4.2 Reduction of Order

In this section, we study the solution of second-order homogeneous differential equation

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = 0. \qquad (1)$$

In particular, we learn how to reduce the eq. to a first-order linear equation.

REDUCTION OF ORDER

Suppose that y_1 is a nontrivial solution of (1) on an interval I. We seek a second solution y_2 so that y_1 and y_2 are linearly independent on I. (If so, we know the general solution of (1) as $y = c_1 y_1 + c_2 y_2$.)

If y_1 and y_2 are linearly independent, then y_2/y_1 is not a constant on I, that is $y_2(x)/y_{(1)} = u(x)$

or $\quad y_2(x) = u(x)\,y_1(x)$.

The function $u(x)$ can be found by substituting $y_2(x) = u(x)\,y_1(x)$ into the given DE. This method is called **reduction of order** because we must solve a linear first-order differential equation to find u.

Example. Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval $I = (-\infty, \infty)$, use reduction of order to find a second solution y_2 that is linearly independent with y_1.

Solution: If $y = u(x)\, y_1(x) = u(x) e^x$, then

$$y' = ue^x + e^x u', \quad y'' = ue^x + 2e^x u' + e^x u''$$

$\Rightarrow$

$$y'' - y = e^x (u'' + 2u') = 0.$$

Since $e^x \neq 0$, the last equation requires $u'' + 2u' = 0$.
Substitute $w = u'$, we have $w' + 2w = 0$, which is a first-order linear DE in w.

Solve: $w' + 2w = 0$.

The integrating factor is $e^{\int 2dx} = e^{2x}$,

$\Rightarrow$ $$\frac{d[we^{2x}]}{dx} = 0$$

$\Rightarrow$ $w = c_1 e^{-2x}$ or $u' = c_1 e^{-2x}$

$\Rightarrow$ $u = -\frac{1}{2} c_1 e^{-2x} + c_2$.

Thus

$$y = u(x) e^x = -\frac{c_1}{2} e^{-x} + c_2 e^x.$$

By choosing $c_2 = 0$ and $c_1 = -2$, we obtain the desired second solution, $y_2 = e^{-x}$.
Because $W(e^x, e^{-x}) \neq 0$, the solutions are linearly independent on $(-\infty, \infty)$. $\square$

GENERAL CASE

We divide (1) by $a_2(x)$ to obtain

$$y'' + P(x) y' + Q(x) y = 0 \qquad (3)$$

where $P(x)$ and $Q(x)$ are continuous on some interval I. Suppose $y_1(x)$ is a known solution of (3) on I and that $y_1(x) \neq 0$ for every $x \in I$.

If we define $y = v(x) y_1(x)$ then

$$y' = v y_1' + y_1 v' , \quad y'' = v y_1'' + 2 y_1' v' + y_1 v''$$

Thus

$$y'' + P y' + Q y = v y_1'' + 2 y_1' v' + y_1 v'' + P(v y_1' + y_1 v') + Q y$$

$$= v[y_1'' + P y_1' + Q y_1] + y_1 v'' + (2 y_1' + P y_1) v'$$

$$= y_1 v'' + (2 y_1' + P y_1) v'.$$

$\Rightarrow$

$$y_1 v'' + (2 y_1' + P y_1) v' = 0 \quad \text{or}$$

$$y_1 w' + (2 y_1' + P y_1) w = 0 \quad (w = v'). \quad (4)$$

The eq (4) are both linear and separable.

$\Rightarrow$

$$\frac{dw}{w} + 2 \frac{y_1'}{y_1} dx + P dx = c$$

$$\ln |w| + 2 \ln |y_1| = -\int P dx + c$$

$$\Leftrightarrow \quad \ln |w y_1^2| = -\int P dx + c$$

or

$$w y_1^2 = c_1 e^{-\int P dx}.$$

Since $w = v'$, we have

$$v = c_1 \int \frac{e^{-\int p\,dx}}{y_1^2}\, dx + c_2.$$

By choosing $c_1 = 1$, $c_2 = 0$, we have

$$y_2 = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2(x)}\, dx. \quad (5)$$

Example: $y_1 = x^2$ is a solution of

$$x^2 y'' - 3xy' + 4y = 0 \quad (6)$$

Find the general solution of the eq on $(0, \infty)$.

Solution:

$$(6) \iff y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0$$

By (2),

$$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4}\, dx$$

$$= x^2 \int \frac{e^{3\ln x}}{x^4}\, dx$$

$$= x^2 \int \frac{dx}{x} = x^2 \ln x.$$

The general solution is

$$y = c_1 y_1(x) + c_2 y_2(x)$$
$$= c_1 x^2 + c_2 x^2 \ln x. \qquad \square$$