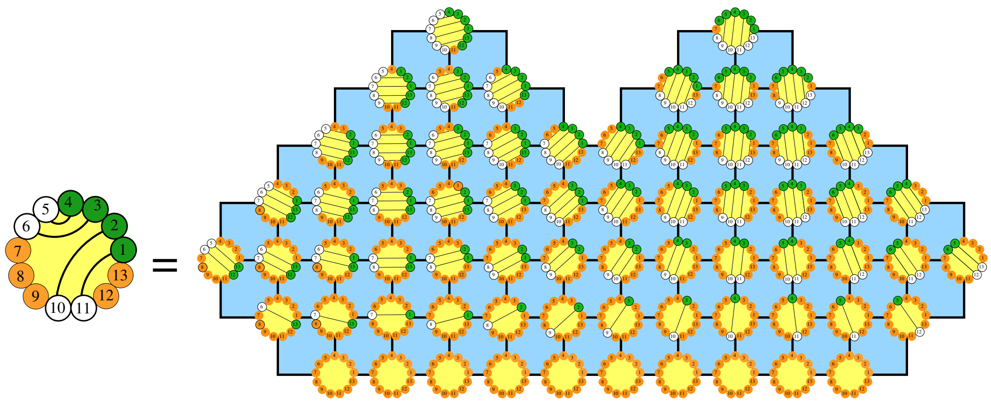

Figure 4: Illustration of the correspondence between a semicontiguous minor (left-hand side) and the domino tilings of a region weighted by central minors (right-hand side). The lattice points on the right-hand side are replaced by the corresponding central minors. This picture first appeared in [Lai19].

using dominoes) of the Aztec diamond of order h is exactly $2^{h(h+1)/2}$. This work has inspired a large body of work in the enumeration of tilings.

We are also interested in a larger family of minors, called *semicontiguous minors*. A *semicontiguous minor* is a circular minor $\det M_A^B$, where *at least one* of A and B is contiguous. In particular, contiguous minors are a special class of semicontiguous minors. Kenyon and Wilson conjectured that

Conjecture 1 (Kenyon–Wilson, Conjecture 3 in [KW14])**.** *Any semicontiguous minor can also be written as the tiling generating function of some region on the square lattice.*

See the illustration of Conjecture 1 in Figure 4. I recently proved this conjecture in [Lai19] by building a special family of regions on the square lattice whose tiling generating functions are given by the semicontiguous minors. In particular, our region is obtained from an *Aztec rectangle* (a natural generalization of the Aztec diamond) or a special union of two Aztec diamonds by trimming the base along a zigzag path determined by the non-contiguous index set.

It would be interesting to know when a general circular minor has the same property as that in Kenyon and Wilson's conjecture:

Open Problem 2. *Characterize the circular minors that can be represented as the tiling generating function of a region on the square lattice.*

I have made significant progress on this problem by finding a large family of circular minors that have this property.

Besides the above open problem, there are still a number of conjectures about electrical networks that I plan to tackle in the next three years.

Two electrical networks are *(electrically) equivalent* if they have the same response matrix. Colin de Verdière [Col] showed that two equivalent electrical networks can be obtained from each other by a sequence of simple transformations. An electrical network is called *critical* if it has the minimal number of edges among its equivalent class. In [KW14], Kenyon and Wilson also posed the following two conjectures about testing the well-connectivity of a critical network.

Conjecture 2 (Conjecture 1 in [KW14])**.** *For a critical network on m edges, one can test the well-connectivity using m noninterlaced minors of the response matrix.*

Conjecture 3 (Conjecture 2 in [KW14])**.** *For a critical network with m edges, there is a 'base set' S of m noninterlaced minors of the response matrix, analogous to the set of the central minors of a well-connected network, such that any nonzero noninterlaced minor of the response matrix is (1) a Laurent polynomial in minors from S, and (2) a positive function of minors from S.*

The *weakly separated sets* were first defined by Leclerc and Zelevinsky [LZ] while studying the q-deformation of the coordinate ring of the flag variety. They posed an interesting purity conjecture: *every maximal by inclusion collection of pairwise weakly separated subsets of* $[n]$ *have the same size*. This conjecture has been proven independently by Danilov–Karzanov–Koshevoy [DKK] and by Oh–Postnikov–Speyer [OPS] using different methods. In [ALT], Alman–Lian–Tran defined the weak separation for circular minors as follows. Two index sets $A, B \subset [n]$ around a circle are *weakly separated* if there are no $a, a' \in A - B$ and $b, b' \in B - A$ such that $a < b < a' < b'$ or $b < a < b' < a'$. Two circular pairs (P, Q) and (R, S) are weakly separated if $P \cup R$ is weakly separated from $Q \cup S$ and $P \cup S$ is weakly separated from $Q \cup R$. They conjecture the following purity conjecture for circular minors.

Conjecture 4 (Conjecture 6.3.4 in [ALT])**.** *Let $\mathcal{C}$ be a set of circular minors for an $n \times n$ response matrix. Then the following are equivalent:*

(1) $\mathcal{C}$ is a minimal positivity test.

(2) The circular pairs corresponding to the minors in $\mathcal{C}$ form a maximal set of pairwise weakly separated circular pairs.

(3) $\mathcal{C}$ is a cluster of $\mathcal{LM}_n$.

Here $\mathcal{LM}_n$ is a certain "*Laurent Phenomenon algebra*" (first introduced by Lam and Pylyavskyy in [LP16]) defined in Section 6.2 of [ALT].

Together with Pavlo Pylyavskyy of the University of Minnesota, I am investigating a purity conjecture that has the conjecture of Leclerc and Zelevinsky above as a special case.

3 Cluster Algebra

Cluster algebra was first introduced by Fomin and Zelevinsky in the 2000s (see e.g. [FZ02, FZ03, FZ07]) as a tool for studying total positivity and dual canonical bases in Lie theory. However, cluster algebra has since taken on a life of its own as a new mathematical field with connections to various areas of mathematics. Especially, cluster algebra and enumeration of *perfect matchings*[3] (equivalently, tilings) are strongly related.

I am working with Gregg Musiker [LM17] on a project at the boundary between cluster algebra and enumeration of perfect matchings. This project is inspired by the work in [Zha, LMNT, Lai16f, Lai17a, Lai17b]. In particular, we consider the dP_3 quiver (see Figure 5) and construct a family of subgraphs of the brane tiling restricted by certain 6-sided oriented non-self-intersecting contours (the direction of a side of a contour depends on the sign of the 'side-length'). See Figure 6 for several examples: the contours are represented by the dotted lines with the side-lengths shown; our 6-sided graphs consist of all vertices and edges of the shaded faces.

Our family of graphs generalizes many known families, including the Aztec Dragons [Ciu03, Zha], Aztec Castles [LMNT], and Dragon regions [Lai16f]. Associate the weight $\frac{1}{x_i x_j}$ to each edge bordering faces labelled by i and j in the brane tiling. For a subgraph G of the brane tiling, we

[3] A perfect matching of a graph G is a collection of disjoint edges covering all vertices of G.

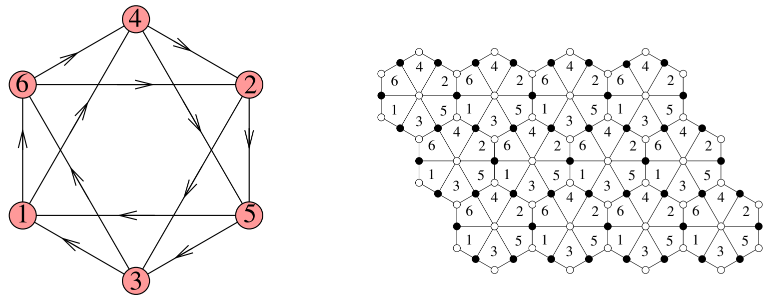

Figure 5: The dP_3 quiver and its associated brane tiling.

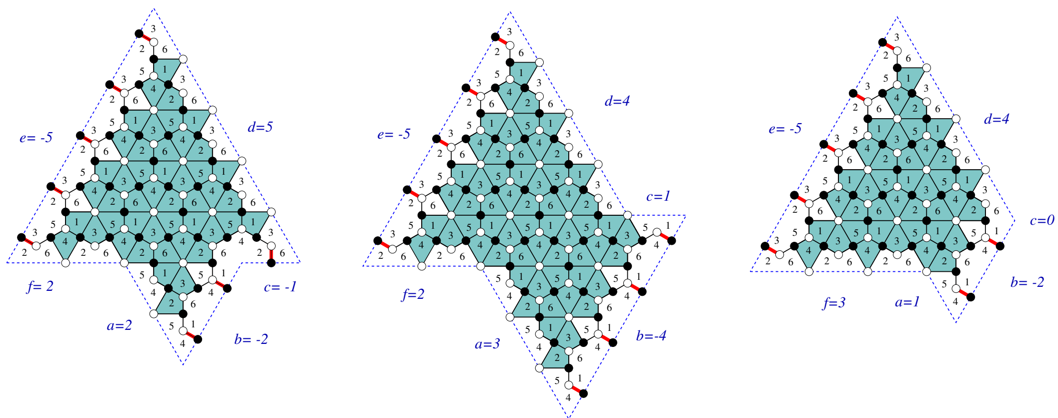

Figure 6: Several examples of the 6-sided graphs for Model 1.

define the *covering monomial* $F(G)$ to be the product of weights of all shaded faces and their neighbor faces. Next, we define the weight $W(G) := F(G)\sum_{\mu} wt(\mu)$, where the sum is taken over all perfect matchings μ of G, and $wt(\mu)$ is the product of weights of edges in μ.

Theorem 3 (Musiker and Lai [LM17])**.** *Let G be any six-sided graph in the above family. The weight $W(G)$ is equal to the cluster variable obtained by a sequence of toric mutations on the dP_3 quiver. Moreover, the latter cluster variable can be written as a closed-form product formula in the variables $x_1, x_2, \ldots, x_6$.*

In the second part of the project, we considered a number of known families of regions from the literature under one roof [LM18]. We showed that many discrete results in the enumeration of tilings/ perfect matchings are actually connected in the cluster algebra point of view. There are actually four different models of the dP_3 quiver, the regions in the first part of the project [LM17] correspond to the first model. The other three models give new regions on different lattices (see Figures 7, 8, and 9 for examples). We showed that these regions can be transformed into each other by using the local move called *urban renewal*. This local move was first introduced by Greg Kuperberg.

We notice that several key arguments in our proofs do not work for the case of *self-intersecting* contours. In particular, our region is not defined when the contour self-intersects. We recently found interesting connections between this case and the recent work of Kenyon and Pemantle [KP] about the double dimer model. A *double dimer* can be viewed as the superposition of two different dimers (with certain boundary conditions), so it is a collection of paths and cycles. We have conjectured in [LM18] that the cluster variables in the case of self-intersecting contours count the

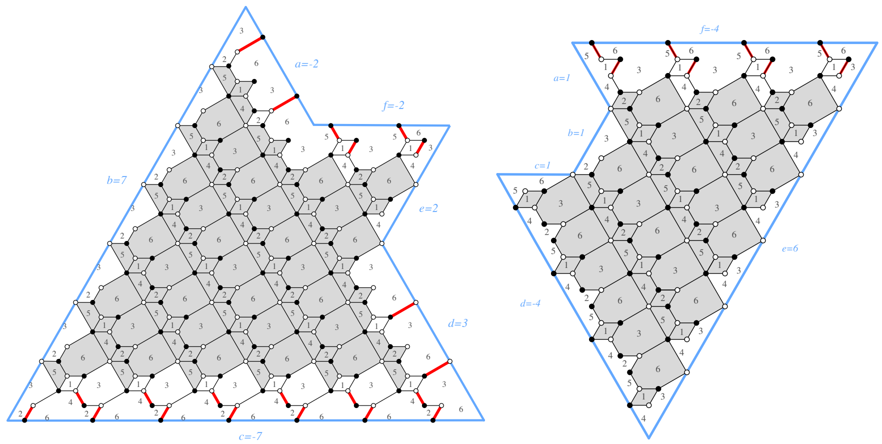

Figure 7: Several examples of the 6-sided graphs for Model 2.

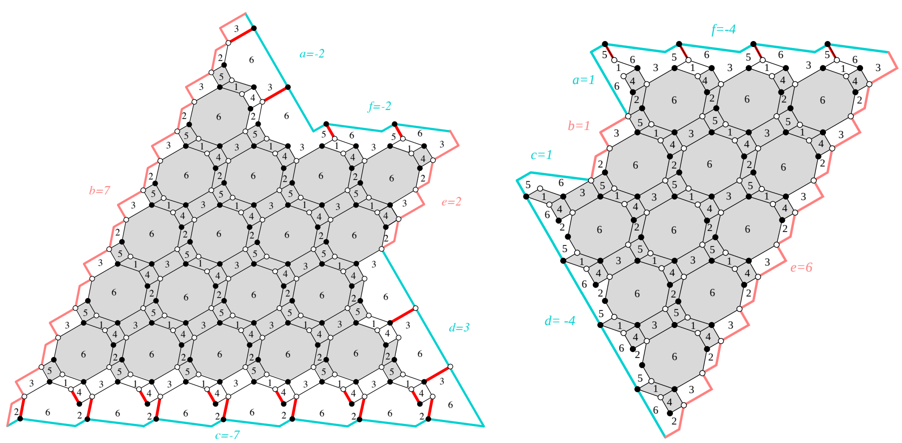

Figure 8: Several examples of the 6-sided graphs for Model 3.

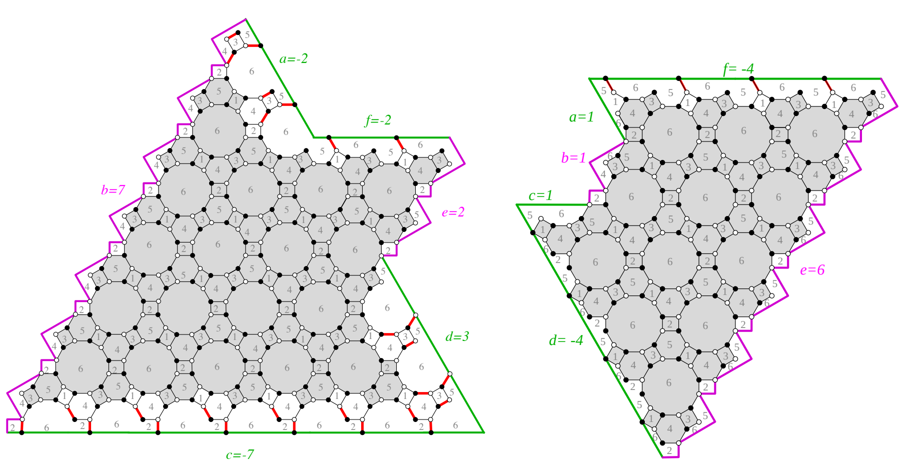

Figure 9: Several examples of the 6-sided graphs for Model 4.

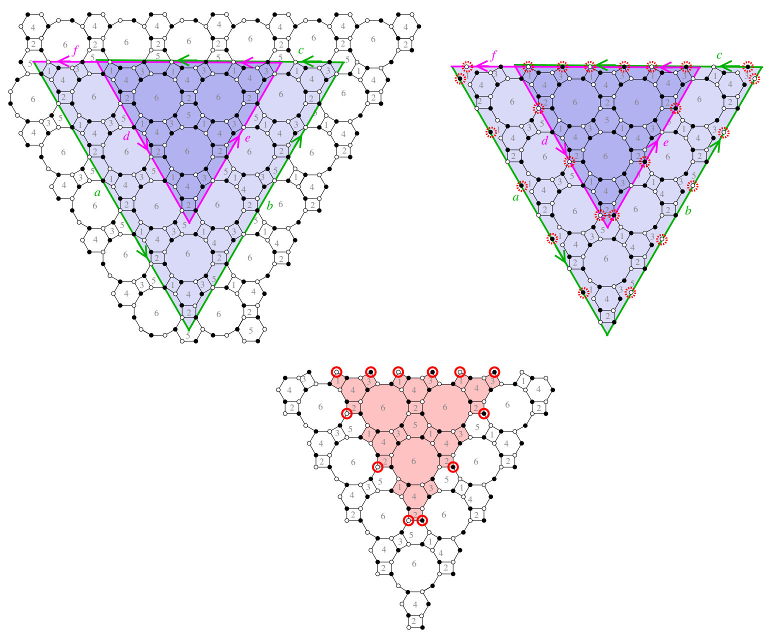

Figure 10: A self-intersecting contour in the lattice of Model 4 (top left), the graph restricted by the self-intersecting contour (top right), and the taut condition for the double-dimer region (lower).

objects in a 'mixed model'. In particular, our objects are the union of a double dimer configuration in some triangular region (with a certain 'taut' condition) and a dimer configuration outside the triangular region. See Figure 10 for an example: the top left picture shows the position of a self-intersecting contour in the lattice of Model 4; the top right picture shows the graphs restricted by the contour (the dark shaded triangle is the double dimer region, and the light shaded region is the ordinary dimer region in the top pictures); the bottom picture shows the taut condition for the double dimer region: the paths in the double dimer configuration can only pass through the circled vertices.

Conjecture 5. *In the case of self-intersecting contours, the weighted sum of the mixed configurations is equal to the cluster variable obtained by a sequence of toric mutations on the dP_3 quiver in Theorem 3.*

There are many more regions in literature whose tilings can be counted by cluster variables of certain quivers.

Open Problem 3. *Can we use different quivers to get the generating functions of the tilings of several related regions, e.g. 'Needle Regions' [Lai16f] and 'Hexagonal Dungeons' [CL14]?*

4 Enumerations of Regions with Central Holes

The tale of tiling enumerations of regions with central 'holes' originally came from a problem posed by James Propp. Here a *hole* is a portion that has been removed from the interior of a region (see Figure 11 for examples; the holes are illustrated by the black portions). Propp's Problem 2 in [Pro99] asks for a tiling formula for an almost-regular hexagon of side lengths[4]

[4]From now on, we always list the side lengths of a hexagon in the clockwise order, starting from the north side.

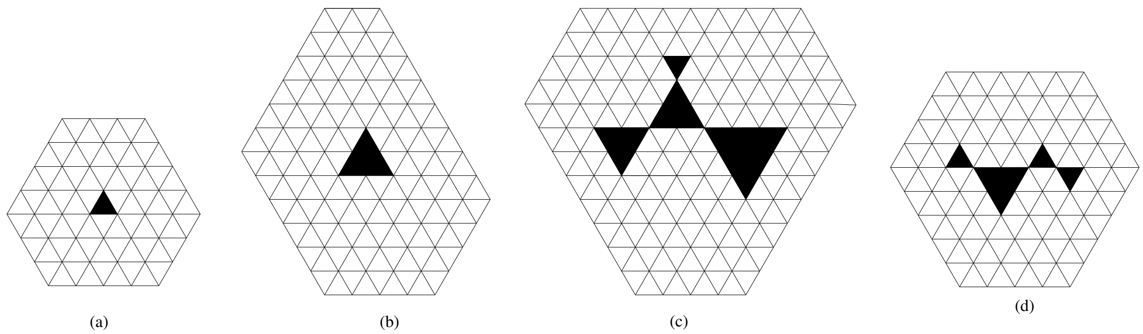

Figure 11: Several hexagons with central holes whose tilings are enumerated by simple product formulas.

$n, n+1, n, n+1, n, n+1$ with the central unit triangle removed (see Figure 11(a) for $n = 3$). This problem has been solved and generalized by a number of authors (see e.g. [Ciu98, HG, OK]).

A milestone in this line of work is the theorem by Ciucu, Eisenkölbl, Krattenthaler and Zare [CEKZ] saying that the tilings of a hexagon are still enumerated by a 'simple' product formula if we remove a triangle with arbitrary side length from the center (a product formula is called '*simple*' if all the factors are linear in parameters). This hexagon with the central hole is called a '*cored hexagon*' (see Figure 11(b) for an example). In 2013, Ciucu and Krattenthaler [CK13] extended the structure of the central triangular hole in the cored hexagons to a cluster of four triangular holes, called a '*shamrock hole*' (see Figure 11(c)). The explicit enumeration of a hexagon with a shamrock hole in the center (called a '*S-cored hexagon*') yields a striking asymptotic result that they mentioned as a '*dual*' of MacMahon's classical theorem. Ciucu [Ciu17] later considered a new structure, called a '*fern*', which is a string of triangles with alternating orientations, and a hexagon with a fern removed from the center, called an '*F-cored hexagon*'. This new region also yields a nice tiling formula and another dual of MacMahon's theorem (illustrated in Figure 11(d)). I recently wrote 5 papers [CL17, Lai18a, Lai17e, Lai18b, Lai18c] investigating the fern structure deeper.

Besides 'holes' (portions removed inside a region), we are also interested in a different type of defect, called '*dents*', which are portions removed from the boundary of a region. There are various results about regions with dents by a number of authors (see e.g. [CLP], [Pro99], [Eis], [CF16], [Lai17c], [Lai17d], [CL17] and the lists of references therein). One of the earliest and most important results in this body of work is the following tiling formula of a *semihexagon with dents* by Cohn, Larsen and Propp. Assume that $a_1, a_2, \dots, a_n$ are non-negative integers. Let $S(a_1, a_2, \dots, a_m)$ denote the upper half of a hexagon of side-lengths $a_2 + a_4 + a_6 + \dots, a_1 + a_3 + a_5 + \dots, a_1 + a_3 + a_5 + \dots, a_2 + a_4 + a_6 + \dots, a_1 + a_3 + a_5 + \dots, a_1 + a_3 + a_5 + \dots$ in which $k := \lfloor \frac{m}{2} \rfloor$ triangles of sides $a_1, a_3, a_5, \dots, a_{2k+1}$ are removed from the base, so that the distance between the i-th and the $(i+1)$-th removed triangles is a_{2i} (see Figure 12 for an example). Cohn, Larsen and Propp [CLP] interpreted semi-strict Gelfand–Tsetlin patterns as lozenge tilings of the dented semihexagon $S(a_1, a_2, \dots, a_m)$, and obtained the following tiling formula

$$s(a_1, a_2, \dots, a_{2l-1}) = s(a_1, a_2, \dots, a_{2l}) = \frac{1}{\mathrm{H}(a_1 + a_3 + a_5 + \dots + a_{2l-1})} \frac{\prod_{\substack{1 \leqslant i < j \leqslant 2l-1 \\ j-i \text{ odd}}} \mathrm{H}(a_i + a_{i+1} + \dots + a_j)}{\prod_{\substack{1 \leqslant i < j \leqslant 2l-1 \\ j-i \text{ even}}} \mathrm{H}(a_i + a_{i+1} + \dots + a_j)}, \tag{3}$$

where $s(a_1, a_2, \dots, a_m)$ denotes the number of tilings of $S(a_1, a_2, \dots, a_m)$, and where $\mathrm{H}(n) :=$

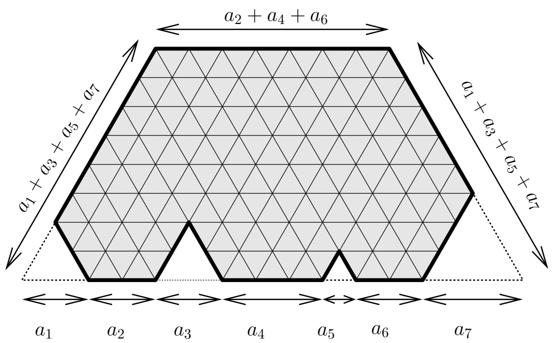

Figure 12: The semihexagon with dents $S(2, 2, 2, 3, 1, 2, 4)$.

$0! \cdot 1! \cdot 2! \dots (n-1)!$ is the 'hyperfactorial' function.

Even though there are a number of elegant enumerations for regions with holes and for regions with dents, there are *not* many known results for regions with *both* holes and dents. In the first part of my recent 97-page-long paper [Lai18c], I enumerated a number of such rare regions. In particular, our regions are hexagons with three aligned ferns removed: one central fern and two side ferns attached to the boundary. I showed that the number of tilings of such a region is always given by a product of the tiling number of a cored hexagon and a simple multiplicative factor. This gives a *multi-parameter* generalization for Ciucu's main theorem in [Ciu17]. This result actually generalizes all known enumerations of regions with central holes listed above, except for the one about S-cored hexagons in [CK13]. Especially, our theorem also implies a new dual of MacMahon's classical theorem on plane partitions that generalizes the dual of Ciucu [Ciu17].

In their 2001 paper [CEKZ], Ciucu, Eisenkölbl, Krattenthaler and Zare also considered the cored hexagons with the triangular hole slightly off center. They conjectured two striking tiling formulas for this case:

Conjecture 6 (Ciucu–Eisenkölbl–Krattenthaler–Zare; Conjecture 1 in [CEKZ])**.** *Let x, y, z, m be nonnegative integers, x, y, z having the same parity. The number of the lozenge tilings of a hexagon with sides $x, y+m, z, x+m, y, z+m$, with an equilateral triangle of side length m removed at 1 unit to the west of the center, equals*

$$\begin{aligned}
&\frac{1}{4}\frac{\mathrm{H}(m+x)\,\mathrm{H}(m+y)\,\mathrm{H}(m+z)\,\mathrm{H}(m+x+y+z)}{\mathrm{H}(m+x+y)\,\mathrm{H}(m+y+z)\,\mathrm{H}(m+z+x)}\\
&\quad\times\frac{\mathrm{H}(m+\lfloor\frac{x+y+z}{2}\rfloor)\,\mathrm{H}(m+\lceil\frac{x+y+z}{2}\rceil)}{\mathrm{H}(m+\frac{x+y}{2}+1)\,\mathrm{H}(m+\frac{y+z}{2})\,\mathrm{H}(m+\frac{z+x}{2}-1)}\\
&\quad\times\frac{\mathrm{H}(\frac{m}{2})^2\,\mathrm{H}(\lfloor\frac{x}{2}\rfloor)\,\mathrm{H}(\lceil\frac{x}{2}\rceil)\,\mathrm{H}(\lfloor\frac{y}{2}\rfloor)\,\mathrm{H}(\lceil\frac{y}{2}\rceil)\,\mathrm{H}(\lfloor\frac{z}{2}\rfloor)\,\mathrm{H}(\lceil\frac{z}{2}\rceil)}{\mathrm{H}(\frac{m}{2}+\lfloor\frac{x}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{x}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{y}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{y}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{z}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{z}{2}\rceil)}\\
&\quad\times\frac{\mathrm{H}(\frac{m}{2}+\lfloor\frac{x+y}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{x+y}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\frac{y+z}{2})^2\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{z+x}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{z+x}{2}\rceil)}{\mathrm{H}(\frac{m}{2}+\lfloor\frac{x+y+z}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{x+y+z}{2}\rceil)\,\mathrm{H}(\frac{x+y}{2}-1)\,\mathrm{H}(\frac{y+z}{2})\,\mathrm{H}(\frac{z+x}{2}+1)}\cdot P_1(x,y,z,m),
\end{aligned} \tag{4}$$

where $P_1(x, y, z, m)$ is the polynomial given by

$$P_1(x,y,z,m)=\begin{cases}(x+y)(x+z)+2xm & \text{if } x \text{ is even,}\\ (x+y)(x+z)+2(x+y+z+m)m & \text{if } x \text{ is odd.}\end{cases} \tag{5}$$

Conjecture 7 (Ciucu–Eisenkölbl–Krattenthaler–Zare; Conjecture 2 in [CEKZ])**.** *Let x, y, z, m be nonnegative integers, x is of parity different from the parity of y and z. The number of the lozenge tilings of a hexagon with sides $x, y+m, z, x+m, y, z+m$, with an equilateral triangle of side length m removed at 3/2 unit to the west of the center, equals*

$$\frac{1}{16}\frac{\mathrm{H}(m+x)\,\mathrm{H}(m+y)\,\mathrm{H}(m+z)\,\mathrm{H}(m+x+y+z)}{\mathrm{H}(m+x+y)\,\mathrm{H}(m+y+z)\,\mathrm{H}(m+z+x)}$$
$$\times\frac{\mathrm{H}(\frac{m}{2})^2\,\mathrm{H}(\lfloor\frac{x}{2}\rfloor)\,\mathrm{H}(\lceil\frac{x}{2}\rceil)\,\mathrm{H}(\lfloor\frac{y}{2}\rfloor)\,\mathrm{H}(\lceil\frac{y}{2}\rceil)\,\mathrm{H}(\lfloor\frac{z}{2}\rfloor)\,\mathrm{H}(\lceil\frac{z}{2}\rceil)}{\mathrm{H}(\frac{m}{2}+\lfloor\frac{x}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{x}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{y}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{y}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{z}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{z}{2}\rceil)}$$
$$\times\frac{\mathrm{H}(\frac{m}{2}+\lfloor\frac{x+y}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{x+y}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\frac{y+z}{2})^2\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{z+x}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{z+x}{2}\rceil)}{\mathrm{H}(\frac{m}{2}+\lfloor\frac{x+y+z}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{x+y+z}{2}\rceil)\,\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor-1)\,\mathrm{H}(\frac{y+z}{2})\,\mathrm{H}(\lceil\frac{z+x}{2}\rceil+1)}$$
$$\times\frac{\mathrm{H}(m+\lfloor\frac{x+y+z}{2}\rfloor)\,\mathrm{H}(m+\lceil\frac{x+y+z}{2}\rceil)}{\mathrm{H}(m+\lceil\frac{x+y}{2}\rceil+1)\,\mathrm{H}(m+\frac{y+z}{2})\,\mathrm{H}(m+\lfloor\frac{z+x}{2}\rfloor-1)}P_2(x,y,z,m) \tag{6}$$

where $P_2(x, y, z, m)$ is the polynomial given by

$$P_2(x,y,z,m)=\begin{cases}((x+y)^2-1)((x+z)^2-1)+4xm(x^2+2xy+y^2+2xz+3yz+z^2 & \\ \quad +2xm+3ym+3zm+2m^2-1) & \text{if } x \text{ is even,}\\ ((x+y)^2-1)((x+z)^2-1)+4(x+y+z+m)m(x^2+xy-1) & \text{if } x \text{ is odd.}\end{cases} \tag{7}$$

Here we extend the define the hyperfactorial to half integers via the gamma function as $\mathrm{H}(n+\frac{1}{2}) := \prod_{i=0}^{n}\Gamma(i+\frac{1}{2})$, recall that $\Gamma(n+\frac{1}{2})=\frac{(2n-1)!!}{2^n}\sqrt{\pi}$.

In the second part of [Lai18c], I considered various regions that are generalizations of the above off-central cored hexagons. I, in fact, provided an extensive list of *thirty* exact enumerations of these 'off-central' regions. Two of my enumerations imply the above decade-old conjectures of Ciucu–Esenkölbl–Krattenthaler–Zare as two very-special cases[5].

I am generalizing my finding in [Lai18c] further by investigating several elegant factorizations for regions with central holes.

As mentioned above, the main theorem in [Lai18c] says that the tiling number of any hexagon with three ferns removed can be factorized into the tiling number of a certain cored hexagon and a simple multiplicative factor. One would ask for a similar factorization for the general case of an *arbitrary number* of ferns. The previous results in [CL17, Lai18c] and our computer data support the existence of such a factorization. Intuitively, we conjecture that the tiling number of a hexagon with an arbitrary number of ferns removed is equal to the tiling number of a new hexagon in which each removed fern is replaced by a single triangle of the same size times a simple multiplicative factor. Let us state in detail our first conjecture in the next paragraph.

Let x, z be nonnegative integers, and $d_1, d_2, \dots, d_{k-1}$ be nonnegative integers whose sum is equal to $x+z$. Assume that $\mathcal{F}_i$ is a given fern of length f_i (the *length* of a fern is the sum of the sides of its triangles), for $i = 1, \dots, k$. Denote by U the sum of sides of all up-pointing triangles in the k ferns $\mathcal{F}_1, \dots, \mathcal{F}_k$, and D the sum of sides of all down-pointing triangles in these ferns. Consider a symmetric hexagon H of side lengths $x+D, z+U, Z+D, x+U, z+D, z+U$. We remove the k ferns $\mathcal{F}_i$'s along the horizontal axis l of the hexagon H such that the distance between the ith

[5]After posting my paper on `arxiv.org`, I was informed by Rosengren that the two conjectures were already proved by him in his 2016 paper [Ros], using a different method, namely lattice path combinatorics and Selberg integral.

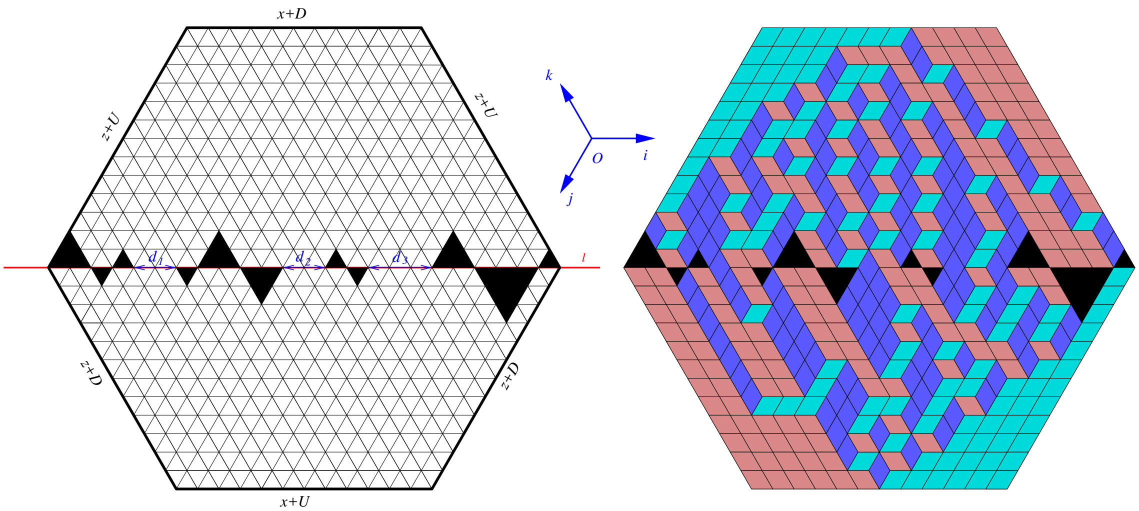

Figure 13: A hexagon with four aligned ferns removed (left) and one of its tilings, visualized as a stack of unit cubes fitting in a special box (right).

and the $(i+1)$th fern is d_i, for $i = 1, 2, \dots, k-1$. Denote by $H_{x,z}(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k;\ d_1, d_2, \dots, d_{k-1})$ the resulting region (see the left picture in Figure 13 for an example).

Conjecture 8 (General Factorization for Hexagons with Ferns Removed)**.**

$$\begin{aligned}\mathrm{M}(H_{x,z}(\mathcal{F}_1, \dots, \mathcal{F}_k; d_1, \dots, d_{k-1})) = &\mathrm{M}(H_{x,z}(\triangle_{f_1}, \triangle_{f_2}, \dots, \triangle_{f_k}; d_1, d_2, \dots, d_{k-1})) \\ &\times \frac{\mathrm{M}(S^+)\,\mathrm{M}(S^-)}{\mathrm{M}(S_0)} \times PP(U, D, z), \end{aligned} \tag{8}$$

where $\mathrm{M}(R)$ *denotes the number of tilings of region* R*, and where* $\triangle_{f_i}$ *denotes the fern consisting of a single up-pointing triangle of side* f_i*.*

UPDATE: Conjecture 8 has been proved and generalized by Ranjan Rohatgi and I.

In the above factorization, we actually find an elegant interpretation for the multiplicative factor as $\frac{\mathrm{M}(S^+)\,\mathrm{M}(S^-)}{\mathrm{M}(S_0)} \times PP(U, D, z)$, where S^+ and S^- are the semihexagons whose dents obtained by dividing the region on the right-hand side along the horizontal axis l, where $S_0 = S(f_1, d_1, f_2, d_2, \dots, d_{k-1}, f_k)$, and $PP(U, D, z)$ is the number of tilings of a hexagon of side lengths U, D, z, U, D, z. The number of tilings of each of the three semihexagons here is given by Cohn–Larsen–Propp's simple product formula (3), and the number of tilings of the hexagon is given by MacMahon's Theorem 1, for $q = 1$.

The above factorization is hard to verify because the tiling numbers of the $H_{x,z}$-type regions are *not* given by simple product formulas. We, in general, do not have any simple exact formula for the number of tilings of an $H_{x,z}$-type region. Even for the simplest case when the number of ferns is 1 and the unique fern consists of only a single triangle, the tiling number is *not* given by a nice formula. The best formula we have in hand is the *complicated* determinantal formula found by Rosengren [Ros].

It is worth noting that several special cases of Conjecture 8 have been verified recently in my joint work with Ciucu [CL17] (for the case of two ferns) and in my solo paper [Lai18c] (for the case of three ferns). Even though the general case of an arbitrary number of ferns should be more complicated, I think that I am on the right track to tackle this conjecture.

Similar to Section 1, can also interpret any lozenge tiling of an $H_{x,z}$-type region R as a monotonic stack of unit cubes fitting in a special box consisting of adjacent rectangular boxes (see the right picture in Figure 13(b) for an example). We define the *tiling generating function* of a region R to be $\mathrm{M}_q(R) := \sum_\pi q^{vol(\pi)}$, where the sum is over all monotonic stacks π corresponding to lozenge tilings of R.

Conjecture 9 (q-Factorization for Hexagons with Ferns Removed)**.** *With the same assumption as Conjecture 8, we have*

$$\mathrm{M}_q(H_{x,z}(\mathcal{F}_1,\dots,\mathcal{F}_k;d_1,\dots,d_{k-1})) = \mathrm{M}_q(H_{x,z}(\Delta_{f_1},\Delta_{f_2},\dots,\Delta_{f_k};d_1,d_2,\dots,d_{k-1})) \times \frac{\mathrm{M}_q(S^+)\,\mathrm{M}_q(S^-)}{\mathrm{M}_q(S_0)} \times PP_q(U,D,z). \tag{9}$$

It is known that the tiling generating function $\mathrm{M}_q(S)$ of a dented semihexagon S is given by the 'natural' q-analog of the s-formula (3), in which each hyperfactorial $\mathrm{H}(n)$ is replaced by the corresponding q-hyperfactorial $\mathrm{H}_q(n)$ (see e.g. [Sta86]). We use the notation $s_q(.)$ for this q-analog of the s-formula.

It is the fact that working on tiling generating functions is much more difficult than working on 'plain' tilings (i.e., unweighted tilings). Therefore, one can say that Conjecture 9 is harder than its unweighted version, Conjecture 8. However, I strongly believe that the method used in my recent work about q-enumerations [Lai17c], [Lai17d], [CL17] is the right approach for Conjecture 9. In particular, we used lattice path combinatorics to transform weighting in the tiling generating function $\mathrm{M}_q(.)$ into a 'more-friendly' one, that does not depend on the choice of tilings. Then, we can use a flexible combination of classical methods, such as determinant calculus and symmetric function, and a newer method, the graphical condensation introduced by Eric Kuo [Kuo04] (usually referred to *Kuo condensation*). As it provides recurrences for tiling generating functions, the later method is very powerful, especially when some conjectural formulas have been found. In the last four years, I have published a number of tiling papers using this method (see e.g. [CL14, CL17], [Lai16f]—[LR18]). See [CF15, CF16, Ciu17, CK13, KW14, LMNT] for additional recent applications of Kuo condensation.

My approach so far is to give inductive proofs for the factorizations in Conjectures 8 and 9. It would be very interesting to find some combinatorial explanation for these factorizations. I suspect that there is a Schur function identity behind these factorizations, as the s-formulas and MacMahon's product formula in our factorization can all be written in terms of some special Schur functions. This was also discussed in Section 9 of my recent joint work with Mihai Ciucu [CL17].

Open Problem 4. *Find an explanation for the factorizations in Conjectures 8 and 9 in terms of a Schur function identity.*

To be more optimistic, a combinatorialist would expect a bijective proof for our factorizations:

Open Problem 5. *Find a bijective proof for Conjectures 8 and 9.*

Even though there is a large body of work on enumeration of lozenge tilings, most of the results focus on regions whose tiling number is given by a simple product formula. When the tiling number is not nice, it is very hard to handle. For instance, when the triangular hole in the cored hexagon is moved away *only* 1 *unit* from the center, the tiling formulas are not simple anymore, one of the factors is not linear in the parameters (see Conjectures 1 and 2 by Ciucu, Eisenkölbl, Krattenthaler and Zare in [CEKZ]). As mentioned above, even these not-too-bad tiling formulas took us more than 10 years to verify (independently by Rosengren [Ros] and by myself [Lai18c]). The above factorizations in Conjectures 8 and 9 would become a new useful tool to handle a new class of problems in the enumeration of tilings. We give below an example in which our factorizations can be used to imply a new result in asymptotic enumeration of tilings, which seemed to be impossible.

In [Ciu17], based on the explicit tiling formula of a hexagon with a single fern removed from its center (called a *F-cored hexagon*), Ciucu obtained a striking asymptotic result, that he called a '*dual*' of MacMahon's Theorem 1[6]. Intuitively, he considered the ratio between the tiling numbers of an F-cored hexagon and a standardized version of it. Ciucu showed that, as the sides of the hexagon get large, the ratio tends to a product of two s-functions in Cohn–Larsen–Propp's formula (3). However, when the removed fern has been moved away from the center, the tiling number of the F-cored hexagon is not given by a simple product formula anymore. As a consequence, Ciucu's asymptotic result is restricted to the case when the removed fern is in the center of the hexagon. By a similar reason, since a nice q-enumeration for the F-cored hexagons seems not to exist, the q-analog for Ciucu's elegant finding has not been found.

I am able to use the factorization in Conjecture 9 to obtain the following new (q-)dual of MacMahon's theorem, in which there are *no* restrictions on the positions of the removed ferns. The following corollary generalizes Ciucu's dual in two different ways.

We assume further that the fern $\mathcal{F}_i$ consists of t_i triangles of sides $a_{i,j}$ as they appear from left to right, for $i = 1, \dots, k$ and $j = 1, \dots, t_i$. Denote by u_i and d_i the sums of sides of all up-pointing and of all down-pointing triangles in the fern $\mathcal{F}_i$.

Corollary 1 (q-dual of MacMahon's theorem). *For $|q| < 1$*

$$\lim_{N\to\infty} \frac{\mathrm{M}_q(H_{xN,zN}(\mathcal{F}_1, \dots, \mathcal{F}_k; \lfloor d_1 N\rfloor, \lfloor d_2 N\rfloor, \dots, \lfloor d_{k-1} N\rfloor))}{\mathrm{M}_q(H_{xN,zN}(\mathcal{F}'_1, \dots, \mathcal{F}'_k; \lfloor d_1 N\rfloor, \lfloor d_2 N\rfloor, \dots, \lfloor d_{k-1} N\rfloor))} = \prod_{i=1}^{k} s^{+}_{q,i} s^{-}_{q,i}, \tag{10}$$

where the fern $\mathcal{F}'_i$ consists of an up-pointing triangle of side u_i followed by a down-pointing triangle of side d_i, and where $s^{+}_{q,i} := s_q(a_{i,1}, a_{i,2}, \dots, a_{i,t_i})$ and $s^{-}_{q,i} := s_q(a_{i,2}, a_{i,3}, \dots, a_{i,t_i})$.

I believe that there are still many more similar factorizations in the domain of regions with central holes. For example, in my current joint project with Mihai Ciucu and Ranjan Rohatgi, we found a factorization for hexagons with three bowties removed. Our result implies a number of known tiling enumerations and q-enumerations, including Ciucu-Krattenthaler's enumeration of S-cored hexagons in [CK13], the two q-enumerations of mine [Lai17c, Lai17d], and my recent joint work with Ranjan Rohatgi [LR17]. This also implies a new q-dual of MacMahon's theorem that generalizes the one by Ciucu and Kratenthaler in [CK13].

We conclude this section by pointing out that there are still many open problems in this topic, including the following problems about the symmetric tilings.

Inspired by the fruitful study of symmetric plane partitions (see e.g. the classical paper [Sta86] and the excellent survey [Kra15] and the lists of references therein), we would like to investigate two symmetry classes of tilings of hexagons with removed ferns: the reflectively symmetric tilings, (i.e. the tilings which are invariant under the reflection over a vertical line), and the centrally symmetric

[6]The name came from the fact that this result investigates the tiling number *outside* a given contour, as opposed to MacMahon's result that gives the number of tilings staying *inside* a hexagonal contour [CK13, Ciu17].

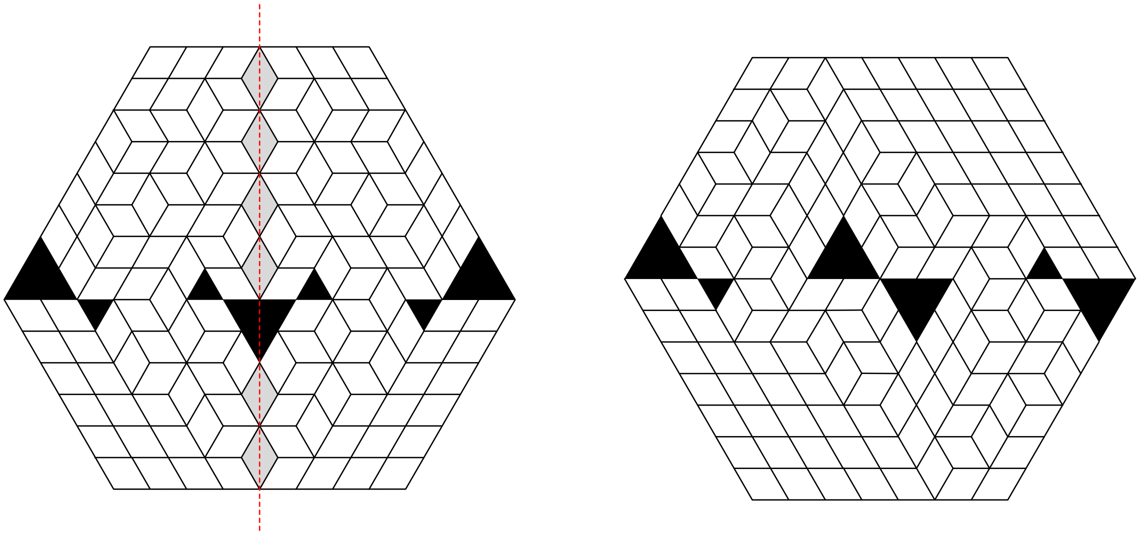

Figure 14: A reflectively symmetric tiling (left) and centrally symmetric tiling (right).

tilings (i.e. the tilings which are invariant under 180° rotations). These two classes correspond respectively to the *transposed-complementary plane partitions* (TCPP) and *self-complementary plane partitions* (SCPP) in the plane partition theory.

As shown in the left picture in Figure 14, a reflectively symmetric tiling of our region must contain all vertical shaded lozenges along the vertical axis. Removal of these lozenges separates the region into two disjoint congruent 'halved hexagons'. This means that any reflectively symmetric tiling is determined by a tiling of some halved hexagon. My previous work in [Lai18a, Lai17e, Lai18b] suggests the existence of a factorization (similar to that in Conjecture 8) for halved hexagons. This leads us to the following open problem:

Open Problem 6. *Find a factorization for reflectively symmetric tilings of hexagons with ferns removed.*

UPDATE: The problem has been solved by me.

Our data shows that the number of centrally symmetric tilings of a hexagon with three removed ferns seems to have a nice prime factorization (see the right picture in Figure 14 for such a symmetric tiling). This suggests the following conjecture:

Conjecture 10. *The number of tilings of the hexagon with three ferns, which are invariant under* 180° *rotations, is given by a simple product formula.*

UPDATE: Conjecture 10 have been solved and generalized by me.

This conjecture implies Stanley's well-known enumeration of SCPP (by setting the three ferns to be empty). As all known proofs of this enumeration use heavy algebraic tools (see [Sta86, Kup, Ste]), it would be interesting to find a combinatorial proof for this enumeration and for Conjecture 10.

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