

Section 2.3. LINEAR EQUATIONS

Defn: A first-order ODE of the form

$$a_1(x) \frac{dy}{dx} + a_0(x) y = g(x) \quad (1)$$

is said to be a **linear equation** in the variable y .

By dividing both sides of (1) by $a_1(x)$, we get a more useful form, the **standard form**, of the linear equation

$$\frac{dy}{dx} + P(x)y = f(x) \quad (2).$$

for example:

$$\frac{dy}{dx} + 2xy = 0$$

and

$$\frac{dy}{dx} = y + 5 \quad \text{are linear and separable.}$$

however

$$\frac{dy}{dx} + y = x \quad \text{is linear, but non separable.}$$

METHOD OF SOLUTION

We want to transform the LHS of (2) into the derivative of a product by multiplying both sides of (2) by a special function $\mu(x)$:

$$\begin{aligned} \frac{dy}{dx} [\mu(x)y] &= \text{LHS} \times \mu(x) \\ &= \left(\frac{dy}{dx} + P(x) \right) \mu(x) \end{aligned}$$

Assume we can find $\mu(x)$ as above, then

$$\frac{dy}{dx} [\mu(x)y] = \mu(x)f(x) \quad (\text{LHS} = \text{RHS})$$

$$\Rightarrow \mu(x)y = \int \mu(x)f(x)$$

$$\Rightarrow y = \mu(x)^{-1} \int \mu(x)f(x) + C \mu(x)^{-1}$$

is a solution.

How to find such function $\mu(x)$?

$$\frac{d}{dx} [\mu(x)y] = LHS \times \mu(x)$$

$$\mu \frac{dy}{dx} + \frac{d\mu}{dx} y = \mu \frac{dy}{dx} + \mu P y$$

We want to have

$$\frac{d\mu}{dx} = \mu P$$

$$\Leftrightarrow \frac{d\mu}{\mu} = P dx$$

$$\Leftrightarrow \ln |\mu(x)| = \int P(x) dx + C_1$$

$$\rightarrow \mu = C_2 e^{\int P(x) dx}$$

The function $\mu(x) = e^{\int P(x) dx}$ is called the **integrating factor** of eq. (2).

Example: Solve $\frac{dy}{dx} - 3y = 0$

We have $P(x) = -3$, thus the integrating factor is

$$\mu(x) = e^{\int (-3) dx} = e^{-3x}$$

Multiply both sides by $\mu(x)$ we get

$$e^{-3x} \frac{dy}{dx} - 3e^{-3x} y = e^{-3x} \cdot 0$$

$$\Leftrightarrow \frac{d}{dx} [e^{-3x} y] = 0$$

$$\Leftrightarrow e^{-3x} y = c \quad \text{or} \quad y = c \cdot e^{3x} \quad \square$$

Example 2: Solve $\frac{dy}{dx} - 3y = 6$

$$\Leftrightarrow \frac{d}{dx} [e^{-3x} y] = 6 \cdot e^{-3x}$$

$$\Leftrightarrow e^{-3x} y = 6 \int e^{-3x} dx$$

$$\Leftrightarrow e^{-3x} y = 6 \left(\frac{e^{-3x}}{-3} \right) + c$$

$$\boxed{y = -2 + c e^{3x}}$$

\square

Note that if P and f are continuous in an interval I , then

$$y = M^{-1} \int M f dx + c M^{-1}$$

$$(4) \quad y = e^{-\int P dx} \int e^{\int P dx} f(x) dx + c \cdot e^{-\int P dx}$$

is a solution of (2), and any solution of (2) defined on I is a member of this family. We call (4) the **general solution** of the equation on I .

In this case the interval of existence and uniqueness I_0 is the entire the interval I .

Example 3: Find the general solution

$$x \frac{dy}{dx} - 4y = x^6 e^x$$

$$\Leftrightarrow \frac{dy}{dx} - \frac{4}{x} y = x^5 e^x \quad (7)$$

$$\Rightarrow P(x) = -\frac{4}{x}, \quad f(x) = x^5 e^x$$

We have P and f are continuous on $I = (0, \infty)$.
Hence the integrating factor

$$\begin{aligned} \mu(x) &= e^{\int P(x) dx} = e^{-4 \int \frac{1}{x} dx} \\ &= e^{-4 \ln x} \quad (\ln x = \ln|x|) \\ &= x^{-4} \quad (\text{since } x > 0) \end{aligned}$$

Thus multiply both sides of (7) by $\mu(x) = x^{-4}$

$$x^{-4} \frac{dy}{dx} - \frac{4}{x^5} y = x e^x$$

$$\frac{d}{dx} [x^{-4} y] = x e^x$$

$$\begin{aligned} x^{-4} y &= \int x e^x dx \\ &= x e^x - e^x + C \end{aligned}$$

$\Leftrightarrow y = x^5 e^x - x^4 e^x + C x^4$
is the general solution defined on $(0, \infty)$.

Example 4. Find the general solution

$$(x^2 - 9) \frac{dy}{dx} + xy = 0$$

$$\Leftrightarrow \frac{dy}{dx} + \frac{x}{x^2 - 9} y = 0$$

$$P(x) = \frac{x}{x^2 - 9}, \quad f(x) = 0$$

We have P and f are continuous on $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$.

Solve on $(3, \infty)$:

$$\begin{aligned} \mu(x) &= e^{\int \frac{x}{x^2 - 9} dx} = e^{\frac{1}{2} \int \frac{dx^2}{x^2 - 9}} \\ &= e^{\frac{1}{2} \ln |x^2 - 9|} \\ &= \sqrt{x^2 - 9} \quad (x^2 > 9 \text{ as } x > 3). \end{aligned}$$

Multiply both sides of the equation by $\mu(x)$:

$$\sqrt{x^2 - 9} \frac{dy}{dx} + \frac{x \sqrt{x^2 - 9}}{x^2 - 9} y = 0$$

$$\Leftrightarrow \frac{d}{dx} [\sqrt{x^2 - 9} y] = 0$$

$$\Leftrightarrow \sqrt{x^2 - 9} y = C \quad \Leftrightarrow y = \frac{C}{\sqrt{x^2 - 9}}$$

We have the same solution on $(-\infty, -3)$.
However, $x \in (-3, 3)$ we get $y = \frac{C}{\sqrt{9 - x^2}}$.

Example: Solve an IVP

$$\frac{dy}{dx} + y = x, \quad y(0) = 4$$

$$\mu(x) = e^{\int dx} = e^x \quad (P=1, f(x)=x).$$

$$\Rightarrow e^x \frac{dy}{dx} + e^x y = x e^x$$

$$\Rightarrow \frac{d}{dx} (e^x y) = x e^x$$

$$\Rightarrow e^x y = x e^x - e^x + C$$

$$\Rightarrow y = x - 1 + C/e^x.$$

Substitute $x=0, y=0$, we get $C=5$.

$$\Rightarrow y = x - 1 + 5e^{-x}.$$

PIECEWISE-LINEAR D.E.

$$\frac{dy}{dx} + P(x)y = f(x)$$

where $f(x)$ is piecewise-defined function.

Example 6: Solve $\frac{dy}{dx} + y = f(x), y(0) = 0$

$$\text{where } f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & x > 1. \end{cases}$$

Consider $0 \leq x \leq 1$. Our eq becomes

$$\frac{dy}{dx} + y = 1$$

$$(P(x)=1, \text{ so } \mu(x) = e^{\int P dx} = e^x)$$

$$\Leftrightarrow \frac{d}{dx} [e^x y] = e^x$$

$$\Leftrightarrow e^x y = e^x + c$$

$$\Leftrightarrow y = 1 + c \cdot e^{-x}$$

Since the initial point $x_0 = 0$ in this interval,

$$0 = 1 + c \cdot e^{-0} \Leftrightarrow c = -1$$

$$\Rightarrow \text{solution is } y = 1 - e^{-x}$$

Consider $x > 1$. Our eq is

$$\frac{dy}{dx} + y = 0$$

$$\Leftrightarrow \frac{d}{dx} [e^x y] = 0$$

$$\Leftrightarrow e^x y = c$$

$$\Leftrightarrow y = c \cdot e^{-x}$$

Thus our solution is

$$y(x) = \begin{cases} 1 - e^{-x} & 0 \leq x \leq 1; \\ c \cdot e^{-x} & x > 1. \end{cases}$$

Reading exercise : ERROR FUNCTION. (pages 60-61)