Section 2.1. Solution Curve Without a Solution

1. DIRECTED FIELDS.

As we have seen in Section 1.2 that whenever \( f(x,y) \) and \( \frac{df}{dy} \) satisfy certain continuity conditions, qualitative question about existence and uniqueness of solution can be answered.

In this section we will see that other qualitative question about properties of solutions:
- How does a solution behave near a certain point?
- How does a solution behave asymptotically?

Recall: A derivative \( \frac{dy}{dx} \) of a differentiable function \( y = y(x) \) gives slopes of tangent lines at points on it graph.

Consider a first-order ODE (in the normal form)

\[
\frac{dy}{dx} = f(x,y) \quad (1)
\]
The function \( f(x,y) \) is usually called the **slope function** or the **rate function**.

If we evaluate \( f \) over a rectangular grid of points in the \( xy \)-plane, and draw a small arrow, called a **lineal element**, at each point \((x,y)\) of the grid with slope \( f(x,y) \). (A line element is usually oriented by increasing direction of \( x \).) Then the collection of all these line elements is called a **direction field** (or a **slope field**) of the ODE \( \frac{dy}{dx} = f(x,y) \).

Visually, the direction field suggests the shape of a family of solution curves of the diff. eq., and consequently, it may be possible to see at glance certain qualitative aspects of the solutions - regions in the plane, e.g., in which a solution has unusual behavior.

A single solution curve passes through a direction field must follow the flow pattern of the field: it tangent to a lineal element when it intersects a point of the grid.

In general, a finer grid gives a better approx of the solution curve.

**Example:** The direction field of the equation

\[
\frac{dy}{dx} = 0.2xy
\]

can tell us when \(|f(x,y)|\) increases as \(|x|\) and \(y\) increase, when \(f(x,y)\) is positive/negative.
This suggest the behavior of a solution curve.

Example: The direction field of the diff. eq.

\[ \frac{dy}{dx} = \sin (x + y) \]

Example: The direction field of

\[ \frac{dy}{dx} = \sin y \]

2. **Autonomous First-Order ODEs.**

An ODE in which the independent variable does not appear explicitly is said to be autonomous. If \( x \) denotes the independent variable, then an autonomous ODE has form

\[ F(y, y', y'', \ldots) = 0. \]

In particular, a first-order autonomous ODE has form

\[ F(y, y') = 0 \]

or

\[ \frac{dy}{dx} = f(y). \quad (2) \]

For example, \[ \frac{dy}{dx} = \sin y \] is autonomous

while \[ \frac{dy}{dx} = 4x y^{1/2} \] is nonautonomous.
All the equations in section 1.3 are autonomous.

\[
\frac{dP}{dt} = kP, \quad \frac{dx}{dt} = kx(\mu+1-x), \quad \frac{dT}{dt} = k(T-T_m)
\]

\[
\frac{dA}{dt} = 12 - \frac{A}{50}, \ldots
\]

**Critical Points.** The zeros of a first-order autonomous DE (2) are very important. We say that a real number \(c\) is a critical point of the eq (2) if it is a zero of \(f\). (\(f(c) = 0\)).

A critical point is also called an equilibrium point or a stationary point. Observe that when we plug \(y(x) = c\) into (2) then both sides of the equation are zero. This means:

"If \(c\) is a critical point of (2), then \(y(x) = c\) is a constant solution of the autonomous DE."

The solution \(y(x) = c\) is called an equilibrium solution.

**Example:** Consider the diff. eq.

\[
\frac{dP}{dt} = P(a-bP)
\]

where \(a, b\) are positive constant.
\( \text{Interval} \quad \text{Sign of } f(p) \quad P(+) \quad \text{Arrow} \\
(\infty, 0) \quad - \quad \Rightarrow \quad \downarrow \\
(0, \frac{a}{b}) \quad + \quad \Rightarrow \quad \uparrow \\
(\frac{a}{b}, \infty) \quad - \quad \Rightarrow \quad \downarrow \\
\Rightarrow \text{Phase portrait of DE:} \\
\quad \quad \downarrow \quad \text{p-axis} \\
\quad \quad \uparrow \quad \frac{a}{b} \\
\quad \quad \downarrow \quad \frac{0}{b} \\
\quad \downarrow \quad 0 \\
\)

**SOLUTION CURVE.**

Without solving an autonomous DE, we can say pretty much about its solution curves.
- Since the function \( f \) in (2) is independent from \( x \), we can consider \( f \) defined for \( -\infty < x < \infty \) or for \( 0 < x < \infty \).
- Since \( f \) and \( f' \) are continuous functions of \( y \) on some interval I of the y-axis, Theorem 1.2.1 hold for some horizontal strip \( R \) corresponding to I, so through any point \((x_0, y_0) \in R\) there passes only one solution curve of (2).
- For sake of discussion, we assume that (2) has exactly two equilibrium \( y(x) = c_1 \) and \( y(x) = c_2 \) for \( c_1 < c_2 \). These solution curves partition
the plane into 3 parts $R_1, R_2, R_3$.

Without proof we can say the follows about nonconstant solution $y(x)$ of (2).

1. If $(x_0, y_0) \in R_i$ (i = 1, 2, 3) and $y(x)$ is a solution whose graph passes through $(x_0, y_0)$, then the graph remain in $R_i$ for all $x$.

2. By the continuity of $f$, we must have either $f(y) > 0$ or $f(y) < 0$ for all $y$ in $R_i$.

3. Since $\frac{dy}{dx} = f(y(x))$ is either positive or negative in $R_i$, $y(x)$ is strictly monotonic.

4. If $y(x)$ is bounded above by a critical point $c_i$, then the graph of $y(x)$ must approach the graph of $y(x) = c_i$ either as $x \to \infty$ or as $x \to -\infty$.

Example: Solution of $\frac{dy}{dx} = (y-1)^2$. 
ATTRACTORS AND REPPELLERS.

Suppose that \( y(x) \) is a nonconstant solution of the autonomous differential equation given in (2) and that \( c \) is a critical point of the DE.

There are 3 types of behavior that \( y(x) \) can exhibit near \( c \).

\[
\begin{align*}
\text{(a)} & & \text{(b)} & & \text{(c)} & & \text{(d)} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & \\
\bullet & & \bullet & & \bullet & & \bullet \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & & & & & \\
& & & & & & \\
\bullet & & \bullet & & \bullet & & \bullet \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & & & & & \\
& & & & & & \\
\bullet & & \bullet & & \bullet & & \bullet
\end{align*}
\]

(a) Two arrows point toward \( c \).

\( \Rightarrow \) All solutions \( y(x) \) of (2) that start from an initial point \((x_0, y_0)\) sufficiently near \( c \) exhibit the asymptotic behavior \( \lim_{x \to \infty} y(x) = c \). We say \( c \) is asymptotically stable, and \( c \) is also called an attractor.

(b) Two arrows point away \( c \).

\( \Rightarrow \) All solutions starting from \((x_0, y_0)\) will move away from \( c \) as \( x \) increases. So, \( x \) is unstable, and \( x \) is called a repeller.

(c) and (d) \( c \) is semi-stable.
AUTONOMOUS DES AND DIRECTION FIELD

⊕ All linear elements along a horizontal strip have the same slope.
⊕ The linear elements along a vertical strip do not vary.

TRANSLATION PROPERTY.

If \( y(x) \) is a solution of \( \frac{dy}{dx} = f(y) \), then \( y_1(x) = y(x-k) \), \( k \) is a constant, is also a solution.

Example: \( y(x) = e^x \) is a solution of \( y' = y \).

Of course \( y(x) = e^x + 6 \) is also a solution.