4.3. Homogeneous Linear Equations with Constant Coefficients.

In this section we will study solution of the homogeneous linear equations

\[ a_n(x) y^{(n)} + \cdots + a_1(x) y' + a_0(x) y = 0 \]

where \( a_n(x), \ldots, a_1(x), a_0(x) \) are constants \( a_n, \ldots, a_1, a_0 \).

Let us start with the first-order eq

\[ ay' + by = 0 \] \hspace{1cm} (1)

We can solve it by separating variables or using integrating factors. However, we can use the following algebraic method:

If we substitute \( y = e^{mx} \), then \( y' = me^{mx} \) and our eq (1) becomes:

\[ a m e^{mx} + b e^{mx} = 0 \]

\[ \Rightarrow (am + b)e^x = 0 \]

\[ \Rightarrow am + b = 0 \]

\[ \Rightarrow y = e^{-b/a}x \] is a solution of (1).

And \( y = c e^{-b/a}x \) is the general solution of (1).

**AUXILIARY EQUATION**

Let us consider next the second-order eq:
\[ ay'' + by' + cy = 0 \quad (2). \]

If we try to find a solution of form \( y = e^{mx} \), then after substituting

\[ y' = me^{mx} \quad \text{and} \quad y'' = m^2 e^{mx} \]

to (2), we get

\[ am^2 e^{mx} + bm e^{mx} + ce^{mx} = 0 \]

or

\[ (am^2 + bm + c) e^{mx} = 0. \]

Thus, \( y = e^{mx} \) is a solution of (2) when \( m \) satisfies

\[ am^2 + bm + c = 0 \quad (3). \]

The eq (3) is called the **auxiliary equation** of the DE (2).

There are 3 cases to distinguish:

**CASE I:** \( b^2 - 4ac > 0 \).

Then (3) has two distinct real roots

\[ m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

and

\[ m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

Then the general solution of (2) is

\[ y = c_1 e^{m_1x} + c_2 e^{m_2x} \]
**CASE II:** \( b^2 = 4ae \)

The auxiliary eq (3) has a repeated root \( m = -\frac{b}{2a} \).

\[ \Rightarrow \text{we have a solution } y_1 = e^{mx}. \]

By reduction of order method (u 4.2) we can find a second solution of (2) as

\[ y_2 = e^{mx} \int e^{mx} \, dx = e^{mx} \int dx = xe^{mx}. \]

The general solution is

\[ y = c_1 e^{mx} + c_2 xe^{mx}. \]

**CASE III:** \( b^2 < 4ae \)

The eq (3) has conjugate complex roots.

\[ m_1 = \alpha + i\beta \text{ and } m_2 = \alpha - i\beta , \]

where \( \alpha \) and \( \beta > 0 \) are real.

Thus \( y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} \) is the general solution of eq(2), however, we would like to avoid complex number. We use Euler's formula

\[ e^{i\theta} = \cos \theta + i \sin \theta. \]

\( \Rightarrow \)

\[ e^{i\beta x} = \cos \beta x + i \sin \beta x \]

\( \Rightarrow \)

\[ e^{i\beta x} + e^{-i\beta x} = 2 \cos \beta x, \quad e^{i\beta x} - e^{-i\beta x} = 2i \sin \beta x. \]
Let us choose \( c_1 = c_2 = 1 \),

\[
Y_1 = e^{(\sigma + i\beta)x} + e^{(\sigma - i\beta)x},
\]

and \( c_1 = 1 \) and \( c_2 = -1 \)

\[
Y_2 = e^{(\sigma + i\beta)x} - e^{(\sigma - i\beta)x}.
\]

Thus

\[
Y_1 = e^{\sigma x} (e^{i\beta x} + e^{-i\beta x}) = 2 e^{\sigma x} \cos \beta x
\]

\[
Y_2 = e^{\sigma x} (e^{i\beta x} - e^{-i\beta x}) = 2i e^{\sigma x} \sin \beta x.
\]

\[
\Rightarrow \frac{1}{2} Y_1 = e^{\sigma x} \cos \beta x
\]

and \( \frac{1}{2} Y_2 = e^{\sigma x} \sin \beta x \) are two real solutions of eq \( \text{(2)} \). Then the solution has form

\[
Y = c_1 e^{\sigma x} \cos \beta x + c_2 e^{\sigma x} \sin \beta x.
\]

Example 1. Solve the following differential equations

\[
a) \ 2y'' - 5y' - 3y = 0
\]

\[
2m^2 - 5m - 3 = (2m+1)(m-3) = 0
\]

\[
m_1 = -\frac{1}{2}, \ m_2 = 3
\]

\[
\Rightarrow \ Y = c_1 e^{-\frac{x}{2}} + c_2 e^{3x}.
\]

\[
b) \ y'' - 10y' + 25y = 0
\]

\[
m^2 - 10m + 25 = (m-5)^2 = 0, \ m_1 = m_2 = m = 5
\]
\[ y = c_1 e^{5x} + c_2 x e^{5x}. \]

\[ (c) \ y'' + 4y' + 7y = 0 \]

\[ m^2 + 4m + 7 = 0, \quad m_1 = -2 + \sqrt{3} i \]
\[ m_2 = -2 - \sqrt{3} i \]

\[ y = c_1 e^{-2x} \cos \sqrt{3} x + c_2 e^{-2x} \sin \sqrt{3} x. \]

**Example:** Solve the IVP:
\[ 4y'' + 4y' + 17y = 0 \]
\[ y(0) = -1, \quad y'(0) = 2 \]

**Solution.** Let \( y = e^{mx} \), the auxiliary eq is
\[ 4m^2 + 4m + 17 = 0 \]

There are two conjugate complex solutions:
\[ m_1 = -\frac{1}{2} + 2i \quad \text{and} \quad m_2 = -\frac{1}{2} - 2i. \]

\[ y = C_1 e^{-x/2} \cos 2x + C_2 e^{-x/2} \sin 2x. \]

Apply \( y(0) = -1 \), we have
\[ -1 = C_1 \cos 0 + C_2 \sin 0 \]

\[ -1 = C_1. \]

We have
\[ y' = -\left( e^{-x/2} \cos 2x \right)' + C_2 \left( e^{-x/2} \sin 2x \right)' \]
\[ = \left( -e^{-x/2} \cos 2x + \frac{1}{2} e^{-x/2} \sin 2x \right) \]
\[ + C_2 \left( -\frac{1}{2} e^{x/2} \sin 2x + 2 e^{x/2} \cos 2x \right) \]

Apply \( y'(0) = 2 \):
\[ 2 \left( C_2 + \frac{1}{2} \right) = 2 \quad (\Rightarrow \quad C_2 = \frac{3}{4}). \]
Solution of the IVP is

\[ y = e^{-\frac{x}{2}} \left(-\cos 2x + \frac{3}{4} \sin 2x\right). \]

**TWO IMPORTANT EQUATIONS**

1. \( y'' + k^2 y = 0 \)
2. \( y'' - k^2 y = 0 \)

By investigating the auxiliary equation \( m^2 + k^2 = 0 \), with two complex roots \( m_1 = ki \) and \( m_2 = -ki \).

\( \Rightarrow \) Solution of (1) is

\[ y = c_1 \cos kx + c_2 \sin kx. \]

Similarly, solution of (2) is

\[ y = c_1 e^{kx} + c_2 e^{-kx}. \]

Note that:

\[ \cosh kx = \frac{e^{kx} + e^{-kx}}{2} \]

and

\[ \sinh kx = \frac{e^{kx} - e^{-kx}}{2}. \]

Then (2) has solution:

\[ y = c_1 \cosh kx + c_2 \sinh kx. \]
HIGHER-ORDER EQUATIONS

In general the $n$th-order homogeneous eq:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

has the auxiliary eq:

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

If this eq has all $n$ real distinct real roots then the general solution has form:

$$y = c_1 e^{m_1 x} + \cdots + c_n e^{m_n x}.$$  

However it is harder to summarize the analogues of cases II and III.

If $m_1$ has multiplicity $k$, then the general solution must contain the linear combination

$$c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x} + \cdots + c_k x^{k-1} e^{m_1 x}.$$  

Finally if the auxiliary eq has complex roots then these roots come with conjugate pairs. That is if

$$m_1 = \alpha + \beta i$$

is a root, then $m_2 = \alpha - \beta i$ is also a root and by Euler's identity

$$c_1 e^{(\alpha + \beta i) x} + c_2 e^{(\alpha - \beta i) x}$$

can be written as

$$c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x.$$  

Example. Solve $y''' + 3y'' - 4y = 0$

The auxiliary eq is:

$$m^3 + 3m^2 - 4m = 0$$

has roots $m_1 = 1$, $m_2 = m_3 = -2$.

The general solution is

$$y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}.$$
Example: Solve \(y^{(4)} + 2y'' + y = 0\).

The auxiliary eq
\[
m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0 \text{ has roots:}
\]
\[m_1 = m_3 = i \quad \text{and} \quad m_2 = m_4 = -i.
\]
The general solution has form
\[
y = C_1 e^{ix} + C_2 e^{-ix} + C_3 x e^{ix} + C_4 x e^{-ix}.
\]
By Euler's identity
\[
C_1 e^{ix} + C_2 e^{-ix} = C_1 \cos x + C_2 \sin x
\]
\[
C_3 x e^{ix} + C_4 x e^{-ix} = C_3 x \cos x + C_4 x \sin x.
\]
Thus the general solution has form
\[
y = C_1 e^{ix} + C_2 e^{-ix} + C_3 x e^{ix} + C_4 x e^{-ix}.
\]

In general if the complex root \(m_1 = \alpha + \beta i\)
has multiplicity \(k\), then its conjugate \(m_2 = \alpha - \beta i\)
is also a root of multiplicity \(k\).

Then the general solution must contain the linear combination of \(2k\) solutions
\[
e^{\alpha x} \cos \beta x, \ x e^{\alpha x} \cos \beta x, \ldots, \ x^{k-1} e^{\alpha x} \cos \beta x,
\]
\[
e^{\alpha x} \sin \beta x, \ x e^{\alpha x} \sin \beta x, \ldots, \ x^{k-1} e^{\alpha x} \sin \beta x.
\]

Example: Solve \(3y''' + 5y'' + 10y' - 4y = 0\)

The auxiliary eq is
\[
3m^3 + 5m^2 + 10m - 4 = 0
\]
We try to find a root of the eq.
+ If an integer root exists, it must be a factor of the last coefficient \( a_0 \). That is it must be \( \pm 1, \pm 2, \) or \( \pm 4 \).

But it is not the case here.

+ We try to find a rational root of form

\[
\frac{p}{q}, \quad p \text{ is a divisor of } a_0, \quad q \text{ is a divisor of } a_3.
\]

\[
\Rightarrow \quad \frac{p}{q} : \quad \pm 1, \pm 2, \pm 3, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}
\]

Checking each of the cases, we get a root \( m_1 = \frac{1}{3} \)

\[
\Rightarrow \quad 3m^2 + 5m^2 + 10m - 4 = (m - \frac{1}{3})(3m^2 + 6m + 12)
\]

The quadratic eq

\[
3m^2 + 6m + 12 = 0
\]

does not have real roots.

\[
m_2 = -1 - \sqrt{3} \quad \text{and} \quad m_2 = -1 + \sqrt{3}
\]

Therefore, the general solution is

\[
y = c_1 e^{\frac{x}{3}} + c_2 e^{-x} \cos \sqrt{3}x + \tau c_3 e^{-x} \sin \sqrt{3}x.
\]