Section 2.2. Seperable Equations

We will study of solution methods withe the simplest differential equations: first-order equations with separable variables.

SOLUTION BY INTEGRATION.

Consider an ODE

$$\frac{dy}{dx} = f(x,y) = g(x) \qquad (1)$$

(i.e. f does not depend on x). One can solve this equation by integration. If $g(x)$ is a continuous function, then take integration both sides of (1) gives

$$y = \int g(x)\,dx = G(x) + c$$

where $G(x)$ is an antiderivative (indefinite integral) of $g(x)$.

For example

① $\frac{dy}{dx} = \sin x$

has solution

$$y = \int \sin x \, dx = -\cos x + c.$$

② $\frac{dy}{dx} = 1 + e^x$ has solution $y = \int (1+e^x)\,dx$

$$= e^x + x + c$$

Definition: A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)\,h(y)$$

is said to be **seprable**, or have **separable variables**.

For example

$$\frac{dy}{dx} = y^4 e^{x^2+4y}$$

is separable, while

$$\frac{dy}{dx} = \sqrt{y} + \ln x$$

is nonseparable.

Observe that by dividing by function $h(y)$, we have

$$\frac{dy}{dx} = g(x)\,h(y)$$

$$\Leftrightarrow \quad \frac{1}{h(y)} \cdot \frac{dy}{dx} = g(x).$$

$$\Leftrightarrow \quad p(y)\,\frac{dy}{dx} = g(x) \qquad \left(p(y) = \frac{1}{h(y)}\right). \quad (2)$$

If $y = \varphi(x)$ is a solution of (2), then

$$p(\varphi(x))\,\varphi'(x) = g(x)$$

$$\Leftrightarrow \quad \int p(\varphi(x))\,\varphi'(x)\,dx = \int g(x)\,dx \qquad (3)$$

But $\frac{dy}{dx} = \varphi'(x)$, so (3) is equivalent to

$$\int p(y)\,dy = \int g(x)\,dx \quad \text{or}$$

$$P(y) = G(x) + C \qquad (4)$$

wher $P(y)$ is the antiderivative of $p(y) = \frac{1}{h(y)}$ and $G(x)$ is the antiderivative of $g(x)$.

Example 1 Solve

$$(1+x)\,dy - y\,dx = 0$$

$$\Leftrightarrow \quad \frac{dy}{y} = \frac{dx}{1+x}$$

$$\Rightarrow \quad \int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\Leftrightarrow \quad \ln|y| = \ln|1+x| + C_1$$

$$\Leftrightarrow \quad |y| = |1+x|\,e^{C_1}$$

$$\Leftrightarrow \quad y = \pm e^{C_1}(1+x)$$

$$= c\,(1+x)$$

Example 2. Solve the IVP

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(2) = -1.$$

$$\Leftrightarrow \quad y\,dy = -x\,dx$$

$$\Rightarrow \quad \int y\,dy = -\int x\,dx$$

$$\Leftrightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + c_1$$

$\Leftrightarrow \quad x^2 + y^2 = c^2$ is an implicit solution

($\Leftrightarrow \quad y = \pm\sqrt{c^2 - x^2}$ is an explicit solution)

Let $x = 1$, $y = -2$, so $1 + 4 = c^2$

$\Leftrightarrow \quad c = \sqrt{5}$.

Since $y < 0$, our solution is $y = -\sqrt{5 - x^2}$

$(-\sqrt{5} < x < \sqrt{5})$.

Example 3. Solve

$$\frac{dy}{dx} = y^2 - 9 \qquad (5)$$

$$\frac{dy}{y^2 - 9} = dx \Leftrightarrow \left[\frac{1}{6} \cdot \frac{1}{y-3} - \frac{1}{6} \cdot \frac{1}{y+3}\right] dy = dx$$

$$\Leftrightarrow \frac{1}{6}\int \frac{dy}{y-3} - \frac{1}{6}\int \frac{dy}{y+3} = dx$$

$$\Leftrightarrow \frac{1}{6} \ln|y-3| - \frac{1}{6} \ln|y+3| = x + c_1$$

$$\Leftrightarrow \quad \ln\left|\frac{y-3}{y+3}\right| = 6x + c_2$$

$$\Leftrightarrow \quad \frac{y-3}{y+3} = \pm e^{6x + c_2} = c\, e^{6x}$$

Solving for y: $\quad y = 3\,\dfrac{1 + c e^{6x}}{1 - c e^{6x}}$. $\qquad (6)$

However, we can write our eq. as

$$\frac{dy}{dx} = (y-3)(y+3)$$

This is an autonomous DE, it has two constant solutions $y=-3$ and $y=3$.
The solution $y=3$ is a member of the family of solution (6), when $c=0$. However, $y=-3$ is a singular solution: $y=-3$ can not be a member of (6).

Example 4. Solve the IVP

$$(e^{2y}-y)\cos x \frac{dy}{dx} = e^{y}\sin 2x \quad , \quad y(0)=0$$

$$\Leftrightarrow \quad \frac{e^{2y}-y}{e^{y}}\, dy = \frac{\sin 2x}{\cos x}\, dx$$

$$\Leftrightarrow \quad (e^{y} - ye^{-y})\,dy = 2\sin x \, dx$$

$$\int (e^{y} - ye^{-y})\,dy = 2\int \sin x\, dx$$

$$e^{y} + ye^{-y} + e^{-y} = -2\cos x + c$$

Substitute $y=0$, $x=0$, we get $c=4$.
$\Rightarrow$ The solution is

$$e^{y} + ye^{-y} + e^{-y} = 4 - 2\cos x .$$

AN INTEGRAL-DEFINED FUNCTION.

Assume that we want to solve the IVP

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0.$$

It is easy to verify that it has a solution given by

$$\boxed{y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt.}$$

Example 5. Solve

$$\frac{dy}{dx} = e^{-x^2}, \quad y(4) = 6$$

$$y(x) = y(4) + \int_4^x e^{-t^2}\,dt$$

$$= 6 + \int_4^x e^{-t^2}\,dt$$