CAUCHY-EULER EQUATIONS

We will study the solution of the following eq.

\[ a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x) \]

where \( a_0, a_1, \ldots, a_n \) are all constants. This equation is known as **Cauchy-Euler equation** to honor two mathematicians: Augustin-Louis Cauchy (1789–1857) and Leonhard Euler (1707–1783).

**METHOD OF SOLUTION FOR HOMOGENEOUS EQ**

We try a solution \( y = x^m \), where \( m \) is to be determined.

\[ = a_k x^k \frac{d^k y}{dx^k} = a_k x^k m(m-1)\cdots(m-k+1) x^{m-k} \]

\[ = a_k m(m-1)\cdots(m-k+1) x^m. \]

Let us work in detail for case \( n = 2 \):

\[ a x^2 \frac{d^2 y}{dx^2} + b x \frac{dy}{dx} + c y = 0 \quad (2) \]

The auxiliary eq. is

\[ a m(m-1) + b m + c = 0 \quad (3) \]
\[ am^2 + (b-a)m + c = 0 \]  

**CASE I:** The eq (3) has two distinct real roots, say \( m_1 \) and \( m_2 \).

In this case \( y_1 = x^{m_1} \) and \( y_2 = x^{m_2} \) form a fundamental set of solutions. Hence (2) has the general solution:

\[ y = C_1 x^{m_1} + C_2 x^{m_2} \]

**Example:** Solve

\[ x^2 \frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 12 = 0 \]

The auxiliary eq is \( m^2 - 7m + 12 = 0 \)

\[ \Rightarrow (m - 3)(m - 4) = 0 \]

\[ m_1 = 3 \quad ; \quad m_2 = 4 \]

\[ \Rightarrow \text{The general solution is} \]

\[ y = C_1 x^3 + C_2 x^4 \]

**CASE II:** Repeated Real Root.

Eq (3) has root \( m_1 = m_2 = \frac{-(b-a)}{2a} \)

We have the first solution \( y_1 = x^{m_1} \), and we would like to find the second solution.

We bring (2) into the standard form

\[ \frac{d^2 y}{dx^2} + \frac{b}{a} \frac{dy}{dx} + \frac{c}{ax^2} y = 0 \]
\[ p(x) = \frac{b}{a x}, \quad \int p(x) \, dx = \int \frac{b}{a x} \, dx = \frac{b}{a} \ln x, \]

Thus
\[ y_2 = x^{m_1} \int \frac{e^{-(b/a) \ln x}}{x^{2m_1}} \, dx \]

(reduction of order formula, Section 4.2, pp 133)
\[ = x^{m_1} \int x^{-b/a} x^{-2m_1} \, dx \]
\[ = x^{m_1} \int x^{-b/a} \frac{(b-a)/a}{x} \, dx \]
\[ = x^{m_1} \int \frac{dx}{x} = x^{m_1} \ln x. \]

The general solution of the Cauchy-Euler eq is:
\[ y = C_1 x^{m_1} + C_2 x^{m_1} \ln x. \]

Example: \[ 3 x^2 \cdot y'' + 9 x \cdot y' + 3 y = 0 \]

The auxiliary eq is:
\[ 3 m^2 + (9 - 3) m + 3 = 0 \]
\[ (\Rightarrow) \quad 3 (m+1)^2 = 0 \]
\[ \Rightarrow \quad \text{The general solution is} \]
\[ y = C_1 x^{-1} + C_2 x^{-1} \ln x. \]
CASE III: CONJUGATE COMPLEX ROOTS

If the auxiliary eq (3) has two conjugate complex roots \( m_1 = \alpha + i\beta, \ m_2 = \alpha - i\beta \) \((\alpha, \beta \text{ are real})\)

Then (2) has the general solution

\[ y = C_1 e^{\alpha + i\beta} + C_2 e^{\alpha - i\beta} \]  

(4)

We would like to find two linearly independent real solutions of (2).

We have

\[ e^{i\beta} = (e^{\ln x})^{i\beta} = e^{i\beta \ln x} \]

By Euler identity:

\[ e^{i\beta} = \cos (\beta \ln x) + i \sin (\beta \ln x) \]

\[ e^{-i\beta} = \cos (\beta \ln x) - i \sin (\beta \ln x) \]

\[ e^{i\beta} + e^{-i\beta} = 2 \cos (\beta \ln x) \]

\[ e^{i\beta} - e^{-i\beta} = 2i \sin (\beta \ln x) \]

Take \( C_1 = C_2 = \frac{1}{2} \) and \( C_1 = \frac{1}{2i}, \ C_2 = -\frac{1}{2i} \) in (4)

We get two particular solutions

\[ y_1 = x^2 \cos (\beta \ln x), \ y_2 = x^2 \sin (\beta \ln x). \]
Thus the general solution is this case can be re-written as
\[ y = x^{\frac{1}{2}} [c_1 \cos(E \ln x) + c_2 \sin(E \ln x)]. \]

**Example:** Solve the following IVP

**Solution:**
\[ 4x^2 y'' + 17 y = 0, \quad y(1) = -1, \quad y'(1) = -\frac{1}{2}. \]

The auxiliary eq is
\[ 4m^2 - 4m + 17 = 0. \]

There are two complex roots: \( m_1 = \frac{1}{2} + 2i \) and \( m_2 = \frac{1}{2} - 2i \). The general solution is
\[ y = x^{\frac{1}{2}} [c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)]. \]

By \( y(1) = -1 \), \( c_1 = -1 \).
By \( y'(1) = \frac{1}{2} \), \( c_2 = 0 \).

\[ \Rightarrow \quad \text{The solution is} \quad y = -x^{\frac{1}{2}} \cos(2 \ln x). \]

**Example** Solve
\[ x^3 y''' + 5x^2 y'' + 7xy' + 8y = 0. \]

**Solution**
We want to find a solution of form \( y = x^m \).

\[ y' = mx^{m-1}, \quad y'' = m(m-1)x^{m-2}, \quad y''' = m(m-1)(m-2)x^{m-3} \]

\[ \Rightarrow \quad \text{the auxiliary eq} \quad (m+2)(m^2+4) = 0 \]
There are 3 roots:

\[ m_1 = -2 \]
\[ m_2 = 2i \]
\[ m_3 = -2i \]

\[ y = c_1 x^{-2} + c_2 \cos(2 \ln x) + c_3 \sin(2 \ln x). \]

\[ \text{NONHOMOGENEOUS EQS} \]

**Example:** Solve

\[ x^2 y'' - 3x y' + 3y = 2x^4 e^x. \]

**Solution:**

The auxiliary eq:

\[ (m-1)(m-3) = 0 \]

\[ y_c = c_1 x + c_2 x^3. \]

We would like to find \( y_p = u_1 y_1 + u_2 y_2 \)

\[ u_1' = \frac{w_2}{w} = -\frac{y_2 f(x)}{W} \]

\[ u_2 = \frac{w_1}{w} = \frac{y_1 f(x)}{W} \]

\[ W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3 \]

\[ u_1' = -x^2 e^x \quad \Rightarrow \quad u_1 = -x^2 e^x + 2x e^x - e^x \]
\[ u_2' = e^x \quad \Rightarrow \quad u_2 = e^x \]
\( y_p = (-x^2e^x + 2xe^x - 2e^x)x + e^x x^2 = 2xe^x - 2xe^x \)

The general solution is

\[ y = y_c + y_p = c_1x + c_2x^3 + 2xe^x - 2xe^x. \]

**REDUCTION TO CONSTANT COEFFICIENTS.**

**Example:** solve  
\[ x^2y'' - xy' + y = (\ln x \text{ on } (0, \infty)) \]

**Solution:** substitute \( x = e^t \), or \( t = \ln x \)

\[
\frac{dy}{dt} = \frac{dx}{dt} \frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt} \quad \text{chain rule}
\]

\[
\frac{d^2y}{dx^2} = \frac{1}{x} \frac{d}{dx} \left( \frac{dy}{dt} \right) + \frac{dy}{dt} \left( -\frac{1}{x^2} \right)
\]

\[
= \frac{1}{x} \left( \frac{d^2y}{dt^2} \frac{1}{x} \right) + \frac{dy}{dt} \left( -\frac{1}{x^2} \right)
\]

\[
= \frac{1}{x^2} \left( \frac{d^2y}{dt^2} - \frac{dy}{dt} \right)
\]

Our eq becomes

\[
\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t \quad (\text{auxiliary eq } m^2 - 2m + 1 = 0)
\]

\[ y_c = c_1e^t + c_2te^t \]

\[ L_1 = D^2 \]

\[ L_1^{-1} D^2 (D-1)^2 y = 0 \]

\[ \Rightarrow \text{we try } y_p = A + Bt \]
\[ -2B + A + B t = t \]

\[ \Rightarrow \quad A = 2, \quad B = 1 \]

Thus

\[ y = y_c + y_p = C_1 e^t + C_2 t e^t + 2 + t \]

Finally, substitute back \( t = \ln x \), we get

\[ y = C_1 x + C_2 x \ln x + 2 + \ln x \]

Remark:

1. The previous example worked for \( x > 0 \) (\( x = e^t \)).
   
   If \( x < 0 \), we substitute \( t = -x \) and use the chain rule to have
   
   \[ \frac{dy}{dx} = \frac{dy}{dt} \]
   
   \[ \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} \]

   Then we can apply the above method.

2. The second-order eq:

   \[ a (x - x_0)^2 \frac{d^2 y}{dx^2} + b (x - x_0) \frac{dy}{dx} + cy = 0 \]

   is also called a Cauchy-Euler eq.