3.1. LINEAR MODELS

GROW AND DECAY

$$\frac{dx}{dt} = Kx, \quad x(t_0) = x_0 \qquad (1)$$

Example 1.

A culture initially has P_0 number of bacteria. At $t = 1h$ the number of bacteria is measured to be $\frac{3}{2} P_0$. If the the rate of grow is propotional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.

Solution:

$$\frac{dP}{dt} - KP = 0$$

Integrating factor $\mu(t) = e^{\int -K\,dt} = e^{-Kt}$.

$$d[\mu P] = 0$$

$$\Leftrightarrow \quad e^{-Kt} P = c \quad \Leftrightarrow \quad P = c.e^{Kt}.$$

We need to find c and K.

$\boxed{t=0:}$ $\qquad P_0 = c$

$\boxed{t=1}$ $\qquad \frac{3}{2} P_0 = P_0 \cdot e^{K}$

$\Leftrightarrow \quad e^{K} = \frac{3}{2} \Leftrightarrow K = \ln\frac{3}{2} \; (\approx 0.4055)$

We need to find t such that

$$P(t) = P_0 \, e^{(\ln\frac{3}{2})t} = 3P_0$$

$$\Leftrightarrow \quad e^{(\ln\frac{3}{2})t} = 3$$

$$\Leftrightarrow \quad \ln\frac{3}{2}\, t = \ln 3$$

$$\Leftrightarrow \quad t = \frac{\ln 3}{\ln 3/2} \approx 2.71\,(h).$$

$\square$

HALF LIFE. In physics the **half-life** is a measure of the stability of a radioactive substance. This is simply the time it takes fore $\frac{1}{2}$ of the atoms in the initial amount A_0 to disintergrate, ore transmute, into the atoms of anothe element.

Example. A breeder reactor converts relatively stable uranium-238 into isotope plotonium-239. After 15 years it is determined that 0.043% of the initial amount A_0 of plutonuum has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

Solution:

$$\frac{dA}{dt} = KA, \qquad A(0) = A_0$$

$$A(t) = A_0 e^{Kt}.$$

If 0.043% of the atoms of A_0 have disintegrated, then 99.957% of the substance remains. We can find K:

$$\Rightarrow \quad 0.99957 A_0 = A_0 e^{15K}$$

$$\Rightarrow \quad K = \frac{1}{5} \ln 0.99957 \approx -0.00002867.$$

Find the half-life t:

$$\frac{1}{2} A_0 = A_0 e^{-0.00002867t}$$

$$\Leftrightarrow \quad \frac{1}{2} = e^{-0.00002867t}$$

$$\Leftrightarrow \quad t = \frac{\ln \frac{1}{2}}{-0.00002867} = \frac{\ln 2}{0.00002867} \approx 24{,}180 \text{ yr.}$$

$\square$

Reading exercise: (Example about Age of Fossil)

NEWTON'S LAW OF COOLING / WARMING

$$\frac{dT}{dt} = k(T - T_m)$$

where T_m is the temperature of the medium around the object.

Example 4. When a cake removed from an oven, its temperature is measured at 300°F. Three minutes later the temterature is 200°F. How long will it take for the cake to cool off to 100°F? The room temper. is 70°F.

Solution:

$$\frac{dT}{dt} = k(T - T_m), \quad T(0) = 300$$

$$\Leftrightarrow \frac{dT}{(T-70)} = k\,dt \quad \text{(Separable eq.)}$$

$$\Leftrightarrow \ln|T-70| = kt + C_1$$

$$\Leftrightarrow T - 70 = c\,e^{kt}$$

$$\Leftrightarrow T = c \cdot e^{kt} + 70$$

$\boxed{t=0}$ $\quad 300 = c + 70 \Rightarrow c = 230$

$\boxed{t=3}$ $\quad 200 = 230 \cdot e^{3k} + 70$

$$\Leftrightarrow e^{3k} = \frac{130}{230} \Leftrightarrow k = \frac{1}{3}\ln\frac{130}{230}$$

$$\approx -0.19018$$

$\boxed{\text{Find } t}$

$$100 = 70 + 230\, e^{-0.19018t}$$

$$\Leftrightarrow \quad e^{-0.19018t} = \frac{30}{230}$$

$$\Leftrightarrow \quad -0.19018t = \ln\frac{3}{23}$$

$$\Leftrightarrow \quad t = \frac{\ln\frac{23}{3}}{0.19018} \approx 10.7 \text{ (mins)}$$

$\square$

MIXTURE:

Example: A large tank holds 200 gls of brine solution. Brine is being pumped into the tank at the rate 4 gal/min. The concentration of the salt in the inflow is 3 lb/gal. The mixture is also pumped out at the rate 4 gal/min.

$$R_{in} = (4 \text{ gal/min})(3 \text{ lb/gal}) = 12 \text{ lb/min}$$

$$R_{out} = \left(\frac{A}{200} \text{ lb/gal}\right)(4 \text{ gal/min})$$

$$= \frac{A}{50} \text{ lb/min}$$

Assume that 50 pounds of salt were dissolved initially in the 200 gallons, how much salt in the tank after a long time?

<u>Solution:</u>

<u>Recall:</u> $$\frac{dA}{dt} = R_{in} - R_{out}$$

$$\frac{dA}{dt} = 12 - \frac{A}{50}$$

$$\Leftrightarrow \quad \frac{dA}{dt} + \frac{A}{50} = 12, \quad A(0) = 50.$$

Integrating factor is $e^{t/50}$

$$\Rightarrow \quad \frac{d}{dt}\left[A e^{t/50}\right] = 12 e^{t/50}$$

$$\Leftrightarrow \quad A \cdot e^{t/50} = 600 e^{t/50} + c.$$

$$\Leftrightarrow \quad A = 600 + c e^{-t/50}.$$

$\boxed{t=0}$ $$50 = 600 + c.$$

$$\Leftrightarrow \quad c = -550$$

So $$A(t) = 600 - 550 e^{-t/50}.$$

Let $t \to \infty$, $A(t) \to 600$. $\square$

<u>It is as we expect</u>, as after a long time the concentration of solution in the tank is equal to the concentration of inflow.

$\Rightarrow$ we have total $(200 \text{ gal})(3 \text{ lb/gal}) = 600 \text{ lb}.$

$\boxed{\text{Example 6}}$ We now assume that the mixture was pumped out at a slower rate of 3 gal/min. Then the liquid will accumulate in the tank at rate

$$(4-3)\ \text{gal/min} = 1\ \text{gal/min}$$

new $R_{out} = \left(\frac{A}{200+t}\right) \cdot 3 = \frac{3A}{200+t}$ (lb/min)

$\Rightarrow$ our eq becomes

$$\frac{dA}{dt} = 12 - \frac{3A}{200+t}$$

$$\Leftrightarrow \quad \frac{dA}{dt} + \frac{3}{200+t} A = 12$$

integrating factor is $e^{\int (3/200+t)\, dt} = (200+t)^3$.

$$\Leftrightarrow \quad \frac{d}{dt}\left((200+t)^3 A\right) = 12(200+t)^3$$

$$\Leftrightarrow \quad (200+t)^3 A = 3(200+t)^4 + c$$

$$\Leftrightarrow \quad A = 3(200+t) + c/(200+t)^3$$

$\boxed{t=0}$: $\quad 50 = 600 + \frac{c}{200^3}$

$$C = -\frac{8 \cdot 10^6}{550}$$

$$\Rightarrow A(t) = 600 + 3t - \frac{\frac{800 \cdot 10^6}{550}}{(200+t)^3} \to \infty$$

as $t \to \infty$.

$\square$