# 7.1. DEFINITION OF THE LAPLACE TRANSFORM.

We can view defferentiation and integration as two **transforms.** They transform a function into another function.

For example

$$\frac{d}{dx} x^n = n x^{n-1} \quad , \quad \frac{d}{dx} e^x = e^x$$

$$\int x^4 \, dx = \frac{1}{5} x^5 + c \quad , \quad \int \cos x \, dx = \sin x + c .$$

Moreover, these two transforms are **linear** :

$$\frac{d}{dx} [\alpha f(x) + \beta g(x)] = \alpha \frac{d}{dx} f(x) + \beta \frac{d}{dx} g(x)$$

$$\int [\alpha f(x) + \beta g(x)] \, dx = \alpha \int f(x) \, dx + \beta \int g(x) \, dx .$$

In this seetion, we will examine a special type of integral transform called the **Laplace transform.** In addition to possessing the linearity property the Laplace transform has many other interesting properties that make it very useful in solving linear IVP's.

# INTEGRAL TRANSFORM

If $f(x,y)$ is a function of two variables, then a definite integral of $f$ with respect to one of the variables leads to a function of the other variable.

For example, $\int_1^2 2xy^2\,dx = 3y^2$.

Similarly, a definite integral such that $\int_a^b K(s,t)f(t)\,dt$ transforms a function $f$ of the variable $t$ into a function $F$ of the variable $s$. In particular, we are interested in the following **integral transform**:

$$\int_0^\infty K(s,t)f(t)\,dt = \lim_{b\to\infty}\int_0^b K(s,t)f(t)\,dt. \qquad (1)$$

If the limit in (1) exists, we say that the integral is **convergent**, and is **divergent** otherwise.

## DEFINITION

The function $K(s,t)$ in (1) is called the **kernel** of the transform. The choice $K(s,t) = e^{-st}$ as a kernel gives us an especially important transform:

**Defn. (Laplace Transform)** Let $f$ be a function defined for $t \geq 0$. Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \qquad (2)$$

is said to be the **Laplace transform** of $f$, provided that the integral converges.

Example 1. Evaluate $\mathcal{L}\{1\}$.

Solution: 
$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\,dt$$
$$= \lim_{b\to\infty} \left.\frac{-e^{-st}}{s}\right|_0^b = \lim_{b\to\infty}\frac{1}{s}\left(-e^{-sb}+1\right)$$
$$= \frac{1}{s}$$
provided that $s>0$. The integral diverges when $s\le 0$. $\square$

Example 2. Evaluate $\mathcal{L}\{t\}$.

Solution: $\mathcal{L}\{t\} = \int_0^\infty e^{-st}\,t\,dt$. Integrating by parts and using $\lim_{t\to\infty} t\,e^{-st} = 0$, for $s>0$, we get
$$\mathcal{L}\{t\} = \int_0^\infty \underbrace{t}_{u}\ \underbrace{e^{-st}\,dt}_{dv}$$
$$= \underbrace{\left.\frac{-te^{-st}}{s}\right|_0^\infty}_{uv} - \int_0^\infty \underbrace{\frac{-e^{-st}}{s}}_{v}\ \underbrace{dt}_{du}$$
$$= \lim_{t\to\infty}\left(-te^{-st}-0\right) + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$
$$= 0 + \frac{1}{s}\left(\frac{1}{s}\right) \qquad \text{(by Example 1)}$$
$$= \frac{1}{s^2} \qquad \square$$

**Example 3:** Evaluate $\mathcal{L}\{e^{at}\}$ $(a \neq 0)$.

**Solution:**

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left. \frac{-e^{-(s-a)t}}{s-a} \right|_0^\infty$$

$$= \frac{1}{s-a}$$

given that $s > a$. For $s \leq a$, the integral diverges. $\square$

**Example 4:** Evaluate $\mathcal{L}\{\sin 2t\}$.

**Solution**

$$\mathcal{L}\{\sin 2t\} = \int_0^\infty e^{-st} \sin 2t\, dt$$

$$= \int_0^\infty \underbrace{\sin 2t}_{u}\ \underbrace{e^{-st} dt}_{dv}$$

$\left(\lim_{t\to\infty} e^{-st} \sin 2t = 0 \text{ for } s>0\right)$

$$= \left. \frac{-e^{-st} \sin 2t}{s} \right|_0^\infty + \frac{2}{s} \int_0^\infty e^{-st} \cos 2t\, dt$$

$$= \frac{2}{s} \int_0^\infty e^{-st} \cos 2t\, dt, \quad s>0$$

$$= \frac{2}{s} \left[ \left. \frac{-e^{-st} \cos 2t}{s} \right|_0^\infty - \frac{2}{s} \int_0^\infty e^{-st} \sin 2t\, dt \right]$$

$\left(\lim_{t\to\infty} e^{-st} \cos 2t = 0, \ s>0\right)$

$$= \frac{2}{s^2} - \frac{4}{s^2} \mathcal{L}\{\sin 2t\}$$

Thus $\mathcal{L}\{2t\} = \dfrac{2}{s^2+4}, \quad s>0$ $\square$

# $\mathcal{L}$ IS A LINEAR TRANSFORM.

For a linear combination of functions $f, g$, we can write:

$$\int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]\,dt = \alpha \int_0^\infty e^{-st} f(t)\,dt + \beta \int_0^\infty e^{-st} g(t)\,dt.$$

Whenever both integrals converge for $s > c$, for some constant $c$. Hence it follows that

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s) \qquad (3).$$

<u>Example:</u>

a) $\mathcal{L}\{1 + 5t\} = \mathcal{L}\{1\} + 5\mathcal{L}\{t\}$

$$= \frac{1}{s} + \frac{5}{s^2} = \frac{s+5}{s^2} \quad (s>0)$$

b) $\mathcal{L}\{4e^{5t} - 10\sin 2t\}$

$$= 4\mathcal{L}\{e^{5t}\} - 10\mathcal{L}\{\sin 2t\}$$

$$= \frac{4}{s-5} - \frac{20}{s^2+4}.$$

c) $\mathcal{L}\{20e^{-3t} + 7t - 9\}$

$$= 20\mathcal{L}\{e^{-3t}\} + 7\mathcal{L}\{t\} - 9\mathcal{L}\{1\}$$

$$= \frac{20}{s+3} + \frac{7}{s^2} - \frac{9}{s}.$$

**Theorem:** (Laplace transforms of Some Basic Functions)

a) $\mathcal{L}\{1\} = \frac{1}{s}$. $(s>0)$

b) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$. $n = 1, 2, 3, \ldots$ $(s>0)$

c) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ $(s>a)$

d) $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$ $(s>0)$

e) $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$ $(s>0)$

f) $\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$ $(s>0)$

g) $\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$ $(s>k)$

SUFFICIENT CONDITIONS FOR EXISTENCE OF $\mathcal{L}$

We say $f$ is **piecewise continuous** on $[0,\infty)$ if, in any interal $0 \le a \le t \le b$, there are at most a finite number of point $t_k$, $k = 1, 2, \ldots, n$ at which $f$ is discontinuous.

A function $f$ is said to be **exponential order** if there exist constants $c$, $M>0$, and $T>0$ s.t.

$$|f(t)| \le M e^{ct} \quad \text{for all } t>T.$$

**Theorem:** If $f$ is piecewise continuous on $[0, \infty)$ and of exponential order, then $\mathcal{L}\{f(t)\}$ exist for $s > c$.

**Proof.**

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st}$$