CAUCHY - EULER EQUATIONS

We will study the solution of the following eq.

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 x \frac{dy}{dx} + a_0 y = g(x) \quad (1)$$

Where $a_0, a_1, \dots, a_n$ are all constants. This equation is known as **Cauchy-Euler equation** to honor two mathematicians Augustin-Louis Cauchy (1789-1857) and Leonhard Euler (1707-1783).

METHOD OF SOLUTION FOR HOMOGENEOUS EQ

We try a solution $y = x^m$, where m is to be determined

$$\Rightarrow a_k x^k \frac{d^k y}{dx^k} = a_k x^k m(m-1)\dots(m-k+1) x^{m-k}$$

$$= a_k m(m-1)\dots(m-k+1) x^m.$$

Let us work in detail for case $n=2$:

$$a x^2 \frac{d^2 y}{dx^2} + b x \frac{dy}{dx} + c y = 0 \quad (2)$$

The auxiliary eq is

$$a m(m-1) + b m + c = 0 \quad (3)$$

$$\Leftrightarrow \quad am^2 + (b-a)m + c = 0 \qquad (3)$$

CASE I: The eq (3) has two distinct real roots, say m_1 and m_2.

In this case $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ form a fundamental set of solutions. Hence (2) has the general solution:

$$y = c_1 x^{m_1} + c_2 x^{m_2},$$

Example: Solve

$$x^2 \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 12 = 0$$

The auxiliary eq is $m^2 - 7m + 12 = 0$

$$\Leftrightarrow (m-3)(m-4) = 0$$

$$m_1 = 3, \quad m_2 = 4$$

$\Rightarrow$ The general solution is

$$y = c_1 x^3 + c_2 x^4. \qquad \square$$

CASE II. Repeated Real Root.

Eq (3) has root $m_1 = m_2 = -\dfrac{(b-a)}{2a}$

We have the first solution $y_1 = x^{m_1}$, and we would like to find the second solution:

We bring (2) into the standard form

$$\frac{d^2y}{dx^2} + \frac{b}{a}\frac{dy}{dx} + \frac{c}{ax^2}y = 0$$

$$P(x) = \frac{b}{ax}, \quad \int P(x)\,dx = \int \frac{b}{ax}\,dx = \frac{b}{a}\ln x,$$

Thus

$$y_2 = x^{m_1} \int \frac{e^{-(b/a)\ln x}}{x^{2m_1}}\,dx$$

(reduction of order formular Section 4.2, pp 133)

$$= x^{m_1} \int x^{-b/a} \cdot x^{-2m_1}\,dx$$

$$= x^{m_1} \int x^{-b/a} \cdot x^{(b-a)/a}\,dx$$

$$= x^{m_1} \int \frac{dx}{x} = x^{m_1} \ln x.$$

The general solution of the Cauchy-Euler eq is:

$$y = C_1 x^{m_1} + C_2 x^{m_1} \ln x.$$

Example: $3x^2 \cdot y'' + 9x \cdot y' + 3y = 0$

The auxiliary eq is:

$$3m^2 + (9-3)m + 3 = 0$$

$$\Leftrightarrow \quad 3(m+1)^2 = 0$$

$\Rightarrow$ The general solution is

$$y = C_1 x^{-1} + C_2 x^{-1} \ln x.$$

$\square$

CASE III: CONJUGATE COMPLEX ROOTS

If the auxiliary eq (3) has two conjugate complex root $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$. (α, β are real, $\beta > 0$)

Then (2) has the general solution

$$y = C_1 x^{\alpha + i\beta} + C_2 x^{\alpha - i\beta} \qquad (4)$$

We would like to find two linearly independent real solution of (2).

We have $x^{i\beta} = \left(e^{\ln x}\right)^{i\beta} = e^{i\beta \ln x}$.

By Euler identity:

$$\begin{cases} x^{i\beta} = \cos(\beta \ln x) + i \sin(\beta \ln x) \\ x^{-i\beta} = \cos(\beta \ln x) - i \sin(\beta \ln x). \end{cases}$$

$$\Rightarrow \begin{cases} x^{i\beta} + x^{-i\beta} = 2\cos(\beta \ln x) \\ x^{i\beta} - x^{-i\beta} = 2i \sin(\beta \ln x) \end{cases}$$

Take $C_1 = C_2 = \frac{1}{2}$ and $C_1 = \frac{1}{2i}$, $C_2 = -\frac{1}{2i}$ in (4)

We get two particular solutions

$$y_1 = x^{\alpha} \cos(\beta \ln x), \quad y_2 = x^{\alpha} \sin(\beta \ln x).$$

Thus the general solution is this case can be re-written as

$$y = x^{\alpha}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)],$$

Example: Solve the following IVP

Solution:

$$4x^2 y'' + 17y = 0, \quad y(1) = -1, \quad y'(1) = -\frac{1}{2}.$$

The auxiliary eq is

$$4m^2 - 4m + 17 = 0.$$

There are two complex roots: $m_1 = \frac{1}{2} + 2i$ and $m_2 = \frac{1}{2} - 2i$. The general solution is

$$y = x^{\frac{1}{2}}[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)].$$

By $y(1) = -1$; $c_1 = -1$.

By $y'(1) = \frac{1}{2}$, $c_2 = 0$.

$\Rightarrow$ The solution is $y = -x^{\frac{1}{2}} \cos(2\ln x)$.

Example Solve

$$x^3 y''' + 5x^2 y'' + 7xy' + 8y = 0.$$

Solution

We want to find a solution of form $y = x^m$.

$\Rightarrow$ $y' = m x^{m-1}$, $y'' = m(m-1) x^{m-2}$

$y''' = m(m-1)(m-2) x^{m-3}$

$\Rightarrow$ the auxiliary eq

$$(m+2)(m^2+4) = 0$$

There are 3 roots:

$$m_1 = -2$$
$$m_2 = 2i$$
$$m_3 = -2i$$

$\Rightarrow$

$$y = c_1 x^{-2} + c_2 \cos(2\ln x) + c_3 \sin(2\ln x).$$

$\square$

NONHOMOGENEOUS EQS

Example: Solve

$$x^2 y'' - 3x y' + 3y = 2x^4 e^x.$$

Solution:

The auxiliary eq:

$$(m-1)(m-3) = 0$$

$\Rightarrow$

$$y_c = c_1 x + c_2 x^3.$$

We would like to find $y_p = u_1 y_1 + u_2 y_2$

$$\Rightarrow \quad u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W}$$

$$u_2 = \frac{W_2}{W} = \frac{y_1 f(x)}{W}$$

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3$$

$$\Rightarrow \quad u_1' = -x^2 e^x \quad \Rightarrow \quad u_1 = -x^2 e^x + 2xe^x - e^x$$
$$u_2' = e^x \qquad\qquad u_2 = e^x$$

$$\Rightarrow y_p = (-x^2e^x + 2xe^x - 2e^x)x + e^x x^3 = 2x^2e^x - 2xe^x$$

The general solution is

$$y = y_c + y_p = c_1 x + c_2 x^3 + 2x^2 e^x - 2x e^x . \quad \Box$$

REDUCTION TO CONSTANT COEFFICIENTS.

Example: Solve
$$x^2 y'' - xy' + y = \ln x \quad \text{on } (0,\infty)$$

Solution:

substitute $x = e^t$, or $t = \ln x$

$$\rightarrow \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} \leftarrow \text{chain Rule}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$
$$= \frac{1}{x}\left(\frac{d^2y}{dt^2}\frac{1}{x}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$
$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Our eq becomes
$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t \qquad (\text{auxiliary eq } m^2 - 2m + 1 = 0)$$

$y_c = c_1 e^t + c_2 t e^t$

$L_1 = D^2$

$$\rightarrow D^2(D-1)^2 y = 0$$

$\Rightarrow$ we try $y_p = A + Bt$

$\Rightarrow \quad -2B + A + Bt = t$

$\Rightarrow \quad A = 2, \; B = 1$

Thus

$$y = y_c + y_p = c_1 e^t + c_2 t e^t + 2 + t$$

Finally substitute back $t = \ln x$, we get

$$y = c_1 x + c_2 x \ln x + 2 + \ln x . \quad \square$$

Remark:

① The previous example worked for $x > 0$ $(x = e^t)$.
If $x < 0$, we substitute $t = -x$ and use the chain rule to have

$$\frac{dy}{dx} = -\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}.$$

Then we can apply the above method.

② The second-order eq:

$$a(x - x_0)^2 \frac{d^2y}{dx^2} + b(x - x_0)\frac{dy}{dx} + cy = 0$$

is also called a Cauchy-Euler eq.