Cyclically Symmetric Tilings of Hexagons with Four Holes

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(Joint work with Ranjan Rohatgi)

Plane partitions

$$\pi = \begin{matrix} \pi_{1,1} & \pi_{1,2} & \cdots & \pi_{1,b-1} & \pi_{1,b} \\ \pi_{2,1} & \pi_{2,2} & \cdots & \pi_{2,b-1} & \pi_{2,b} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \pi_{a,1} & \pi_{a,2} & \cdots & \pi_{a,b-1} & \pi_{a,b} \end{matrix}$$

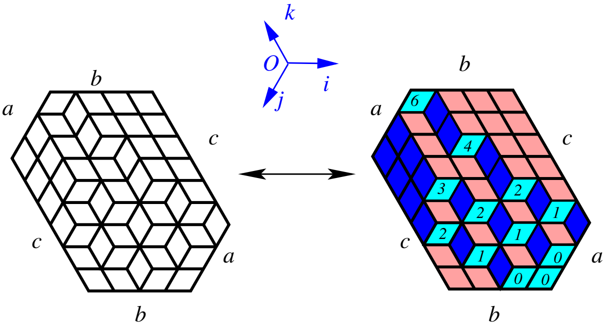
where $\pi_{i,j} \leq \pi_{i-1,j}$ and $\pi_{i,j} \leq \pi_{i,j-1}$.

Plane partitions

6	4	2	1
3	2	1	0
2	1	0	0

Plane partitions as stacks of unit cubes

6	4	2	1
3	2	1	0
2	1	0	0



MacMahon's Theorem

Theorem (MacMahon ∼1900)

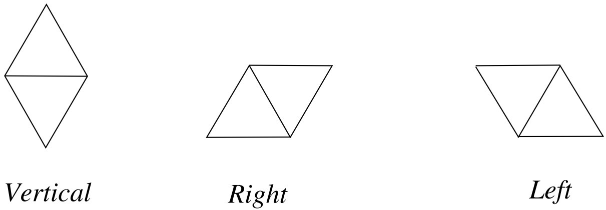
$$\sum_{\pi} q^{|\pi|} = PP_q(a,b,c) = \frac{\mathrm{H}_q(a)\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(a+b+c)}{\mathrm{H}_q(a+b)\,\mathrm{H}_q(b+c)\,\mathrm{H}_q(c+a)},$$

where the sum is taken over all plane partitions π fitting in an $a \times b \times c$ box, and $|\pi|$ is the volume of π.

Definition:

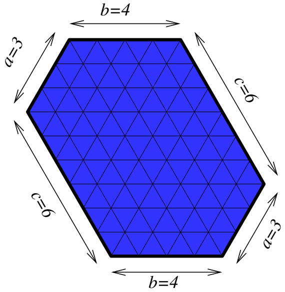
- q-integer $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$
- q-factorial $[n]_q! = [1]_q[2]_q \ldots [n]_q$,
- q-hyperfactorial $\mathrm{H}_q(n) = [0]_q![1]_q! \ldots [n-1]_q!$.

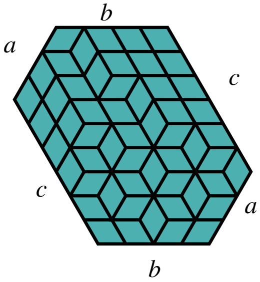
Lozenge tilings



- A lozenge (or unit rhombus) is the union of two adjacent unit equilateral triangles.
- A lozenge tiling of a region R on the triangular lattice is a covering of the region by lozenges, such that there are **no** gaps or overlaps.

Lozenge tilings





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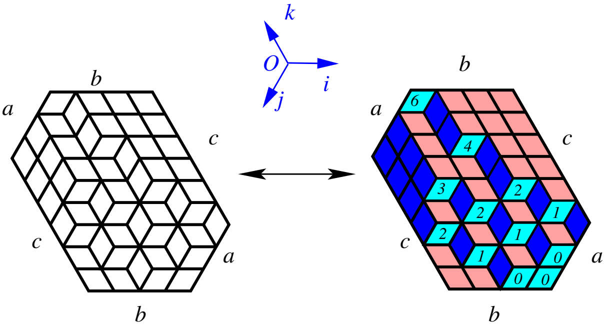
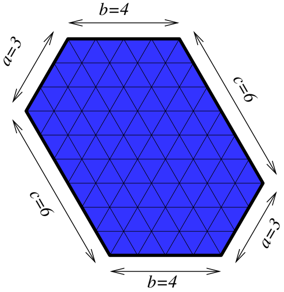


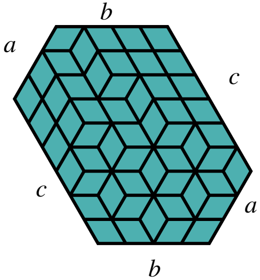
MacMahon's Theorem revisited

Theorem (MacMahon's Theorem for $q = 1$)

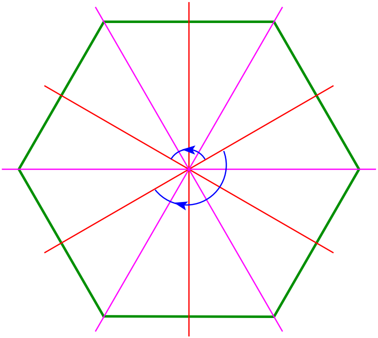
$$\mathrm{T}(Hex(a,b,c)) = PP(a,b,c) = \frac{\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)\,\mathrm{H}(a+b+c)}{\mathrm{H}(a+b)\,\mathrm{H}(b+c)\,\mathrm{H}(c+a)},$$

where $\mathrm{T}(R)$ *denotes the number of lozenge tilings of the region* R.



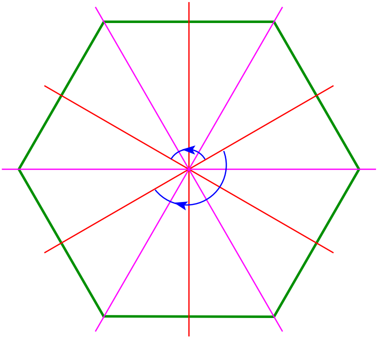


Symmetric Plane Partition



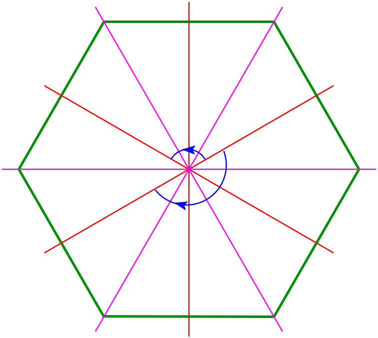
- There are 10 symmetry classes of plane partitions

Symmetric Plane Partition



- There are 10 symmetry classes of plane partitions
- Each corresponds to a symmetry class of lozenge tilings of a hexagon

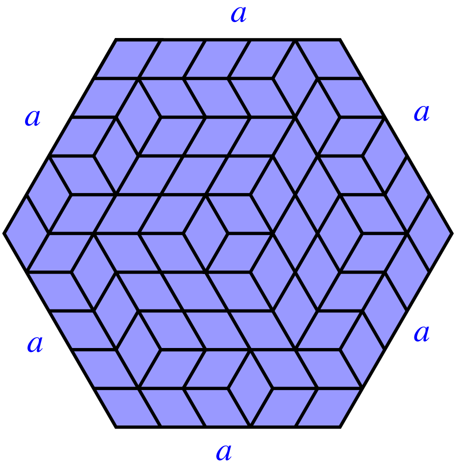
Symmetric Plane Partition



- There are 10 symmetry classes of plane partitions
- Each corresponds to a symmetry class of lozenge tilings of a hexagon
- Each is enumerated by a simple product formula

Cyclically Symmetric Plane Partition

Cyclically symmetric plane partitions correspond to lozenge tilings invariant under 120° rotations.



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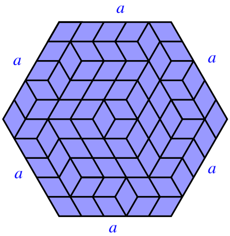

Conjecture (I.G. Macdonald)

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{a} \frac{1-q^{3i-1}}{1-q^{3i-2}} \prod_{1\leq i<j\leq a} \frac{1-q^{3(2i+j-1)}}{1-q^{3(2i+j-2)}} \prod_{1\leq i<j,k\leq a} \frac{1-q^{3(i+j+k-1)}}{1-q^{3(i+j+k-2)}}$$

Macdonald's Conjecture

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- George Andrews (1980) found a determinant for the generating function
- George Andrews verified for $q = 1$

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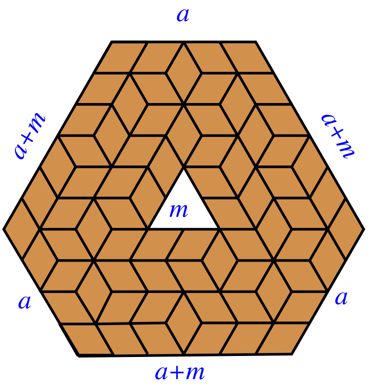
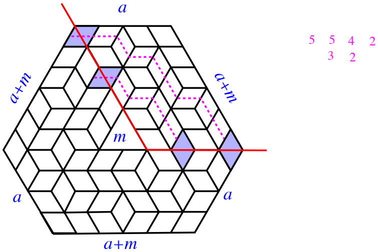
- George Andrews (1980) found a determinant for the generating function
- George Andrews verified for $q = 1$
- Mills–Robbins–Rumsey (1982) evaluated Andrews' determinant

Cyclically Symmetric Tilings of Cored Hexagons



1. Ciucu–Krattenthaler (2000): Formula for cyclically symmetric tilings of a cored-hexagon
2. Krattenthaler (2006): Bijection between cyclically symmetric tilings of a cored-hexagon with hole of size 2 and descending plane partitions.



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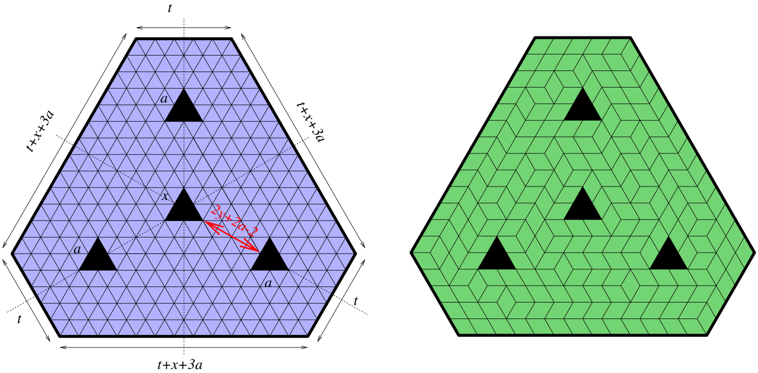
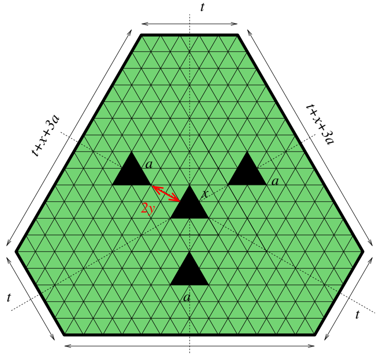
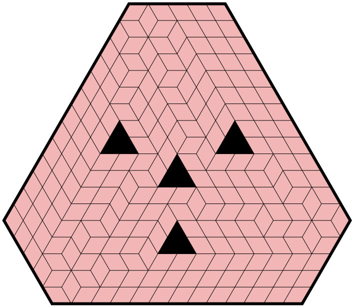
Cyclically Symmetric Tilings of Hexagons with Four Holes: Type I



$$\mathcal{H}_{t,y}(a,x)$$

Cyclically Symmetric Tilings of Hexagons with Four Holes: Type II





$$\overline{\mathcal{H}}_{t,y}(a,x)$$

Explicit Formula: Type I

$$P_1(x,y,z,a) := \frac{1}{2^{y+z}} \prod_{i=1}^{y+z} \frac{(2x+6a+2i)_i[2x+6a+4i+1]_{i-1}}{(i)_i[2x+6a+2i+1]_{i-1}}$$
$$\times \prod_{i=1}^{a} \frac{(z+i)_{y+a-2i+1}(x+y+2z+2a+2i)_{2y+2a-4i+2}}{(i)_y(y+2z+2i-1)_{y+2a-4i+3}}$$
$$\times \prod_{i=1}^{a} \frac{(x+3i-2)_{y-i+1}(x+3y+2i-1)_{i-1}}{(2z+2i)_{y+2a-4i+1}(x+y+z+2a+i)_{y+a-2i+1}}.$$

Pochhammer symbol $(x)_n$:

$$(x)_n = \begin{cases} x(x+1)\dots(x+n-1) & \text{if } n>0; \\ 1 & \text{if } n=0; \\ \dfrac{1}{(x-1)(x-2)\dots(x+n)} & \text{if } n<0. \end{cases}$$

Explicit Formula: Type I

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$$\times \prod_{i=1}^{a} \frac{(z+i)_{y+a-2i+1}(x+y+2z+2a+2i)_{2y+2a-4i+2}}{(i)_y(y+2z+2i-1)_{y+2a-4i+3}}$$

$$\times \prod_{i=1}^{a} \frac{(x+3i-2)_{y-i+1}(x+3y+2i-1)_{i-1}}{(2z+2i)_{y+2a-4i+1}(x+y+z+2a+i)_{y+a-2i+1}}.$$

'Skipping' Pochhammer symbol $[x]_n$:

$$[x]_n = \begin{cases} x(x+2)(x+4)\dots(x+2(n-1)) & \text{if } n>0; \\ 1 & \text{if } n=0; \\ \dfrac{1}{(x-2)(x-4)\dots(x+2n)} & \text{if } n<0. \end{cases}$$

Explicit Formula: Type I

$$P_2(x,y,z,a) := \frac{[x+3y]_a(x+2y+z+2a)_a}{2^{2(ay+z)}[x+3y+2z+2a+1]_a}$$

$$\times \prod_{i=1}^{y+z} \frac{(2x+6a+2i-2)_{i-1}[2x+6a+4i-1]_i}{(i)_i[2x+6a+2i-1]_{i-1}}$$

$$\times \prod_{i=1}^{a} \frac{(z+i)_{y+a-2i+1}(x+y+2z+2a+2i-1)_{2y+2a-4i+3}}{(i)_y(y+2z+2i-1)_{y+2a-4i+3}}$$

$$\times \prod_{i=1}^{a} \frac{(x+3i-2)_{y-i}(x+3y+2i-1)_{i-1}}{(2z+2i)_{y+2a-4i+1}(x+y+z+2a+i-1)_{y+a-2i+2}}.$$

Explicit Formula: Type I

$$F_1(x,y,z,a) = \frac{1}{2^{ya+z}} \frac{[x+y+2z+2a+1]_y}{[x+y+2a-1]_y} \frac{\prod_{i=1}^{\lfloor \frac{a}{3} \rfloor}(x+3y+6i-3)_{3a-9i+1}}{\prod_{i=1}^{\lfloor \frac{a-1}{3} \rfloor}(x+3y+6i-2)}$$

$$\times \prod_{i=1}^{y+z} \frac{i!(x+3a+i-3)!(2x+6a+2i-4)_i(x+3a+2i-2)_i(2x+6a+3i-4)}{(x+3a+2i-2)!(2i)!}$$

$$\times \prod_{i=1}^{a-1}(x+3i-2)_{y-i+1}(x+y+2z+2a+2i)_{2y+2a-4i}$$

$$\times \prod_{i=1}^{y} \frac{[2i+3]_{z-1}(x+3a+3i-5)_{2y+z-a-4i+5}}{(a+i+1)_{z-1}(i)_{a+1}[2i+3]_{a-2}[2x+6a+6i-7]_{z+2y-4i+3}}.$$

Explicit Formula: Type I

$$F_2(x,y,z,a) = \frac{1}{2^{y(a+2)+2a+z+1}} \frac{\prod_{i=1}^{\lfloor \frac{y+1}{3} \rfloor} (x+3i-2)_{3y-9i+4}}{\prod_{i=1}^{\lfloor \frac{y}{3} \rfloor} (x+3y-6i)}$$
$$\times \prod_{i=1}^{y+z} \frac{i!(x+3a+i-1)!(2x+6a+2i)_i(x+3a+2i)_i(x+3a+3i)}{(x+3a+2i)!(2i)!}$$
$$\times \prod_{i=1}^{y} \frac{[2i+3]_{z-1}(x+y+2a+2i-1)_{y+z-3i+2}(x+y+2z+2a+2i)_{2y+2a-4i+3}}{(a+i+2)_{z-1}(i)_{a+2}[2i+3]_{a-1}[2x+6a+6i-1]_{2y+z-4i+2}}.$$

Enumeration for Type I

Theorem (L.–Rohatgi 2017)

For non-negative integers y, t, a, x

$$\mathrm{CS}(\mathcal{H}_{2t+1,y}(2a,2x)) = 2^{2t+4a+1} P_1(x+1,y,t-y,a) P_2(x+1,y,t-y,a),$$

$$\mathrm{CS}(\mathcal{H}_{2t,y}(2a,2x)) = 2^{2t+4a} P_1(x+1,y,t-y-1,a) P_2(x+1,y,t-y,a),$$

$$\mathrm{CS}(\mathcal{H}_{2t+1,y}(2a+1,2x)) = 2^{2t+4a+3} F_1(x+1,y,t-y,a+1) F_2(x+1,y,t-y,a),$$

$$\mathrm{CS}(\mathcal{H}_{2t,y}(2a+1,2x)) = 2^{2t+4a+2} F_1(x+1,y,t-y-1,a+1) F_2(x+1,y,t-y,a).$$

When the middle hole has odd side, the number of tilings is not round.

Enumeration for Type II

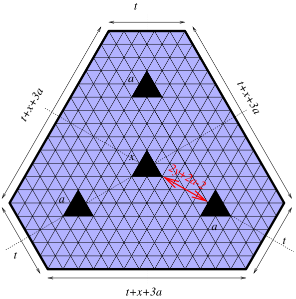
Theorem (L.–Rohatgi 2017)

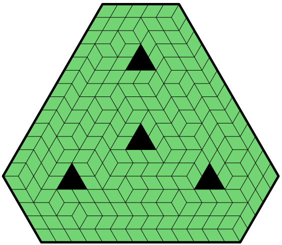
Assume that y, t, a, x are non-negative integers. Then

$$\begin{aligned}
\mathrm{CS}(\overline{\mathcal{H}}_{2t+1,y}(2a,2x)) &\\
&= 2^{2t+4a+1} E_1(x+1, y-1, t-y+2, a) E_2(x+1, y, t-y+1, a),\\
\mathrm{CS}(\overline{\mathcal{H}}_{2t,y}(2a,2x)) &\\
&= 2^{2t+4a} E_1(x+1, y-1, t-y+1, a) E_2(x+1, y, t-y+1, a),\\
\mathrm{CS}(\overline{\mathcal{H}}_{2t+1,y}(2a,2x+1)) &\\
&= 2^{2t+4a+1} E_3(x+2, y-1, t-y+2, a) E_4(x+1, y, t-y+1, a),\\
\mathrm{CS}(\overline{\mathcal{H}}_{2t,y}(2a,2x+1)) &\\
&= 2^{2t+4a} E_3(x+2, y-1, t-y+1, a) E_4(x+1, y, t-y+1, a).
\end{aligned}$$

When the satellite holes have odd side, the number of tilings is not round.

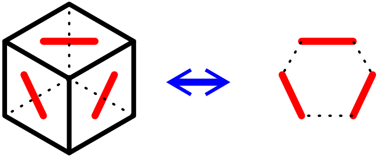
Proof (Sketched)





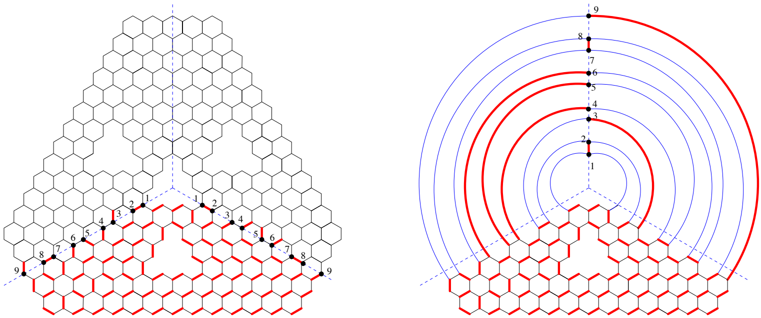
$$\mathrm{CS}(\mathcal{H}_{2t+1,y}(2a,2x)) = \quad 2^{2t+4a+1} P_1(x+1, y, t-y, a) P_2(x+1, y, t-y, a)$$

Proof (Sketched)



$$\mathrm{CS}(\mathcal{H}_{2t+1,y}(2a,2x)) = \mathrm{CS}(G)$$

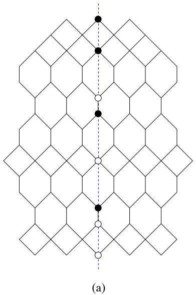
Proof (Sketched)



(a) (b)

$$\mathrm{CS}(\mathcal{H}_{2t+1,y}(2a,2x)) = \mathrm{CS}(G) = \mathrm{M}(Ob(G))$$

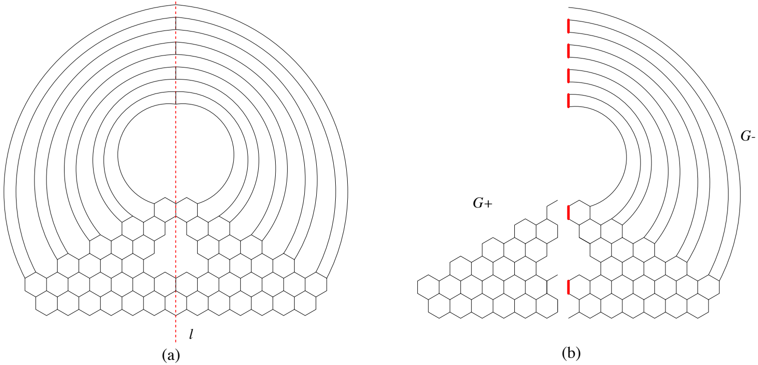
Proof (Sketched)



Lemma (Ciucu's Factorization Theorem)

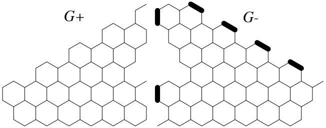
$$M(G) = 2^k\, M(G^+)\, M(G^-).$$

Proof (Sketched)

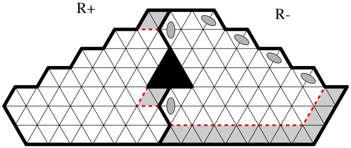


$$\mathrm{CS}(\mathcal{H}_{2t+1,y}(2a,2x)) = \mathrm{CS}(G) = \mathrm{M}(Ob(G))$$
$$= 2^k\,\mathrm{M}(Ob(G)^+)\,\mathrm{M}(Ob(G)^-)$$

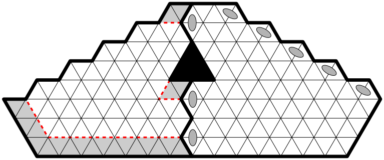
Proof (Sketched)



(a)



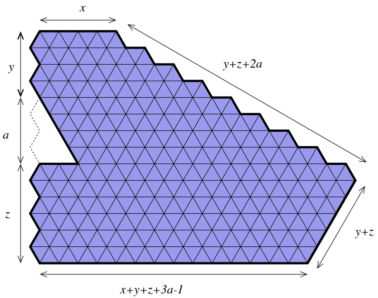
(b)



(c)

$$\begin{aligned}\mathrm{CS}(\mathcal{H}_{2t+1,y}(2a,2x)) = \mathrm{CS}(G) &= \mathrm{M}(Ob(G)) \\ &= 2^k\,\mathrm{M}(Ob(G)^+)\,\mathrm{M}(Ob(G)^-) \\ &= 2^k\,\mathrm{T}(\mathcal{R}^+)\,\mathrm{T}(\mathcal{R}^-)\end{aligned}$$

Proof (Sketched)



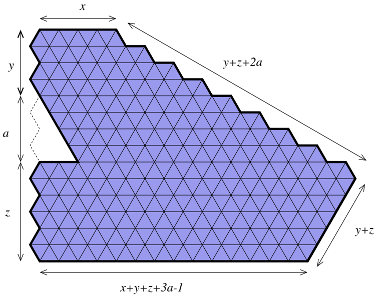
Theorem

For any non-negative integers x, y, z, a

$$\mathrm{T}(\mathcal{R}_{x,y,z}(a)) = P_1(x, y, z, a) \quad (1)$$

$$\mathrm{T}(\overline{\mathcal{R}}_{x,y,z}(a)) = P_2(x, y, z, a) \quad (2)$$

Proof (Sketched)



Theorem

For any non-negative integers x, y, z, a

$$\mathrm{T}(\mathcal{R}_{x,y,z}(a)) = P_1(x, y, z, a) \quad (3)$$

$$\mathrm{T}(\overline{\mathcal{R}}_{x,y,z}(a)) = P_2(x, y, z, a) \quad (4)$$

Kuo Condensation

Theorem (Kuo 2004)

Let R be a region on triangular lattice with $\#\triangle = \#\triangledown + 1$. Let u, v, w, s be four unit triangles appearing in a cyclic order on the boundary of R, such that $u, v, w = \triangle$'s and $s = \triangledown$. Then

$$\mathrm{T}(R - \{v\})\,\mathrm{T}(R - \{u, w, s\}) = \mathrm{T}(R - \{u\})\,\mathrm{T}(R - \{v, w, s\}) + \mathrm{T}(R - \{w\})\,\mathrm{T}(R - \{u, v, s\}).$$

Proof (Sketched)

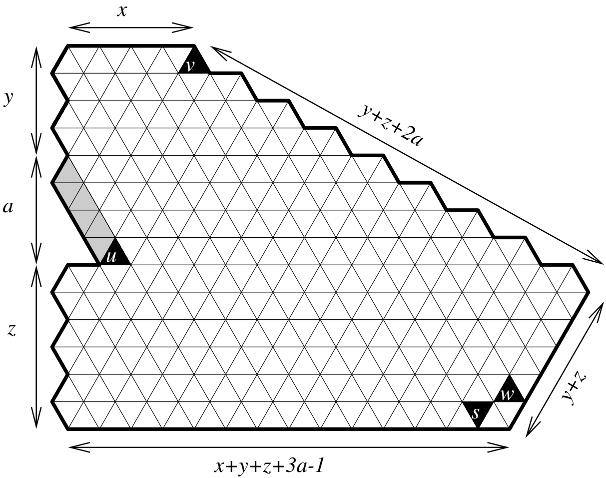
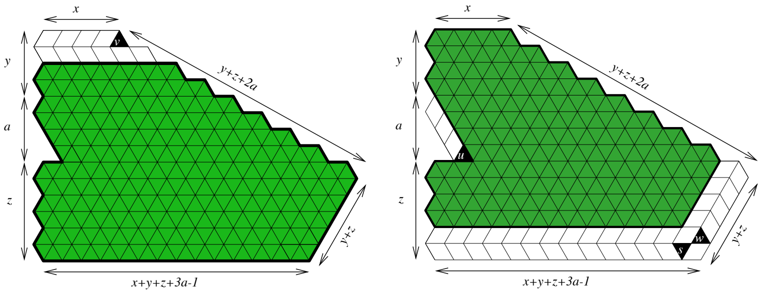


Figure: How to apply Kuo condensation

Create a Recurrence



$$\mathsf{T}(R - \{v\}) = \mathsf{T}(\mathcal{R}_{x+3,y,z}(a-1))$$

$$\mathsf{T}(R - \{u, w, s\}) = \mathsf{T}(\mathcal{R}_{x,y,z-1}(a))$$

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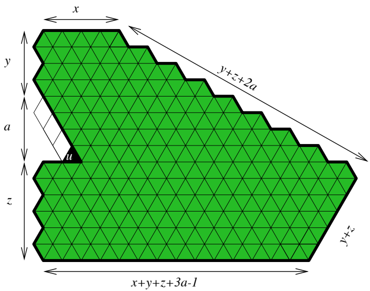

$$\mathrm{T}(R - \{u\}) = \mathrm{T}(\mathcal{R}_{x,y,z}(a))$$

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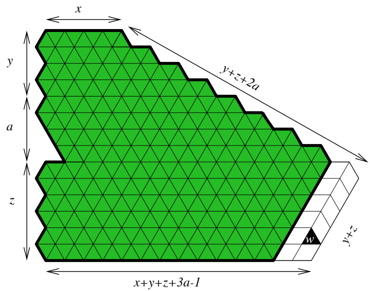

$$\mathsf{T}(R-\{u\}) = \mathsf{T}(\mathcal{R}_{x,y+1,z}(a-1))$$

$$\mathsf{T}(R-\{v,w,s\}) = \mathsf{T}(\mathcal{R}_{x+3,y-1,z-1}(a))$$

Create a Recurrence

Theorem (Kuo 2004)

Let R be a region on triangular lattice with $\#\triangle = \#\triangledown + 1$. Let u, v, w, s be four unit triangles appearing in a cyclic order on the boundary of R, such that $u, v, w = \triangle$'s and $s = \triangledown$. Then

$$\mathrm{T}(R - \{v\})\,\mathrm{T}(R - \{u, w, s\}) = \mathrm{T}(R - \{u\})\,\mathrm{T}(R - \{v, w, s\}) + \mathrm{T}(R - \{w\})\,\mathrm{T}(R - \{u, v, s\}).$$

$$\mathrm{T}(\mathcal{R}_{x+3,y,z}(a-1))\,\mathrm{T}(\mathcal{R}_{x,y,z-1}(a)) = \mathrm{T}(\mathcal{R}_{x,y,z}(a))\,\mathrm{T}(\mathcal{R}_{x+3,y,z-1}(a-1)) + \mathrm{T}(\mathcal{R}_{x,y+1,z}(a-1))\,\mathrm{T}(\mathcal{R}_{x+3,y-1,z-1}(a)).$$

Create a Recurrence

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- $P_1(x, y, z, a)$ satisfies the same recurrence

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$$\mathrm{T}(\mathcal{R}_{x+3,y,z}(a-1))\,\mathrm{T}(\mathcal{R}_{x,y,z-1}(a)) = \mathrm{T}(\mathcal{R}_{x,y,z}(a))\,\mathrm{T}(\mathcal{R}_{x+3,y,z-1}(a-1)) + \mathrm{T}(\mathcal{R}_{x,y+1,z}(a-1))\,\mathrm{T}(\mathcal{R}_{x+3,y-1,z-1}(a)).$$

- $P_1(x, y, z, a)$ satisfies the same recurrence
- The identity follows by induction

Open questions

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- The whole numbers of tilings of $\mathcal{H}_{t,y}(a,x)$ and $\overline{\mathcal{H}}_{t,y}(a,x)$?

Thank you!

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Explicit Formula: Type II

$$E_1(x,y,z,a) = \frac{1}{2^{y+z-1}} \prod_{i=1}^{a} \frac{(2x+2i)_i[2x+4i+1]_{i-1}}{(i)_i[2x+2i+1]_{i-1}}$$
$$\times \prod_{i=1}^{y+z-1} \frac{(2x+6a+2i)_i[2x+6a+4i+1]_{i-1}}{(i)_i[2x+6a+2i+1]_{i-1}}$$
$$\times \prod_{i=1}^{a} \frac{(z+i)_{y+a-2i+1}(2x+3y+2z+4a+2i-3)_{y+2a-4i+1}}{(2i)_{y-1}(y+2z+2i-1)_{y+2a-4i+2}}$$
$$\times \prod_{i=1}^{a} \frac{(x+a+i)_{y+a-2i}(2x+3y+3a+3i-3)_{a-i}}{(x+y+z+2a+i-1)_{y+a-2i}(2x+3a+3i)_{a-i}}. \quad (5)$$

Explicit Formula: Type II

$$E_2(x,y,z,a) = \frac{1}{2^{a+2y+2z-2+\lfloor \frac{a}{2} \rfloor}} \frac{(x+a)_{\lfloor \frac{a+1}{2} \rfloor}(x+2\lfloor \frac{y}{2} \rfloor + \lfloor \frac{y-1}{2} \rfloor + z + 2a)_a}{(x+y+a-1)_a(x+y+z+2a-1)_a}$$

$$\times \prod_{i=1}^{a} \frac{(2x+2i-2)_{i-1}[2x+4i-1]_i}{(i)_i[2x+2i-1]_{i-1}}$$

$$\times \prod_{i=1}^{y+z-1} \frac{(2x+6a+2i-2)_{i-1}[2x+6a+4i-1]_i}{(i)_i[2x+6a+2i-1]_{i-1}}$$

$$\times \frac{(2x+3y+3a-3)_a[2x+2\lfloor \frac{y}{2} \rfloor + 4\lfloor \frac{y-1}{2} \rfloor + 2z + 4a + 1]_a}{[2x+4y+2z+4a-3]_a[2x+4a-2\lfloor \frac{a+1}{2} \rfloor + 1]_{\lfloor \frac{a+1}{2} \rfloor}}$$

$$\times \prod_{i=1}^{a} \frac{(z+i)_{y+a-2i+1}(2x+3y+2z+4a+2i-3)_{y+2a-4i+1}}{(2i)_{y-1}(y+2z+2i-1)_{y+2a-4i+2}}$$

$$\times \prod_{i=1}^{a} \frac{(x+a+i)_{y+a-2i}(2x+3y+3a+3i-3)_{a-i}}{(x+y+z+2a+i-1)_{y+a-2i}(2x+3a+3i)_{a-i}}. \qquad (6)$$

$$E_3(x,2k,z,a) = 2^{\lfloor \frac{a+1}{2} \rfloor \lfloor \frac{z+1}{2} \rfloor + \lfloor \frac{a}{2} \rfloor \lfloor \frac{z}{2} \rfloor - a - z - 2k + 1}$$

$$\times \prod_{i=1}^{2k-1+a+z} \frac{i!(x+i-2)!(2x+2i-2)_i(x+2i-1)_i(2x+3i-2)}{(x+2i-1)!(2i)!}$$

$$\times \prod_{i=1}^{\lceil \frac{a-1}{3} \rceil} [2x+6k+2f(a+i-1)-1]_{f(a-3i+2)-1} \prod_{i=1}^{\lceil \frac{a-2}{3} \rceil} (x+3k+f($$

$$\times \prod_{i=1}^{\lfloor \frac{a+1}{2} \rfloor} [2x+6k+2z+4a+4i-5]_{\lfloor \frac{z}{2} \rfloor + a - 5i + 4} (x+3k+z+2a+$$

$$\times \prod_{i=1}^{\lfloor \frac{a}{2} \rfloor} [2x+6k+2z+4a+4i-3]_{\lfloor \frac{z+1}{2} \rfloor + a - 5i + 1} (x+3k+z+2a+$$

$$\times \prod_{i=1}^{a} \frac{[2k+2i-1]_{z+a-2i+1}(k+i)_{z+a-2i+1}}{(i)_{z+a-2i+1}[2x+4k+2a+2i-3]_{z+a-2i+1}(x+4k+z+2}$$

(7)

where $f(x) = 2\lfloor \frac{x+1}{2} \rfloor + \lfloor \frac{x}{2} \rfloor$ and

Explicit Formula: Type II

$$E_4(x,y,z,a) = \frac{E_3(x,y,z,a)}{K(x,y,z,a)}, \tag{9}$$

where

$$K(x,2k,z,a) = 2^{k+\lfloor\frac{z+1}{2}\rfloor+\lfloor\frac{a+1}{2}\rfloor-1}$$
$$\times \frac{\prod_{i=1}^{k+\lfloor\frac{z}{2}\rfloor+a}(2x+6i-5)\prod_{i=1}^{\lfloor\frac{z+1}{2}\rfloor}(x+3k+3a+3i-}{\prod_{i=1}^{\lfloor\frac{a+1}{2}\rfloor}(2x+6k+6\lfloor\frac{a}{2}\rfloor+6i-5)\prod_{i=1}^{k+z+\lfloor\frac{a}{2}\rfloor}(2x+3k+3\lfloor\frac{a+1}{2}}$$
(10)

and

$$K(x,2k+1,z,a) = 2^{k+\lfloor\frac{z}{2}\rfloor+\lfloor\frac{a}{2}\rfloor}$$
$$\times \frac{\prod_{i=1}^{k+\lfloor\frac{z+1}{2}\rfloor+a}(2x+6i-5)\prod_{i=1}^{\lfloor\frac{z}{2}\rfloor}(x+3k+3a+3}{\prod_{i=1}^{\lfloor\frac{a}{2}\rfloor}(2x+6k+6\lfloor\frac{a+1}{2}\rfloor+6i-5)\prod_{i=1}^{k+z+\lfloor\frac{a+1}{2}\rfloor}(x+3k+}$$
(11)