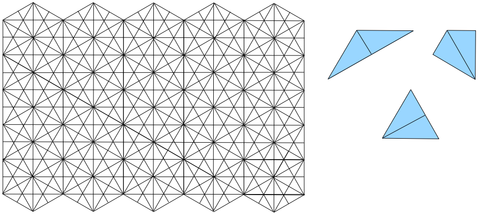
Proof of Blum's Conjecture on Hexagonal Dungeons.

Tri Lai

Combinatorics Seminar, Indiana University, October 14, 2013

Terminology

- A lattice divides the plane into elementary regions, called **cells**. A (lattice) **region** is a connected union of cells.



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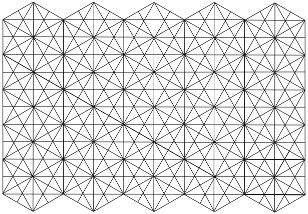

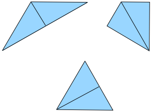

Terminology

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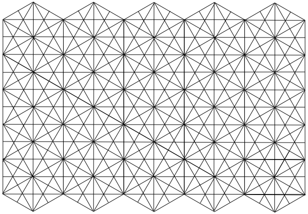
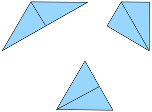
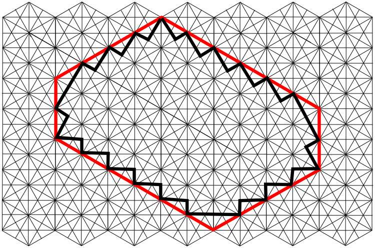
- A **tile** is a union of two cells that share an edge.
- A **tiling** of a region is a covering of the region by tiles so that there are no gaps or overlaps. Denote by $M(R)$ the number of tilings of the region R.





Hexagonal region: Region definition

- Consider the lattice obtained from the triangular lattice by drawing in all the altitudes in all the unit triangles



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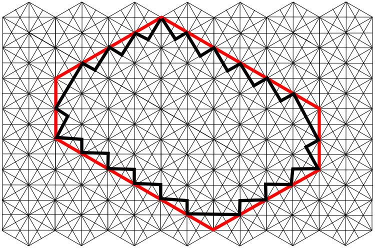
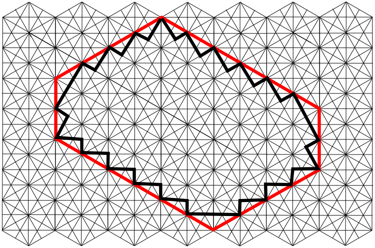
- Hexagonal contour of side-lengths a, $2a$, b, a, $2a$, b.



Hexagonal region: Region definition

- Consider the lattice obtained from the triangular lattice by drawing in all the altitudes in all the unit triangles
- Hexagonal contour of side-lengths a, $2a$, b, a, $2a$, b.
- Hexagonal dungeon of sides a, $2a$, b, a, $2a$, b, denoted by $HD_{a,2a,b}$.



Blum's Conjecture

Conjecture (Matt Blum 1999)

Assume that a and b are two positive integers so that $b \geq 2a$. *Then the number of tilings of the hexagonal dungeon* $HD_{a,2a,b}$ *is* $13^{2a^2}14^{\lfloor \frac{a^2}{2} \rfloor}$.

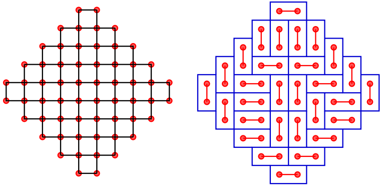
Perfect matchings and dual graph

- A **perfect matching** of G is a set of edges such that each vertex is adjacent to exactly one selected edge.
- Denote by $\mathrm{M}(G)$ the number of perfect matchings of the graph G.

Perfect matchings and dual graph

- A **perfect matching** of G is a set of edges such that each vertex is adjacent to exactly one selected edge.
- Denote by $\mathrm{M}(G)$ the number of perfect matchings of the graph G.
- The **dual graph** G of a region R is the graph whose vertices are cells in R and whose edges connect two adjacent cells.

Bijection between tilings and perfect matchings



We have a bijection between the tilings of R and the perfect matchings of G

Graph Splitting Lemma

Lemma

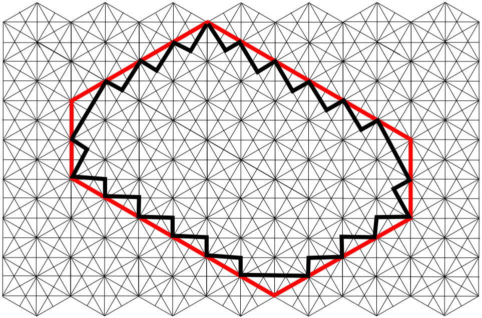
Let $G = (V_1, V_2, E)$ be a bipartite graph. Color the vertices in V_1 and V_2 by black and white respectively. Assume that the induced subgraph H of G satisfies following two conditions:

1. *Separating Condition: There are no edges of G connecting black vertex in H and white vertex in $G - H$.*
2. *Balancing Condition: The numbers of black and white vertices in H are equal.*

Then

$$\mathrm{M}(G) = \mathrm{M}(H)\,\mathrm{M}(G - H) \tag{1}$$

Blum's Conjecture: Refinement



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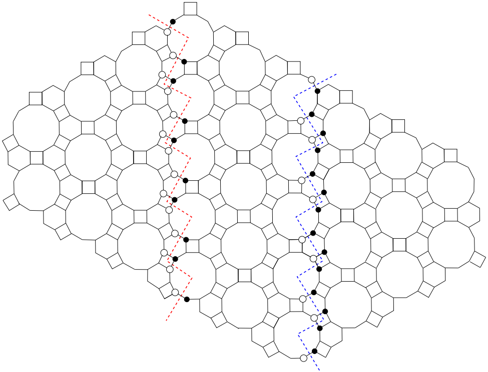
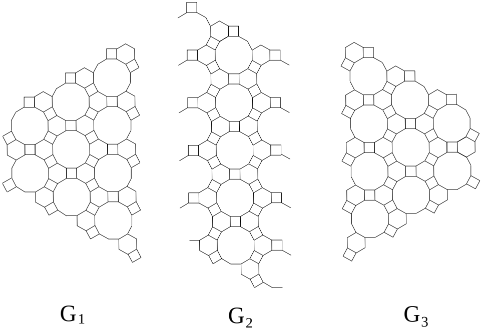


Blum's Conjecture: Refinement

$$\mathrm{M}(G) = \mathrm{M}(G_1)\,\mathrm{M}(G_2)\,\mathrm{M}(G_3) = \mathrm{M}(G_1)^2 = 13^{2a^2}14^{\lfloor \frac{a^2}{2} \rfloor}.$$



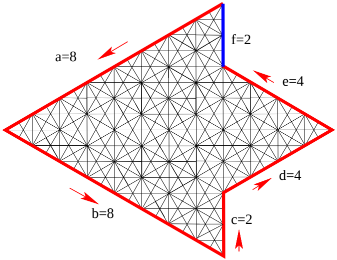
Blum's Conjecture: Refinement

Conjecture (Blum's conjecture: Refined)

$$\mathrm{M}(G_1) = 13^{a^2} 14^{\lfloor \frac{a^2}{4} \rfloor}$$

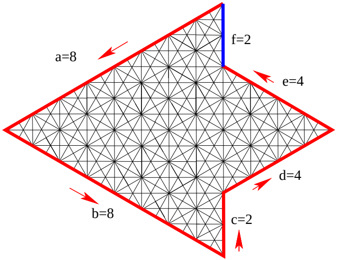
Main result: Definition of regions

- a, b and c are nonnegative integers. $\mathcal{C}(a,b,c)$ is a six-sided lattice contour



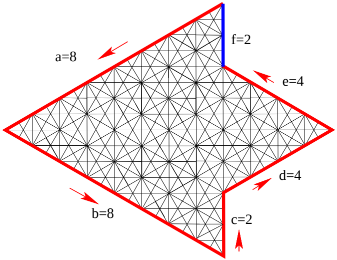
Main result: Definition of regions

- a, b and c are nonnegative integers. $\mathcal{C}(a, b, c)$ is a six-sided lattice contour
- a units southwest, b units southeast, c units north, d units northeast, and e units northwest.



Main result: Definition of regions

- a, b and c are nonnegative integers. $\mathcal{C}(a, b, c)$ is a six-sided lattice contour
- a units southwest, b units southeast, c units north, d units northeast, and e units northwest.
- Choose e so that the ending point is on the same vertical lattice line as the starting point. $a + e = b + d$.



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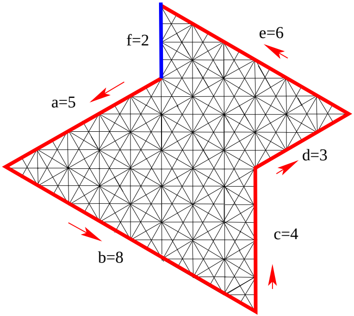
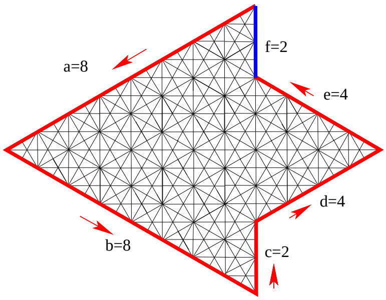


Main result: Definition of regions

- If e above the initial point, then it requires $a + b = 2c + d + e + 2f$ to close the contour.



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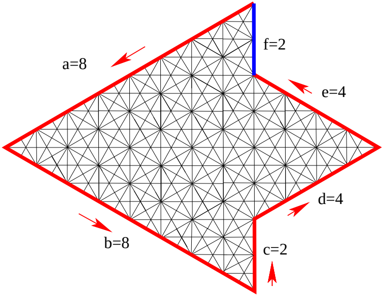
- If e above the initial point, then it requires $a+b=2c+d+e+2f$ to close the contour.
- Since $e=b+d-a$, we get $f=a-c-d$.



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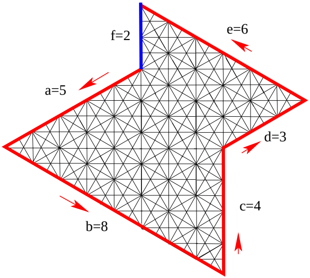
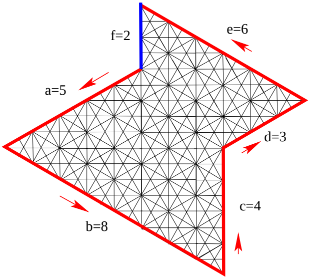


Main result: Definition of regions

- If e below the initial point, then it requires $a + b = 2c + d + e - 2f$ to close the contour.
- Since $e = b + d - a$, we get $f = -a + c + d$. So $f = |a - c - d|$.



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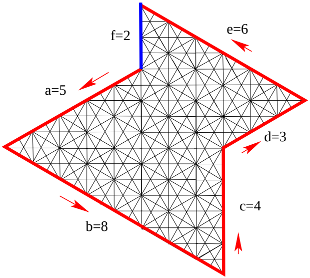
- Since $e = b + d - a$, we get $f = -a + c + d$. So $f = |a - c - d|$.
- Moreover, $d = 2b - a - 2c$.



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- $e = b + d - a = 3b - 2a - 2c$

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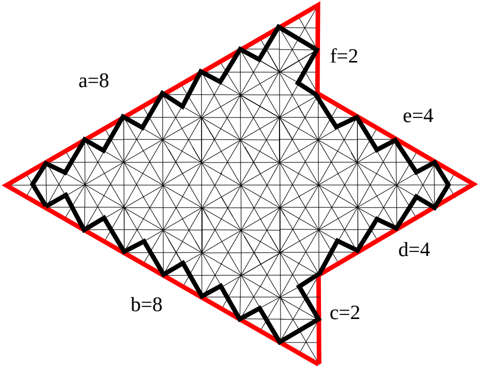
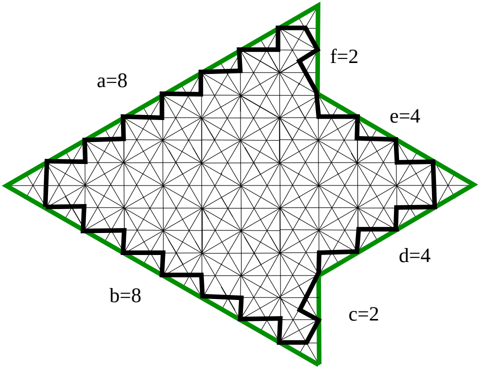
- $d = 2b - a - 2c.$
- $e = b + d - a = 3b - 2a - 2c$
- $f = |a - c - d| = |2b - 2b + c|$

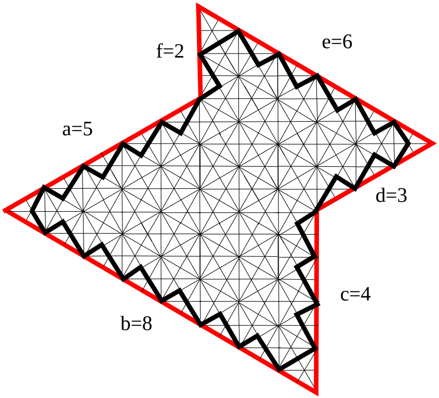
Main result: Definition of region $D_{a,b,c}$ when $a > c + d$.



Main result: Definition of region $E_{a,b,c}$ when $a > c + d$.

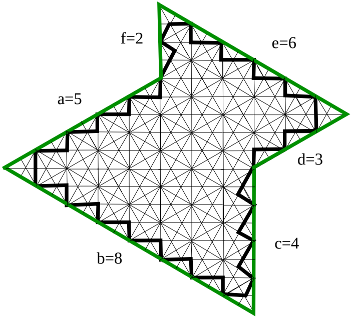


Main result: Definition of region $D_{a,b,c}$ when $a \leq c + d$.

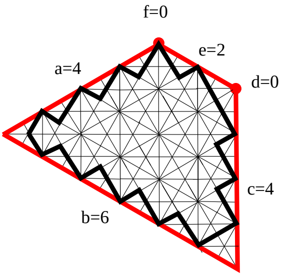


Main result: Definition of region $E_{a,b,c}$ when $a \leq c + d$.

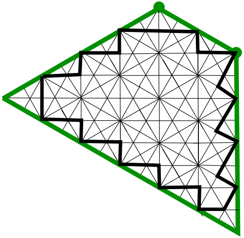
The balancing condition between the cells classes of the region $D_{a,b,c}$ and $E_{a,b,c}$ requires $d = 2b - a - 2c$.



Main result: Definition of regions when $d = f = 0$.



(a)



(b)

Main result: Definition of function $\phi(a,b,c)$

Define two functions ϕ by setting

$$\phi(a,b,c) := h(a,b,c)13^{g(a,b,c)}14^{f(a,b,c)} \tag{2}$$

where

$$g(a,b,c) = (b-a)(b-c) + \left\lfloor \frac{(a-c)^2}{3} \right\rfloor, \tag{3}$$

$$f(a,b,c) = \left\lfloor \frac{(a-b+c)^2}{4} \right\rfloor, \tag{4}$$

$$h(a,b,c) = \begin{cases} 2 & \text{if } 3b+a-c \equiv 4 \pmod 6 \\ 3 & \text{if } 3b+a-c \equiv 1 \pmod 6 \\ 5 & \text{if } 3b+a-c \equiv 5 \pmod 6 \\ 1 & \text{otherwise,} \end{cases} \tag{5}$$

Main result: Definition of function $\psi(a, b, c)$

Define

$$\psi(a,b,c) := p(a,b,c)13^{g(a,b,c)}14^{f(a,b,c)}, \tag{6}$$

where

$$p(a,b,c) = \begin{cases} 2 & \text{if } 3b+a-c \equiv 2 \pmod 6 \\ 3 & \text{if } 3b+a-c \equiv 5 \pmod 6 \\ 5 & \text{if } 3b+a-c \equiv 1 \pmod 6 \\ 1 & \text{otherwise.} \end{cases} \tag{7}$$

Main result

Theorem

Assume that a, b, and c are three nonnegative integers satisfying $b \geq 2$, $d = 2b - a - 2c \geq 0$ *and* $e = 3b - 2a - 2c \geq 0$. *Then*

$$\mathrm{M}(D_{a,b,c}) = \phi(a, b, c); \qquad \mathrm{M}(E_{a,b,c}) = \psi(a, b, c) \tag{8}$$

Corollary

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Assume that a and b are two positive integers so that $b \geq 2a$. Then the number of tilings of the hexagonal dungeon $HD_{a,2a,b}$ is $13^{2a^2}14^{\lfloor \frac{a^2}{2} \rfloor}$.

Proof.

As the refinement, we need to show $\mathrm{M}(G_1) = 13^{a^2}14^{\lfloor \frac{a^2}{4} \rfloor}$. However, G_1 is isomorphic to the dual graph of $D_{2a,3a,2a}$. Thus, $\mathrm{M}(G_1) = \phi(2a,3a,2a) = 13^{a^2}14^{\lfloor \frac{a^2}{4} \rfloor}$, since $h(2a,3a,2a) = 1$, $g(2a,3a,2a) = a^2$, $f(2a,3b,2a) = \lfloor \frac{a^2}{4} \rfloor$. □

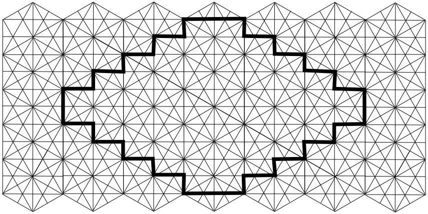
Corollary

Corollary (Ciucu 2001)

The number of tilings of an Aztec Dungeon is always a power of 13 or twice a power of 13.

Proof.

Aztec Dungeon of order n is $E_{n+1,n+1,0}$. □



Recurrence 1

Lemma

Let a, b and c be nonnegative integers so that $b \geq 5$ *and* $c \geq 2$. *Let* $d = 2b - a - 2c$, *and assume in addition that* $a \geq c + d + 1$. *Then*

$$\mathrm{M}(D_{a,b,c})\,\mathrm{M}(D_{a-3,b-3,c-2}) = \mathrm{M}(D_{a-2,b-1,c})\,\mathrm{M}(D_{a-1,b-2,c-2}) + \mathrm{M}(D_{a-1,b-1,c-1})\,\mathrm{M}(D_{a-2,b-2,c-1}),$$

$$\mathrm{M}(E_{a,b,c})\,\mathrm{M}(E_{a-3,b-3,c-2}) = \mathrm{M}(E_{a-2,b-1,c})\,\mathrm{M}(E_{a-1,b-2,c-2}) + \mathrm{M}(E_{a-1,b-1,c-1})\,\mathrm{M}(E_{a-2,b-2,c-1}).$$

Kuo's Graphical Condensation

Combinatorial interpretation of *Desnanot-Jacobi identity* in determinant of matrices.

$$\det(M)\det(M_{1,k}^{1,k}) = \det(M_1^1)\det(M_k^k) - \det(M_1^k)\det(M_k^1).$$

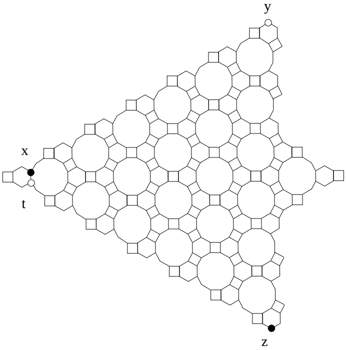
Theorem (Kuo 2004)

Let G be a planar bipartite graph, and let V_1 and V_2 be the two vertex classes. Assume that $|V_1| = |V_2|$. Let x, y, z and t be four vertices appear in a cyclic order on a face of G. Assume in addition that $x, z \in V_1$ and $y, t \in V_2$. Then

$$\mathrm{M}(G)\,\mathrm{M}(G - \{x,y,z,t\}) = \mathrm{M}(G - \{x,y\})\,\mathrm{M}(G - \{z,t\}) + \mathrm{M}(G - \{t,x\})\,\mathrm{M}(G - \{y,z\}).$$

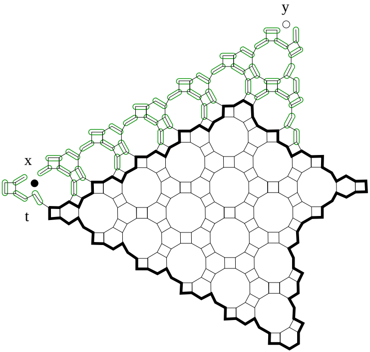
Proof of Recurrence 1

Let G is the dual graph of $D_{a,b,c}$, apply Kuo's condensation to G with the four vertices x, y, z, t.



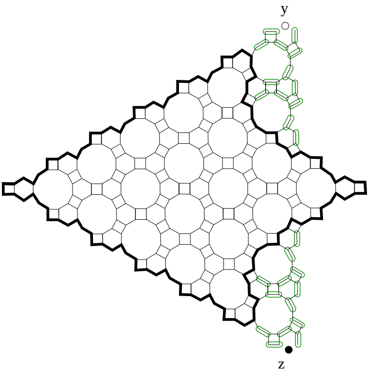
Proof of Recurrence 1

$$M(G - \{x, y\}) = M(D_{a-2,b-1,c}). \tag{9}$$



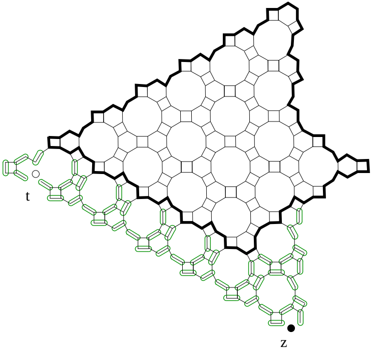
Proof of Recurrence 1

$$\mathrm{M}(G - \{y, z\}) = \mathrm{M}(D_{a-1,b-1,c-1}) \tag{10}$$



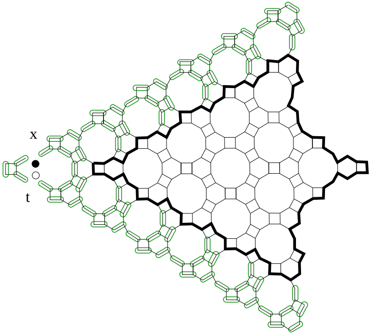
Proof of Recurrence 1

$$M(G - \{z, t\}) = M(D_{a-1,b-2,c-2}) \tag{11}$$



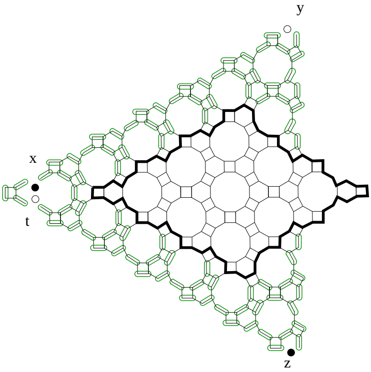
Proof of Recurrence 1

$$M(G - \{t, x\}) = M(D_{a-2,b-2,c-1}) \qquad (12)$$



Proof of Recurrence 1

$$M(G - \{x, y, z, t\}) = M(D_{a-3,b-3,c-2}) \tag{13}$$



Recurrence 2

Lemma

Let a, b and c be nonnegative integers satisfying $a \geq 2$, $b \geq 4$, $2b - a - 2c \geq 2$, *and* $3b - 2a - 2c \geq 2$.
(a). *If* $c \geq 1$, *then*

$$\mathrm{M}(D_{a,b,c})\,\mathrm{M}(D_{a-2,b-2,c}) = \mathrm{M}(D_{a-1,b-1,c})^2 + \mathrm{M}(D_{a,b,c+1})\,\mathrm{M}(D_{a-2,b-2,c-1})$$

$$\mathrm{M}(E_{a,b,c})\,\mathrm{M}(E_{a-2,b-2,c}) = \mathrm{M}(E_{a-1,b-1,c})^2 + \mathrm{M}(E_{a,b,c+1})\,\mathrm{M}(E_{a-2,b-2,c-1}).$$

Recurrence 3

Lemma

(b). *If* $c = 0$, *then*

$$\mathrm{M}(D_{a,b,0})\,\mathrm{M}(D_{a-2,b-2,0}) = \mathrm{M}(D_{a-1,b-1,0})^2 + \mathrm{M}(D_{a,b,1})\,\mathrm{M}(D_{e,d,1})$$

$$\mathrm{M}(E_{a,b,0})\,\mathrm{M}(E_{a-2,b-2,0}) = \mathrm{M}(E_{a-1,b-1,0})^2 + \mathrm{M}(E_{a,b,1})\,\mathrm{M}(E_{e,d,1}),$$

where, as usual, $d = 2a - b - 2c$ *and* $e = 3a - 2b - 2c$.

Recurrence 4

Lemma

Assume that a, b, c *are three nonnegative integers satisfying* $a \geq 2$, $b \geq 5$ *and* $c \geq 2$. *As usual, let* $d = 2b - a - 2c$. *Assume in addition that* $a \leq c + d$.

(a). *If* $d \geq 1$, *then*

$$\mathrm{M}(D_{a,b,c})\,\mathrm{M}(D_{a-2,b-3,c-2}) = \mathrm{M}(D_{a-1,b-1,c})\,\mathrm{M}(D_{a-1,b-2,c-2}) + \mathrm{M}(D_{a-2,b-2,c-1})\,\mathrm{M}(D_{a,b-1,c-1})$$

$$\mathrm{M}(E_{a,b,c})\,\mathrm{M}(E_{a-2,b-3,c-2}) = \mathrm{M}(E_{a-1,b-1,c})\,\mathrm{M}(E_{a-1,b-2,c-2}) + \mathrm{M}(E_{a-2,b-2,c-1})\,\mathrm{M}(E_{a,b-1,c-1}).$$

Recurrence 5

Lemma

(b). *If $d = 0$, then*

$$T(D_{a,b,c})\,M(D_{a-2,b-3,c-2}) = M(E_{c,b-1,a-1})\,M(D_{a-1,b-2,c-2}) + M(D_{a-2,b-2,c-1})\,M(D_{a,b-1,c-1})$$

$$M(E_{a,b,c})\,M(E_{a-2,b-3,c-2}) = M(D_{c,b-1,a-1})\,M(E_{a-1,b-2,c-2}) + M(E_{a-2,b-2,c-1})\,M(E_{a,b-1,c-1}).$$

Recurrences for $\phi(a, b, c)$ and $\psi(a, b, c)$

Lemma

Two functions $\phi(a, b, c)$ and $\psi(a, b, c)$ satisfy the Recurrences 1 – 5.

Base case

All $D_{a,b,c}$ and $E_{a,b,c}$ regions satisfying one of the following conditions:

(i) $\mathcal{P}(a, b, c) \leq 14$ (20 regions)

(ii) $b \leq 4$ (43 regions)

(iii) $c + d \leq 2$ (10 regions)

This cases can be verified by `vaxmacs` written by David Wilson. This software is available at `http://dbwilson.com/vaxmacs/`.

Induction step

1. Assume that the statement is true for any D- and E-type regions with perimeter less than k ($k \geq 16$).
2. We will show the statement is true for any D- and E-type region with perimeter k.
3. By the (ii) and (iii) in base cases, we can assume $b \geq 5$ and $c + d \geq 3$.

Induction step: Case I: $a \leq c + d$ and $a \geq 2$

Case $c \geq 1$ and $d \geq 2$. Use the recurrence 2.

1. $\mathcal{P}(a-2, b-2, c)$, $\mathcal{P}(a-1, b-1, c)$, $\mathcal{P}(a, b, c+1)$, $\mathcal{P}(a-2, b-2, c-1)$ are less than $\mathcal{P}(a, b, c)$ by 8, 4, 4, and 4 units, respectively.

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1. $\mathcal{P}(a-2, b-2, c)$, $\mathcal{P}(a-1, b-1, c)$, $\mathcal{P}(a, b, c+1)$, $\mathcal{P}(a-2, b-2, c-1)$ are less than $\mathcal{P}(a, b, c)$ by 8, 4, 4, and 4 units, respectively.
2. $M(D_{a-2,b-2,c}) = \phi(a-2, b-2, c)$, $M(D_{a-1,b-1,c}) = \phi(a-1, b-1, c)$,
3. $M(D_{a,b,c+1}) = \phi(a, b, c+1)$, $M(D_{a-2,b-2,c-1}) = \phi(a-2, b-2, c-1)$,

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2. $M(D_{a-2,b-2,c}) = \phi(a-2, b-2, c)$, $M(D_{a-1,b-1,c}) = \phi(a-1, b-1, c)$,
3. $M(D_{a,b,c+1}) = \phi(a, b, c+1)$, $M(D_{a-2,b-2,c-1}) = \phi(a-2, b-2, c-1)$,
4. $M(D_{a,b,c})$ and $\phi(a, b, c)$ satisfy the recurrence 2.

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1. $\mathcal{P}(a-2, b-2, c)$, $\mathcal{P}(a-1, b-1, c)$, $\mathcal{P}(a, b, c+1)$, $\mathcal{P}(a-2, b-2, c-1)$ are less than $\mathcal{P}(a, b, c)$ by 8, 4, 4, and 4 units, respectively.
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3. $\mathrm{M}(D_{a,b,c+1}) = \phi(a, b, c+1)$, $\mathrm{M}(D_{a-2,b-2,c-1}) = \phi(a-2, b-2, c-1)$,
4. $\mathrm{M}(D_{a,b,c})$ and $\phi(a, b, c)$ satisfy the recurrence 2.
5. $\mathrm{M}(D_{a,b,c}) = \phi(a, b, c)$, and similarly $\mathrm{M}(E_{a,b,c}) = \psi(a, b, c)$.

Induction step: Case I. $a \leq c + d$ and $a \geq 2$

1. Case $a \geq 2$, $c = 0$ and $d \geq 2$. Use the recurrence 3.

Induction step: Case I. $a \leq c + d$ and $a \geq 2$

1. Case $a \geq 2$, $c = 0$ and $d \geq 2$. Use the recurrence 3.
2. Case $a \geq 2$, $c \geq 2$ and $d \geq 1$. Use the recurrence 4.
3. Case $a \geq 2$, $c \geq 2$ and $d \geq 1$. Use the recurrence 5.

Induction step: Case II. $a \leq c + d$ and $a \leq 1$

1. $f = c + d - a \geq 3 - 1 = 2$

Induction step: Case II. $a \leq c + d$ and $a \leq 1$

1. $f = c + d - a \geq 3 - 1 = 2$
2. Reflect $D_{a,b,c}$ about the side b of the contour we get $E_{f,e,d}$.

Induction step: Case II. $a \leq c + d$ and $a \leq 1$

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2. Reflect $D_{a,b,c}$ about the side b of the contour we get $E_{f,e,d}$.
3. Reflect $E_{a,b,c}$ about the side b of the contour we get $D_{f,e,d}$.

Induction step: Case II. $a \leq c + d$ and $a \leq 1$

1. $f = c + d - a \geq 3 - 1 = 2$
2. Reflect $D_{a,b,c}$ about the side b of the contour we get $E_{f,e,d}$.
3. Reflect $E_{a,b,c}$ about the side b of the contour we get $D_{f,e,d}$.
4. $E_{f,e,d}$ and $D_{f,e,d}$ satisfy the condition of Case I.

Induction step: Case II. $a \leq c + d$ and $a \leq 1$

1. $f = c + d - a \geq 3 - 1 = 2$
2. Reflect $D_{a,b,c}$ about the side b of the contour we get $E_{f,e,d}$.
3. Reflect $E_{a,b,c}$ about the side b of the contour we get $D_{f,e,d}$.
4. $E_{f,e,d}$ and $D_{f,e,d}$ satisfy the condition of Case I.
5. $\mathrm{M}(D_{a,b,c}) = \mathrm{M}(E_{f,e,d}) = \psi(f, e, d) = \phi(a, b, c)$

Induction step: Case II. $a \leq c + d$ and $a \leq 1$

1. $f = c + d - a \geq 3 - 1 = 2$
2. Reflect $D_{a,b,c}$ about the side b of the contour we get $E_{f,e,d}$.
3. Reflect $E_{a,b,c}$ about the side b of the contour we get $D_{f,e,d}$.
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5. $\mathrm{M}(D_{a,b,c}) = \mathrm{M}(E_{f,e,d}) = \psi(f, e, d) = \phi(a, b, c)$
6. $\mathrm{M}(E_{a,b,c}) = \mathrm{M}(D_{f,e,d}) = \phi(f, e, d) = \psi(a, b, c)$

Induction step: Case III. $a \geq c + d + 1$ and $e \geq d$

1. If $c \geq 2$, use recurrence 1.

Induction step: Case III. $a \geq c+d+1$ and $e \geq d$

1. If $c \geq 2$, use recurrence 1.
2. If $c = 1$, use recurrence 3.

Induction step: Case III. $a \geq c + d + 1$ and $e \geq d$

1. If $c \geq 2$, use recurrence 1.
2. If $c = 1$, use recurrence 3.
3. If $c = 0$, use recurrence 4.

Induction step: Case IV. $a \geq c + d + 1$ and $d \geq e$

1. Reflect the regions $D_{a,b,c}$ and $E_{a,b,c}$ over the horizontal line passing the west vertex, we get $D_{a,b,f}$ and $E_{a,b,f}$.

Induction step: Case IV. $a \geq c + d + 1$ and $d \geq e$

1. Reflect the regions $D_{a,b,c}$ and $E_{a,b,c}$ over the horizontal line passing the west vertex, we get $D_{a,b,f}$ and $E_{a,b,f}$.
2. If $c \geq 1$, then $D_{a,b,f}$ and $E_{a,b,f}$ satisfy the condition of Case III.

Induction step: Case IV. $a \geq c + d + 1$ and $d \geq e$

1. Reflect the regions $D_{a,b,c}$ and $E_{a,b,c}$ over the horizontal line passing the west vertex, we get $D_{a,b,f}$ and $E_{a,b,f}$.
2. If $c \geq 1$, then $D_{a,b,f}$ and $E_{a,b,f}$ satisfy the condition of Case III.
3. If $c = 0$, then $D_{a,b,f}$ and $E_{a,b,f}$ satisfy the condition of Case I or Case II.

Induction step: Case IV. $a \geq c + d + 1$ and $d \geq e$

1. Reflect the regions $D_{a,b,c}$ and $E_{a,b,c}$ over the horizontal line passing the west vertex, we get $D_{a,b,f}$ and $E_{a,b,f}$.
2. If $c \geq 1$, then $D_{a,b,f}$ and $E_{a,b,f}$ satisfy the condition of Case III.
3. If $c = 0$, then $D_{a,b,f}$ and $E_{a,b,f}$ satisfy the condition of Case I or Case II.
4. $\mathrm{M}(D(a, b, c)) = \mathrm{M}(D(b, c, f)) = \phi(b, c, f) = \phi(a, b, c)$

Induction step: Case IV. $a \geq c + d + 1$ and $d \geq e$

1. Reflect the regions $D_{a,b,c}$ and $E_{a,b,c}$ over the horizontal line passing the west vertex, we get $D_{a,b,f}$ and $E_{a,b,f}$.
2. If $c \geq 1$, then $D_{a,b,f}$ and $E_{a,b,f}$ satisfy the condition of Case III.
3. If $c = 0$, then $D_{a,b,f}$ and $E_{a,b,f}$ satisfy the condition of Case I or Case II.
4. $\mathrm{M}(D(a,b,c)) = \mathrm{M}(D(b,c,f)) = \phi(b,c,f) = \phi(a,b,c)$
5. $\mathrm{M}(E(a,b,c)) = \mathrm{M}(E(b,c,f)) = \psi(b,c,f) = \psi(a,b,c)$

Conclusion

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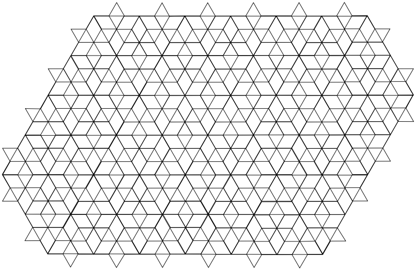
- Sometimes, it is easier to prove a generalization of a conjecture than a conjecture itself.
- The graphical condensation method does not work for the family of hexagonal dungeons (the family is too small), and we need to extend to the families of $D_{a,b,c}$ and $E_{a,b,c}$ regions

Future project

- Consider the weighted version of the conjecture.



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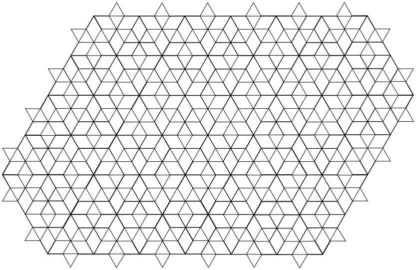
- Consider the weighted version of the conjecture.
- Prove a conjecture about trimmed Aztec rectangle in the lattice.



Future project

- Consider the weighted version of the conjecture.
- Prove a conjecture about trimmed Aztec rectangle in the lattice.
- Consider several variants of the hexagonal dungeons.