1. Spring/Mass systems: Free Undamped Motion.

Hooke’s Law.
Assume that a flexible spring is suspended vertically from a rigid support and then a mass \( m \) is attached to it free end.

\[
\begin{align*}
\text{unstretched} & \quad \text{equilibrium position} \\
\text{equilibrium position} & \quad mg - ks = 0 \\
\text{motion} & \quad l + s
\end{align*}
\]

By Hooke’s law, the spring itself exerts a restoring force \( F \) opposite to the direction of elongation and proportional to the amount of elongation \( s \).

\[
F = -ks
\]

Where \( k > 0 \) is a constant called the spring constant.

For example: if a mass weighing 10 pounds stretches a spring \( \frac{1}{2} \) foot, then

\[
F = 10 = k \left( \frac{1}{2} \right) \Rightarrow k = 20 \text{ lb/ft}.
\]
NEWTON’S SECOND LAW.

When a mass $m$ is attached to the lower end of a spring (of negligible mass), it stretches the spring by an amount $s$ and attains an equilibrium position at which its weight $W$ is balanced by the restoring force $F$. Thus

$$W = mg = ks = F.$$  

Now suppose the mass is set in motion by giving it an displacement and an initial velocity. Assume that the motion takes place in a vertical line, that the displacements $x(t)$ of the mass are measured along this line such that $x=0$ corresponds to the equilibrium position, and that the displacements measured below the equilibrium are positive.

We apply Newton’s second law of motion:

The net or resultant force on a moving body of mass $m$ is given by $\sum F_x = ma$, where

$$a = \frac{d^2x}{dt^2}$$

is its acceleration.

If we assume further that the mass vibrates free of all external forces, then Newton’s law gives:

$$m \frac{d^2x}{dt^2} = -k(x+s) + mg = -kx + mg - ks$$

Thus, $F_1 = -k(x+s)$ is the restoring force.
DE OF FREE UNDAMPED MOTION

\[ \frac{d^2 x}{dt^2} + \omega^2 x = 0 \quad (2) \]

( \omega^2 = \frac{k}{m} ).

This equation is said to describe simple harmonic motion or free undamped motion. Two obvious initial conditions associated with (2) are:

\[ x(0) = x_0 \]

and

\[ x'(0) = x_1. \]

If \( x_0 > 0, x_1 < 0 \), the mass starts from the point below the equilibrium position with an upward velocity. When \( x_1 = 0 \), the mass is said to be released from rest.

To solve (2), we note that the auxiliary eq is \( m^2 + \omega^2 = 0 \). The general solution is

\[ x(t) = C_1 \cos \omega t + C_2 \sin \omega t. \quad (3) \]

The period of motion described by (3) is

\[ T = \frac{2\pi}{\omega}. \]

The natural frequency of the motion is

\[ f = \frac{1}{T} = \frac{\omega}{2\pi}. \]
The circular frequency is \( \omega = \sqrt{\frac{k}{m}} \).

**Example:** A mass weighing 2 pounds stretches a spring 6 inches. At \( t = 0 \) the mass is released from a point 8 inches below the equilibrium position with an upward velocity \( \frac{4}{3} \) ft/s. Determine the equation of motion.

**Solution:**
\[
X(0) = x_0 = 8 \text{ in } = \frac{2}{3} \text{ ft}.
\]
\[
X'(0) = x_1 = -\frac{4}{3} \text{ ft/s}
\]
\[
m = \frac{W}{g} = \frac{2}{32} = \frac{1}{16} \text{ slug}.
\]
From Hooke's law
\[
2 = k(\frac{1}{2}) \implies k = 4 \text{ lb/ft}.
\]
\[
\implies \quad \frac{1}{16} \frac{d^2x}{dt^2} = -4x
\]
\[
or \quad \frac{d^2x}{dt^2} + 64x = 0
\]
\[
\implies \quad \text{general solution is}
\]
\[
X = C_1 \cos 8t + C_2 \sin 8t
\]
From the initial conditions:
\[
C_1 = \frac{2}{3}, \quad C_2 = -\frac{1}{6}
\]
\[
\implies \quad X(t) = \frac{2}{3} \cos 8t - \frac{1}{6} \sin 8t
\]
ALTERNATIVE FORM OF $x(t)$. 

When $c_1, c_2 \neq 0$, the amplitude $A = \sqrt{c_1^2 + c_2^2}$ of the free vibrations is not obvious from inspection of equation (3): $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$.

\[ \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \]

Hence it is often convenient to convert a solution of form (3) to the simpler form

\[ x(t) = A \sin (\omega t + \phi) \]  \hspace{1cm} (6)

where $A = \sqrt{c_1^2 + c_2^2}$ and $\phi$ is a phase angle defined by

\[ \begin{align*}
\sin \phi &= \frac{c_1}{A} \\
\cos \phi &= \frac{c_2}{A}
\end{align*} \]

\[ \tan \phi = \frac{c_1}{c_2}. \]

To see (6) and (3) are equivalent, we apply the addition rule of the sine function:

\[ A \sin (\omega t + \phi) = A \sin \omega t \cos \phi + A \cos \omega t \sin \phi \]

\[ = (A \sin \phi) \cos \omega t + (A \cos \phi) \sin \omega t \]

\[ = c_1 \cos \omega t + c_2 \sin \omega t. \]
Example: We can write the solution
\[ x(t) = \frac{2}{3} \cos 8t - \frac{1}{6} \sin 8t \]
in the previous example as
\[ x(t) = A \sin (8t + \phi) \]
where
\[ A = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{6}\right)^2} = \sqrt{\frac{17}{36}} \approx 0.7 \text{ ft}. \]
\[ \tan \phi = -4 \Rightarrow \phi = \tan^{-1}(-4) = -1.326 \text{ rad}. \]
However, this is not phase angle, since \( \tan^{-1}(4) \) is located in the fourth quadrant, therefore contradicts the fact that \( \sin \phi > 0 \) and \( \cos \phi < 0 \) (\( \phi > 0 \)).
Hence
\[ \phi = \pi - 1.326 = 1.816 \text{ rad}. \]
Thus
\[ x(t) = \frac{\sqrt{17}}{6} \sin(8t + 1.816). \]
The period is
\[ T = \frac{2\pi}{8} = \frac{\pi}{4} \text{ s}. \]

Sometimes, the eq (3) is also written in the form
\[ x(t) = A \cos (\omega t - \phi') \]  
where
\[ \sin \phi' = \left(\frac{c_2}{A}\right)^2 \]
\[ \cos \phi' = \frac{c_1}{A} \]
\[ \tan \phi' = \frac{c_2}{c_1}. \]
DOUBLE SPRING SYSTEM:

Suppose two parallel springs, with constants $k_1$ and $k_2$, are attached into a common rigid support and then a single mass $m$:

Then the **effective spring constant** of the system is

$$k_{\text{eff}} = k_1 + k_2.$$

On the other hand, suppose that the two spring and a single mass $m$ are in series (see figure):

Then the effective spring constant is

$$k_{\text{eff}} = \frac{k_1 k_2}{k_1 + k_2}$$

(or $\frac{1}{k_{\text{eff}}} = \frac{1}{k_1} + \frac{1}{k_2}$).
2. FREE DAMPED MOTION

In the previous model, we assumed that there are no forces acting on the moving mass. It is somehow unrealistic.

In the study of mechanics, damping forces acting on a body are proportional to (a power of) the instantaneous velocity.

Newton's second law says that

\[ m \frac{d^2x}{dt^2} = -Kx - \beta \frac{dx}{dt} \quad (10) \]

where \( \beta > 0 \) is a damping constant, and the "-" sign is a consequence of the fact that the damping force acts in opposite direction to the motion.

The DE of free damped motion is:

\[ x'' + \frac{\beta}{m} x' + \frac{K}{m} x = 0 \]

or

\[ x'' + 2\lambda x' + \omega^2 x = 0 \quad (11) \]

( \( \lambda = \frac{\beta}{2m} \), \( \omega^2 = \frac{K}{m} \)).

CASE I: \( \lambda^2 - \omega^2 > 0 \) the auxiliary of (11) has two distinct real roots

\[ m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}, \quad m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2} \]
\[ x(t) = c_1 e^{m_{1t}} + c_2 e^{m_{2t}} \]

\[ = e^{-\lambda t} \left( c_1 e^{\sqrt{\omega^2 - \lambda^2} t} + c_2 e^{-\sqrt{\omega^2 - \lambda^2} t} \right) \]

**Case II:** \( \lambda^2 - \omega^2 < 0 \)

\( m_1 = m_2 = -\lambda \).

\[ x(t) = c_1 e^{m_{1t}} + c_2 e^{m_{2t}} \]

\[ = e^{-\lambda t} \left( c_1 + c_2 t \right) \]

**Case III:** \( \lambda^2 - \omega^2 < 0 \)

\( m_1 = -\lambda + \sqrt{\omega^2 - \lambda^2} i \)

\( m_2 = -\lambda - \sqrt{\omega^2 - \lambda^2} i \)

\[ x(t) = e^{-\lambda t} \left( (c_1 \cos \sqrt{\omega^2 - \lambda^2} t + c_2 \sin \sqrt{\omega^2 - \lambda^2} t) \right) \]

Case I is called a **overdamped system**, Case II is a **critically damped system**, Case III is a **underdamped system**.

**Example:** Solve the IVP

\[ x'' + 5x' + 4x = 0, \quad x(0) = 1, \quad x'(0) = 1. \]

The auxiliary eq is \( m^2 + 5m + 4 = 0 \)

\( m_1 = -1, \quad m_2 = -4 \)

\[ \Rightarrow \text{the solution has form} \]

\[ y = c_1 e^{-t} + c_2 e^{-4t} \]
\[ \begin{align*} X(0) &= 1 \\ c_1 + c_2 &= 1 \\ \hline \end{align*} \]

\[ \begin{align*} X'(0) &= 1 \\ X'(t) &= -c_1 e^{-t} - 4c_2 e^{-4t} \\ \Rightarrow -c_1 - 4c_2 &= 1 \\ \Rightarrow c_1 &= \frac{5}{3} \\ c_2 &= -\frac{2}{3} \\ \text{Thus} \\ X(t) &= \frac{5}{3} e^{-t} - \frac{2}{3} e^{-4t}. \end{align*} \]

**Example:** A mass weighing 8 pounds stretches a spring 2 feet. Assume that a damping force numerically equal to 2 times the instantaneously velocity acts on the system. Determine the equation of motion if the mass is initially released from the equilibrium position with an upward velocity of 3 ft/s.

**Solution:**

From Hooke's law we see that

\[ 8 = K \frac{32}{4} \Rightarrow K = 4 \text{ lb/ft}. \]

\[ \text{W} = mg \text{ gives } m = \frac{8}{4} = 1 \text{ slug}. \]

The differential equation of the motion is then

\[ \frac{1}{4} \frac{d^2x}{dt^2} = -4x - 2 \frac{dx}{dt} \]

or

\[ \frac{d^2x}{dt^2} + 8 \frac{dx}{dt} + 16x = 0, \quad (17) \]

The auxiliary for (17) is

\[ m^2 + 8m + 16 = (m + 4)^2 = 0 \]

so \( m_1 = m_2 = -4 \).

Hence the system is critically damped, and

\[ X(t) = c_1 e^{-4t} + c_2 t e^{-4t}. \]
\( X(0) = 0 \quad c_1 = 0 \)

\( X'(0) = -3 \quad X'(t) = c_2 e^{-4t} - 4(c_2 + t e^{-4t}) \)

\[ \Rightarrow c_2 = -3. \]

Thus

\[ X(t) = -3 + e^{-4t}. \]

**Example:** A mass weighing 16 pounds is attached to a 5-foot-long spring. At equilibrium the spring measures 8.2 feet. If the mass is initially released from rest at a point 2 feet above the equilibrium position, find the displacement \( x(t) \) if it is further known that the surrounding medium offers a resistance numerically equal to the instantaneous velocity.

**Solution:**

The elongation of the spring after the mass is attached is \( s = 8.2 - 5 = 3.2 \) ft.

From the Hooke’s Law that \( 16 = K(3.2) \)

\[ K = 5 \text{ lb/ft}. \]

In addition, \( m = \frac{16}{32} = \frac{1}{2} \) slugs.

\[ \Rightarrow \text{ Our DE:} \]

\[ \frac{1}{2} x'' = -5x - x' \]

or

\[ x'' + 2x' + 10x = 0 \]

The auxiliary \( m^2 + 2m + 10 = 0 \) has roots

\[ m_1 = -1 + 3i \]

\[ m_2 = -1 - 3i \]

\[ \Rightarrow x(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t). \]
The initial conditions are:
\[ x(0) = -2 \]
\[ x'(0) = 0 \]

\[ \therefore \]
\[ c_1 = -2 \]
\[ c_2 = -\frac{2}{3} \]

Thus
\[ x(t) = e^{-t}\left(-2\cos 3t - \frac{2}{3}\sin 3t\right). \]

**Alternative form of** \( x(t) \)

We can write any motion
\[ x(t) = e^{-\lambda t}\left(c_1\cos \sqrt{\omega^2 - \lambda^2} t + c_2\sin \sqrt{\omega^2 - \lambda^2} t\right) \]

in the alternative form

\[ x(t) = A e^{-\lambda t}\sin(\sqrt{\omega^2 - \lambda^2} t + \phi) \]

where \( \phi \) and \( A \) are defined by

\[ A = \sqrt{c_1^2 + c_2^2} \]
\[ \sin \phi = \frac{c_1}{A}, \quad \cos \phi = \frac{c_2}{A}, \quad \tan \phi = \frac{c_1}{c_2}. \]

The coefficient \( A e^{-\lambda t} \) is called the damped amplitude. The number \( \frac{2\pi}{\sqrt{\omega^2 - \lambda^2}} \) is called the quasi period, and \( \frac{\sqrt{\omega^2 - \lambda^2}}{2\pi} \) is the quasi frequency.
3. DRIVEN MOTION.

We now assume that there is a driving force causing an oscillatory vertical motion of the support of the spring. The DE of our motion is now

\[ m\ddot{x} = -kx - \beta v + f(t) \]

\[ x'' + 2\lambda x' + \omega^2 x = F(t) \]

\[ F(t) = \frac{f(t)}{m}, \quad \lambda = \frac{\beta}{m}, \quad \omega = \frac{k}{m} \]

Example: There is a mass \( m = \frac{1}{5} \) slug attached to a spring with constant \( k = 2 \) lb/ft. The motion is damped with the damping constant \( \beta = 1.2 \) and is driven by an external periodic (\( T = \frac{\pi}{2} \)) force beginning at \( t = 0 \). The force function \( f(t) = 5\cos 4t \). The mass is initially released from rest \( \frac{1}{2} \) ft below the equilibrium. Find the eq of the motion.

Solution: DE:
\[ \frac{1}{5} x'' + 1.2 x' + 2x = 5 \cos 4t \]

or
\[ x'' + 6 x' + 10x = 25 \cos 4t \]

Step 1: \[ x_c = e^{-3t} (4 \cos t + (2 \sin t) \]

Step 2: \[ L_1 = D^2 + 16 \]
\[ x = x_c + A \cos 4t + B \sin 4t \]
We now find $X_p = A \cos 4t + B \sin 4t$.

$$X_p'' + 6X_p' + 10X_p = (6A + 24B) \cos 4t + (-24A - 6B) \sin 4t = 25 \cos 4t$$

Thus

$$-6A + 24B = 25$$

$$-24A - 6B = 0$$

$$\Rightarrow A = \frac{25}{102}, \quad B = \frac{50}{51}.$$  

$$X(t) = c_1 e^{-3t} \cos 4t + c_2 e^{-3t} \sin 4t - \frac{25}{102} \cos 4t + \frac{50}{51} \sin 4t.$$  

TRANIENT AND STEADY TERMS

When $F$ is a periodic function, such as $F = F_0 \sin \omega t$ or $F_0 = \cos \omega t$,

the general solution of

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = F(t)$$

for $\lambda > 0$ is the sum of a non-periodic function $X_c$ and a periodic function $X_p$. Moreover $X_c(t)$ dies off as time increases (i.e. $\lim_{t \to \infty} X_c(t) = 0$). Thus for large value of time, the displacements of the mass are closely approximated by $X_p(t)$. The $X_c(t)$ is called the transient term and $X_p(t)$ is called the steady term.
Example: The solution of the IVP

\[
\frac{d^2x}{dt^2} + 2 \frac{dx}{dt} + 2x = 4 \cos t + 2 \sin t
\]

\[x(0) = 0, \quad x'(0) = \lambda_1\]

is given by

\[x(t) = (x_1 - 2) e^{-t} \left( \sin t + 2 \sin t \right)\]

\text{transient} \quad \text{steady}.

DE O F DRIVEN MOTION WITHOUT DAMPING

Example: Solve the IVP

\[x'' + \omega^2 x = F_0 \sin \delta t\]

\[x(0) = 0\]

\[x'(0) = 0\]

where \(F_0\) is a constant, and \(\omega \neq \delta\).

Solution: The complementary function is

\[x_C(t) = c_1 \cos \omega t + c_2 \sin \omega t.\]

To obtain a particular solution, we assume

\[x_p(t) = A \cos \delta t + \beta \sin \delta t, \quad \text{so that}\]

\[x_p'' + \omega^2 x_p = A (\omega^2 - \delta^2) \cos \delta t + \]

\[\beta (\omega^2 - \delta^2) \sin \delta t = F_0 \sin \delta t.\]

\[= A = 0\]

\[\beta = \frac{F_0}{(\omega^2 - \delta^2)}.\]

\[x_p = \frac{F_0}{\omega^2 - \delta^2} \sin \delta t.\]
The general solution is

\[ X(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{\omega^2 - \delta^2} \sin \delta t \]

Working on the initial conditions, we have

\[ c_1 = 0 \]
\[ c_2 = -\frac{\delta F_0}{\omega (\omega^2 - \delta^2)} \]

\[ \Rightarrow X(t) = \frac{F_0}{\omega (\omega^2 - \delta^2)} (-\delta \sin \omega t + \omega \sin \delta t) \tag{*} \]

**PURE RESONANCE:**

Although eq (\( \star \)) is not definite when \( \delta = \omega \), we can still find the limiting value of its when \( \delta \to \omega \).

\[ X(t) = \lim_{\delta \to \omega} \frac{-\delta \sin \omega t + \omega \sin \delta t}{\omega (\omega^2 - \delta^2)} \]

\[ = F_0 \lim_{\delta \to \omega} \frac{-\sin \omega t + \omega t \cos \omega t}{-2 \omega \delta} \]

\[ = F_0 \frac{-\sin \omega t + \omega t \cos \omega t}{-2 \omega^2} \]

\[ = \frac{F_0}{2\omega^2} \sin \omega t - \frac{F_0}{2\omega} t \cos \omega t. \]

As suspected, when \( t \to \infty \) the displacements become large; in fact, \( |X(t_n)| \to \infty \) when

\[ t_n = \frac{n \pi}{\omega}, \quad n = 1, 2, \ldots \]
The phenomenon that we have just described is known as pure resonance.

READING EXERCISE: SERIES CIRCUIT ANALOGUE.