7.2. Inverse Transforms and Transforms of Derivatives.

1. INVERSE TRANSFORMS

If $F(s) = \mathcal{L}\{f(t)\}$, then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

For example:

$$\mathcal{L}\{1\} = \frac{1}{s} \iff 1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2} \iff t = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3} \iff e^{-3t} = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}.$$

Theorem: (Some Inverse Transform)

(a) $1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$

(b) $t^n = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, n = 1, 2, 3 \ldots,$

(c) $e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$

(d) $\sin kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$

(e) $\cos kt = \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathcal{L}^{-1}\left\{\dfrac{k}{s^2-k^2}\right\}$

(g) $\cosh kt = \mathcal{L}^{-1}\left\{\dfrac{s}{s^2-k^2}\right\}$.

Example: Evaluate (a) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^5}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+7}\right\}$

Solution: a) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^5}\right\} = \dfrac{1}{4!}\mathcal{L}^{-1}\left\{\dfrac{4!}{s^5}\right\} = \dfrac{1}{4!}t^4$.

b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+7}\right\} = \dfrac{1}{\sqrt{7}}\mathcal{L}^{-1}\left\{\dfrac{\sqrt{7}}{s^2+7}\right\}$

$= \dfrac{1}{\sqrt{7}}\sin(\sqrt{7}\,t)$.

$\mathcal{L}^{-1}$ IS A LINEAR TRANSFORM:

(1) $\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}$

Example: Evaluate $\mathcal{L}^{-1}\left\{\dfrac{-2s+6}{s^2+4}\right\}$.

Solution:

$$\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} = -2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= -2\cos 2t + \frac{6}{2}\sin 2t$$

$\square$

Example: Evaluate $\mathcal{L}^{-1}\left\{\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$

Solution: There exist unique real constant A, B, C so that

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$\Leftrightarrow \quad s^2+6s+9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2).$$

Take $s=1$: Terms with B and C are zero

$$\Leftrightarrow \quad 16 = A(1-2)(1+4)$$

$$\Leftrightarrow \quad A = -\frac{16}{5}$$

Take $s=2$: $25 = B(1)(6)$

$$\Leftrightarrow \quad B = \frac{25}{6}$$

Take $s=-4$: $1 = C(-5)(-6)$

$$\Leftrightarrow \quad C = \frac{1}{30}$$

Thus: $$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{-16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}. \quad \square$$

2. TRANSFORMS OF DERIVATIVES.

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\}.$$

given f' is continuous for $t \geq 0$, and $e^{-st}f(t) \to 0$ as $t \to \infty$.

Similarly:

$$\mathcal{L}\{f''(t)\} = -f'(0) + s\mathcal{L}\{f'(t)\}$$
$$= s^2 F(s) - sf(0) - f'(0)$$

In like manner, it can be shown that

$$\mathcal{L}\{f'''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0).$$

Theorem: If $f, f', \dots, f^{(n-1)}$ are continuous on $[0,\infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots$$
$$- \dots - f^{(n-1)}(0),$$

where $F(s) = \mathcal{L}\{f(t)\}$.

SOLVING LINEAR ODE's

Assume that we want to solve an IVP

$$(2) \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = g(t)$$

$$y(0) = y_0, \; y'(0) = y_1, \dots, \; y^{(n-1)}(0) = y_{n-1}$$

Apply the Laplace to eq (2):

$$(3) \quad a_n \mathcal{L}\left\{\frac{d^n y}{dt}\right\} + a_{n-1} \mathcal{L}\left\{\frac{d^{n-1} y}{dt}\right\} + \dots + a_0 \mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

Apply our Theorem:

$$a_n [s^n \mathbf{Y}(s) - s^{n-1} y(0) - \dots - y^{(n-1)}(0)]$$
$$+ a_{n-1} [s^{n-1} \mathbf{Y}(s) - s^{n-2} y(0) - \dots - y^{(n-2)}(0)] + \dots \quad (4)$$
$$+ a_0 \mathbf{Y}(s) = G(s)$$

where $Y(s) = \mathcal{L}\{y(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$.

Eq (4) can be written as:

$$P(s) Y(s) = Q(s) + G(s)$$

$\Rightarrow$

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)} \qquad (5)$$

Here $P(s) = a_n s^n + \dots + a_0$, $Q(s)$ is a polynomial in s of deg $\leq n-1$.

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

Example: Solve the IVP

$$\frac{dy}{dt} + 3y = 13 \sin 2t \quad , \quad y(0) = 6.$$

Solution: Apply Laplace transform to the eq:

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 3\mathcal{L}\{y\} = 13\mathcal{L}\{\sin 2t\}$$

We have:

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) = sY(s) - 6$$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

$$\Rightarrow \quad sY(s) - 6 + 3Y(s) = \frac{26}{s^2+4}$$

$$\Leftrightarrow \quad (s+3)\,Y(s) = 6 + \frac{26}{s^2+4}$$

$$\Leftrightarrow \quad Y(s) = \frac{6}{s+3} + \frac{26}{(s^2+4)(s+3)}$$

$$= \frac{6s^2+50}{(s+3)(s^2+4)}.$$

Write:

$$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

$$\Leftrightarrow \quad 6s^2+50 = A(s^2+4) + (Bs+C)(s+3).$$

Let $s = -3$: $A = 8$

Compare coeffs: $A + B = 6$, $0 = 3B + C$

$\Rightarrow$ $B = -2$, $C = 6$.

Thus

$$Y(s) = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{8}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}$$

$$= 8\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= 8e^{-3t} - 2\cos 2t + 3\sin 2t. \quad \square$$

<u>Example</u>: Solve an IVP

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5.$$

<u>Solution</u>:

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-4t}\}$$

$$\Leftrightarrow s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$\Leftrightarrow Y(s)(s^2 - 3s + 2) = s + 2 + \frac{1}{s+4}$$

$$\Leftrightarrow Y(s) = \frac{s+2}{s^2-3s+2} + \frac{1}{(s^2-3s+2)(s+4)}$$

$$= \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$$

(By previous example)

$$= -\frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}$$

$$\Leftrightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}. \quad \square$$