4.6 Variation of Parameters.

The method of undetermined coefficients has two weaknesses:

+ The coefficients must be constants
+ The function $g(x)$ must be of several special types.

In this section, we investigate a new method to find a particular solution of nonhomogeneous eqs, that has no such restriction on it. This method, due to Joseph Louis Lagrange (1736-1813), is known as variation of parameters.

**LINEAR FIRST-ORDER DEs.**

\[
\frac{dy}{dx} + P(x)y = f(x) \quad (1)
\]

assuming that $P$ and $f$ are continuous on an interval $I$. By integrating factor the general solution is

\[
y = c_1 e^{-\int P(x) dx} + e^{-\int P(x) dx} \int e^{\int P(x) dx} f(x) dx
\]

\[
(\mu(x) = e^{\int P(x) dx})
\]

\[
\frac{d}{dx} [\mu(x) y] = \mu(x) f(x)
\]

\[
(\Rightarrow \mu(x) y = \int \mu(x) f(x) dx + c_1)
\]
In this formula:

\[ y = c_1 e^{-\int p(x) \, dx} + e^{-\int p(x) \, dx} \int e^{\int p(x) \, dx} f(x) \, dx \]

we have \(-\int p(x) \, dx\)
\(y_c = c_1 e^{-\int p(x) \, dx}\) is the general solution of the homogeneous eq corresponding to (1), i.e.

\[ \frac{dy}{dx} + p(x) y = 0 \] \(\text{ (2)}\)

and \(y_p = e^{-\int p(x) \, dx} \int e^{\int p(x) \, dx} f(x) \, dx \) \(\text{ (3)}\)

is a particular solution of (1). This is consistent to what we discuss in the beginning of the chapter.

We now find the particular solution \(y_p\) by a new method known as **variation of parameter**.

1. Assume that \(y_1\) is a known solution of the homogeneous eq (2).
   
   It is easy to see that \(y_1 = e^{-\int p(x) \, dx}\) is such a solution. And \(c_1 y_1(x)\) is the general sol.

2. Variation of parameters consists of finding a particular solution of (1) of the form
   
   \[ y_p = U_1(x) y_1(x) \]
   
   (i.e. we replace the parameter \(c_1\) by a function \(U_1\))
Substituting \( y_p = u_1 y_1 \) into (1), we get

\[
\frac{d}{dx} \left( u_1 y_1 \right) + p(x) u_1 y_1 = f(x)
\]

\( \implies \)

\[
 u_1 \frac{dy_1}{dx} + y_1 \frac{du_1}{dx} + p(x) u_1 y_1 = f(x)
\]

\( \implies \)

\[
u_1 \left[ \frac{dy_1}{dx} + p(x) y_1 \right] - y_1 \frac{du_1}{dx} = f(x) = 0
\]

\( \implies \)

\[
y_1 \frac{du_1}{dx} = f(x).
\]

By separating variables, we find \( u_1 \):

\[
du_1 = \frac{f(x)}{y_1} \, dx
\]

\( \implies \)

\[
u_1 = \int \frac{f(x)}{y_1} \, dx.
\]

\[
y_p = u_1 y_1 = y_1 \int \frac{f(x)}{y_1(x)} \, dx
\]

That is identical to (3) as \( y_1 = e^{-\int p(x) \, dx} \).

**Linear Second-Order DEs**

\[
y'' + p(x) y' + q(x) y = f(x) \quad (4)
\]

(We assume that \( p, q, f \) are continuous on an interval \( I \))
It is not hard to obtain the general solution
\[ y_c = C_1 y_1 + C_2 y_2 \]
of the associated homogeneous equation.

We would like to find a particular solution \( y_p \) of (4) in the form
\[ y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x). \]  
We have
\[ y'_p = u_1 y'_1 + u_1' y_1 + u_2 y'_2 + u_2' y_2 \]
\[ y''_p = u_1 y''_1 + y'_1 u_1' + y_1 u_1'' + u_1' y'_1 
+ u_2 y''_2 + u_2' y_2 + y_2 u_2'' + u_2' y'_2 \]
Substituting (5), (6), (7) into (4), we get
\[ y_p'' + P(x) y_p' + Q(x) y_p = u_1 \left[ y'_1 + P y_1' + Q y_1 \right] 
+ u_2 \left[ y'_2 + P y_2 + Q y_2 \right] 
+ y_1 u_1'' + u_1 y_1' + y_2 u_2'' + u_2 y_2' 
+ P[y_1 u_1' + y_2 u_2'] 
+ y_1' u_1' + y_2' u_2' 
= \frac{d}{dx} \left[y_1 u_1' + y_2 u_2' \right] + P[y_1 u_1' + y_2 u_2'] 
+ y_1' u_1' + y_2' u_2' 
= f(x). \]

We can make a further assumption that
\[ y_1 u_1' + y_2 u_2' = 0. \]
Thus (8) is reduced to
\[ y_1 u_1' + y_2' u_2' = f(x). \]
Thus we have a system with in \( u_1' \) and \( u_2' \):

\[
\begin{align*}
Y_1 u_1' + Y_2 u_2' &= 0 \\
Y_1' u_1' + Y_2' u_2' &= f(x)
\end{align*}
\]

Apply Cramer's Rule:

\[
\begin{align*}
u_1' &= \frac{W_1}{W} = -\frac{Y_2 f(x)}{W} \quad (9) \\
u_2' &= \frac{W_2}{W_1} = \frac{Y_1 f(x)}{W}
\end{align*}
\]

where

\[
W = \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & Y_2 \\ f(x) & Y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} Y_1 & 0 \\ Y_1' & f(x) \end{vmatrix}
\]

(10)

Example 1: Solve

\[ y'' - 4y' + 4y = \left(x + 1\right) e^{2x} \]

Solution:

From the auxiliary eq. \( m^2 - 4m + 4 = 0 \), we have \( y_c = (e^{2x} + c_2 x e^{2x}) \).

\[ y_1 = e^{2x}, \quad y_2 = x e^{2x} \]

Next we compute the Wronskian

\[
W(e^{2x}, x e^{2x}) = \begin{vmatrix} e^{2x} & x e^{2x} \\ 2 x e^{2x} & 2 x e^{2x} + e^{2x} \end{vmatrix} = e^{4x}
\]

\[
W_1 = -\left(x + 1\right) x e^{4x} \quad W_2 = \left(x + 1\right) e^{4x}
\]
By (9): \[ u_1' = -\frac{(x+1)x e^{4x}}{e^{4x}} = -xe^{-x} \]
\[ u_2' = \frac{(x+1)e^{4x}}{e^{4x}} = x+1 \]

\[ \Rightarrow \]
\[ u_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2 \]
\[ u_1 = \frac{1}{2}x^2 + x \]
\[ \text{We do not need to introduce constant parameters here!} \]

\[ \Rightarrow y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)e^{2x} \]
\[ = \frac{1}{6}x^3 e^{2x} + \frac{1}{2}x^2 e^{2x}. \]

and
\[ y = y_c + y_p = C_1 e^{2x} + C_2 x e^{2x} + \frac{1}{6}x^3 e^{2x} + \frac{1}{2}x^2 e^{2x}. \]

**Example:** Solve
\[ 4y'' + 36y = \csc 3x \]

**Solution:**
Put the equation in the standard form
\[ y'' + 9y = \frac{1}{4} \csc 3x \]

The auxiliary eq.
\[ m^2 + 9m = 0 \]

has two complex roots \( m_1 = 3i \) and \( m_2 = -3i \)

the complementary function
\[ y_c = C_1 \cos 3x + C_2 \sin 3x \]

we apply variation of parameters with \( y_1 = \cos 3x, \]
\[ y_2 = \sin 3x, \] and \( f(x) = \frac{1}{4} \csc x. \)
We obtain
\[ W(\cos 3x, \sin 3x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3 \sin 3x & 3 \cos 3x \end{vmatrix} = 3 \]

\[ W_1 = \begin{vmatrix} 1 & \sin 3x \\ \frac{1}{4} \csc 3x & 3 \cos 3x \end{vmatrix} = -\frac{1}{4} \]

\[ W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3 \sin 3x & \frac{1}{4} \csc 3x \end{vmatrix} = \frac{1}{4} \frac{\cos 3x}{\sin 3x} \]

\[ \Rightarrow \quad U_1' = \frac{W_1}{W} = -\frac{1}{12}, \quad U_2' = \frac{W_2}{W} = \frac{1}{12} \frac{\cos 3x}{\sin 3x} \]

Integrating \( U_1' \) and \( U_2' \) gives

\[ U_1 = -\frac{1}{12} x \quad \text{and} \quad U_2 = \frac{1}{36} \ln |\sin 3x| \]

Thus the particular solution is

\[ Y_p = U_1 Y_1 + U_2 Y_2 = -\frac{1}{2} x \cos 3x + \frac{1}{36} (\sin 3x) \ln |\sin 3x| \]

The general solution of the eq is

\[ Y = Y_c + Y_p = C_1 \cos 3x + C_2 \sin 3x - \frac{1}{12} x \cos 3x + \frac{1}{36} (\sin 3x) \ln |\sin 3x| \]

Note that: The method of undetermined coefficients does NOT work for the function \( g(x) = \csc x \).
Remark: When computing the indefinite integrals of \( u_1' \) and \( u_2' \), we do not need to introduce any constant. This is because
\[
y = y_c + y_p = c_1 y_1 + c_2 y_2 + (u_1 + a_1) y_1 + (u_2 + a_2) y_2
= (c_1 + a_1) y_1 + (c_2 + a_2) y_2 + u_1 y_1 + u_2 y_2
= c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2.
\]

**INTEGRAL-DEFINED FUNCTIONS**

We can write \( u_1 \) and \( u_2 \) as integral-defined functions
\[
u_1 = -\int_{x_0}^{x} \frac{y_2(t)f(t)}{W(c_t)} \, dt \quad \text{and} \quad \nu_2 = \int_{x_0}^{x} \frac{y_1(t)f(t)}{W(c_t)} \, dt.
\]

Here we assume that the integrand is continuous on the interval \([x_0, x]\).

**Example.** Solve \( y'' - y = \frac{1}{x} \).

**Solution:** The auxiliary eq: \( m^2 - 1 = 0 \) has two roots \( m_1 = 1, m_2 = -1 \). Thus the complementary function \( y_c = c_1 e^x + c_2 e^{-x} \). Now \( W(e^x, e^{-x}) = -2 \) and
\[
u_1 = \frac{1}{2} \int_{x_0}^{x} \frac{e^t}{t} \, dt \quad \text{and} \quad \nu_2 = -\frac{1}{2} \int_{x_0}^{x} \frac{e^t}{t} \, dt.
\]
Since the integrals here are not elementary, we are forced to write

\[ y_p = \frac{1}{2} e^x \int_{x_0}^x \frac{e^t}{t} \, dt - \frac{1}{2} e^{-x} \int_{x_0}^x \frac{e^t}{t} \, dt \]

and so

\[ y = y_c + y_p = c_1 e^x + c_2 e^{-x} + \frac{1}{2} e^x \int_{x_0}^x \frac{e^t}{t} \, dt - \frac{1}{2} e^{-x} \int_{x_0}^x \frac{e^t}{t} \, dt. \]

**Higher-Order Equations**

\[ y^{(n)} + p_{n-1}(x) y^{n-1} + \cdots + p_1(x) y' + p_0(x) y = f(x) \]

If \( y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \) is the complementary function, then a particular solution is

\[ y_p(x) = u_1(x) y_1(x) + \cdots + u_n y_n(x), \]

where \( u_k' (k = 1, \ldots, n) \) are determined by the system:

\[ y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' = 0 \]

\[ y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' = 0 \]

\[ \vdots \]

\[ y_1^{(n-1)} u_1' + y_2^{(n-1)} u_2' + \cdots + y_n^{(n-1)} u_n' = f(x) \]
The general form of Cramer's Rule gives

\[ u'_k = \frac{W_k}{W}, \quad k = 1, 2, \ldots, n \]

\[ W = W (y_1, \ldots, y_n) \]

\( W_k \) is obtained by replacing the \( k \)-th column in \( W \) by the column consisting of \((0, 0, \ldots, 0, f(cx))\).

In particular, for \( n = 3 \), we have

\[ u'_1 = \frac{W_1}{W}, \quad u'_2 = \frac{W_2}{W}, \quad u'_3 = \frac{W_3}{W}, \]

where

\[ W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix} \]

\[ W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y'_2 & y'_3 \\ f(x) & y''_2 & y''_3 \end{vmatrix} = y_2 y'_3 f(x) - y'_2 y_3 f(x) \]

\[ W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y'_1 & 0 & y'_3 \\ y''_1 & f(x) & y''_3 \end{vmatrix} = y_1 y'_3 f(x) - y'_1 y_3 f(x) \]

\[ W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y'_1 & y'_2 & 0 \\ y''_1 & y''_2 & f(x) \end{vmatrix} = y_1 y'_2 f(x) - y'_1 y_2 f(x) \]