3.1. **LINEAR MODELS**

**GROW AND DECAY**

\[ \frac{dx}{dt} = Kx, \quad x(t_0) = x_0 \quad \text{(1)} \]

**Example 1.**

A culture initially has \( P_0 \) number of bacteria. At \( t = 1 \)h the number of bacteria is measured to be \( \frac{3}{2} P_0 \). If the rate of growth is proportional to the number of bacteria \( P(t) \) present at time \( t \), determine the time necessary for the number of bacteria to triple.

**Solution:**

\[ \frac{dp}{dt} - Kp = 0 \]

Integrating factor \( \mu(t) = e^{\int -K \, dt} = e^{-Kt} \).

\[ d[\mu p] = 0 \]

\[ e^{-Kt}p = c \Rightarrow p = c \cdot e^{Kt} \]

We need to find \( c \) and \( K \).
\[ t = 0 : \quad p_0 = c \]
\[ t = 1 : \quad \frac{3}{2} p_0 = p_0 \cdot e^k \]

\[ e^k = \frac{3}{2} \quad \Rightarrow \quad k = \ln \frac{3}{2} \approx 0.4055 \]

We need to find \( t \) such that

\[ P(t) = p_0 \cdot e^{k\cdot t} = 3p_0 \]

\[ e^{(\ln \frac{3}{2})t} = 3 \]

\[ \ln \frac{3}{2} t = \ln 3 \]

\[ t = \frac{\ln 3}{\ln \frac{3}{2}} \approx 2.71 \text{ (h).} \]

**Half-Life.** In physics the **half-life** is a measure of the stability of a radioactive substance. This is simply the time it takes for \( \frac{1}{2} \) of the atoms in the initial amount \( A_0 \) to disintegrate, or transmute, into the atoms of another element.

**Example.** A breeder reactor converts relatively stable uranium-238 into isotope plutonium-239. After 15 years it is determined that 0.0452% of the initial amount \( A_0 \) of plutonium has disintegrated. Find the half-life of this isotope if the rate
of disintegration is proportional to the amount remaining.

**Solution:**

\[
\frac{dA}{dt} = KA, \quad A(0) = A_0
\]

\[A(t) = A_0 e^{Kt}.\]

If 0.043\% of the atoms of \(A_0\) have disintegrated, then 99.957\% of the substance remains. We can find \(K\):

\[\Rightarrow 0.99957A_0 = A_0 e^{15K}\]

\[\Rightarrow K = \frac{1}{15} \ln 0.99957 \approx -0.00002867.\]

Find the half-life \(t\):

\[
\frac{1}{2} A_0 = A_0 e^{-0.00002867 t}
\]

\[\Rightarrow \frac{1}{2} = e^{-0.00002867 t}\]

\[\Rightarrow t = \frac{\ln \frac{1}{2}}{-0.00002867} = \frac{\ln 2}{0.00002867} \approx 24,180 \text{ yr.}\]

Reading exercise: (Example about Age of Fossil)
NEWTON'S LAW OF COOLING/WARMING

\[ \frac{dT}{dt} = K(T - T_m) \]

where \( T_m \) is the temperature of the medium around the object.

**Example 4.** When a cake removed from an oven, its temperature is measured at 300°F. Three minutes later the temperature is 200°F. How long will it take for the cake to cool off to 100°F? The room temperature is 70°F.

**Solution:**

\[ \frac{dT}{dt} = K(T - T_m), \quad T(0) = 300 \]

\[ \Rightarrow \frac{dT}{(T - 70)} = Kdt \quad \text{(separable eq.)} \]

\[ \Rightarrow \ln|T - 70| = kt + C_1 \]

\[ \Rightarrow T - 70 = Ce^kt \]

\[ \Rightarrow t = C\cdot e^{kt} + 70 \]

\[ t = 0 \]

\[ 300 = C + 70 \Rightarrow C = 230 \]

\[ t = 3 \]

\[ 200 = 230\cdot e^{3k} \]

\[ \Rightarrow e^{3k} = \frac{130}{230} \Rightarrow k = \frac{1}{3} \ln \left( \frac{130}{230} \right) \approx -0.19018 \]
Find $t$

$$100 = 70 + 230 e^{-0.19018t}$$

(⇒) $e^{-0.19018t} = \frac{30}{230}$

(⇒) $-0.19018t = \ln \frac{30}{230}$

(⇒) $t = \frac{\ln \frac{230}{3}}{0.19018} \approx 10.7$ (mins)

**MIXTURE**:

Example: A large tank holds 200 gals of brine solution. Brine is being pumped into the tank at the rate 4 gal/min. The concentration of the salt in the inflow is 3 lb/gal. The mixture is also pumped out at the rate 4 gal/min.

$$R_{in} = (4 \text{ gal/min}) (3 \text{ lb/gal}) = 12 \text{ lb/min}$$

$$R_{out} = \left( \frac{A}{200} \text{ lb/gal} \right) (4 \text{ gal/min})$$

$$= \frac{A}{50} \text{ lb/min}$$

Assume that 50 pounds of salt were dissolved initially in the 200 gallons, how much salt in the tank after a long time?
Solution.

Recall: \( \frac{dA}{dt} = R \text{in} - R \text{out} \)

\( \frac{dA}{dt} = 12 - \frac{A}{50} \)

\( \Rightarrow \frac{dA}{dt} + \frac{A}{50} = 12, \quad A(0) = 50. \)

Integrating factor is \( e^{\frac{t}{50}} \)

\( \Rightarrow d \left[ A e^{\frac{t}{50}} \right] = 12 e^{\frac{t}{50}} \)

\( \Rightarrow A e^{\frac{t}{50}} = 600 e^{\frac{t}{50}} + C. \)

\( \Rightarrow A = 600 + C e^{-\frac{t}{50}}. \)

\[ t = 0 \]

\( 50 = 600 + C. \)

\( \Rightarrow C = -550 \)

So \( A(t) = 600 - 550 e^{-\frac{t}{50}}. \)

Let \( t \to \infty \), \( A(t) \to 600. \)

It is as we expect, as after a long time the concentration of solution in the tank is equal to the concentration of inflow.

So we have total \( (200 \text{ gal}) (3.15 \text{ lb/gal}) = 630 \text{ lb}. \)
Example 6: We now assume that the mixture was pumped out at a slower rate of 3 gal/min. Then the liquid will accumulate in the tank at rate

\[(4 - 3) \text{ gal/min} = 1 \text{ gal/min}\]

new \( R_{out} = \left( \frac{A}{200 + t} \right) \cdot 3 = \frac{3A}{200 + t} \text{ (lb/min)} \]

\[ \Rightarrow \text{our eq becomes} \]

\[ \frac{dA}{dt} = 12 - \frac{3A}{200 + t} \]

\[ \Rightarrow \frac{dA}{dt} + \frac{3}{200 + t} A = 12 \]

integrating factor is \( \exp \left( \int \frac{3}{200 + t} \, dt \right) = (200 + t)^3 \).

\[ \Rightarrow \frac{d}{dt} \left( (200 + t)^3 A \right) = 12 (200 + t)^3 \]

\[ \Rightarrow (200 + t)^3 A = 3 (200 + t)^4 + C \]

\[ \Rightarrow A = \frac{3(200 + t)}{(200 + t)^3} + \frac{C}{(200 + t)^3} \]

\[ \| t = 0 \| : \quad 50 = 600 + \frac{C}{200} \]

\[ \Rightarrow C = -8 \times 10^6 \]

\[ \Rightarrow A(t) = 600 + 3t - \frac{8 \times 10^6}{(200 + t)^3} \to \infty \text{ as } t \to \infty. \]
SERIES CIRCUITS.

Consider an LR-series circuit:

\[
\begin{array}{c}
\text{E} \\
L \\
R \\
\end{array}
\]

By Kirchhoff’s second law we have

\[L \frac{di}{dt} + Ri = E(t) \quad (7)\]

Similar an RC-series circuit give

\[R \frac{dq}{dt} + \frac{1}{C} q = E(t) \quad (8)\]

**Example 7.** A 12-volt battery is connected to a series circuit in which the inductance is \(\frac{1}{2}\) henry and the resistance is 10 ohms. Determine the current \(i\) if the initial current is 0.

**Solution:**

\[\frac{1}{2} \frac{di}{dt} + 10i = 12, \quad i(0) = 0\]
Integrating factor $\mu(t) = e^{20t}$

$$\frac{d}{dt} \left[ e^{20t} i \right] = 24 e^{20t}$$

$$\Rightarrow i(t) = \frac{6}{5} + C e^{-20t}$$

$$\begin{array}{l}
\text{\underline{$i(0)$}} \\
\quad C = -\frac{6}{5}
\end{array}$$

$$\Rightarrow i(t) = \frac{6}{5} - \frac{6}{5} e^{-20t}.$$