4.2 Reduction of Order

In this section, we study the solution of second-order homogeneous differential equation
\[ a_2(x) y'' + a_1(x) y' + a_0(x) y = 0. \quad (1) \]
In particular, we learn how to reduce the eq. to a first-order linear equation.

REDUCTION OF ORDER

Suppose that \( y_1 \) is a nontrivial solution of (1) on an interval \( I \). We seek a second solution \( y_2 \) so that \( y_1 \) and \( y_2 \) are linearly independent on \( I \). (If so, we know the general solution of (1) as \( y = c_1 y_1 + c_2 y_2 \).)

If \( y_1 \) and \( y_2 \) are linearly independent, then \( \frac{y_2}{y_1} \) is not a constant on \( I \), that is \( \frac{y_2(x)}{y_1(x)} = u(x) \) or \( y_2(x) = u(x) y_1(x) \).

The function \( u(x) \) can be found by substituting \( y_2(x) = u(x) y_1(x) \) into the given DE. This method is called reduction of order because we must solve a linear first-order differential equation to find \( u \).

**Example.** Given that \( y_1 = e^x \) is a solution of \( y'' - y = 0 \) on the interval \( I = (-\infty, \infty) \), use reduction of order to find a second solution \( y_2 \) that is linearly independent with \( y_1 \).
Solution: If \( y = u(x) \), \( y_1(x) = u(x) e^x \), then

\[
y' = u e^x + e^x u', \quad y'' = u e^x + 2 e^x u' + e^x u''
\]

\[
\Rightarrow y'' - u = e^x (u'' + 2 u') = 0.
\]
Since \( e^x \neq 0 \), the last equation requires \( u'' + 2 u' = 0 \).
Substitute \( w = u' \), we have \( w' + 2w = 0 \), which is a first-order linear DE in \( w \).

Solve: \( w' + 2w = 0 \).

The integrating factor is \( e^{\int 2 \, dx} = e^{2x} \),

\[
\Rightarrow \frac{d}{dx} \left[ w e^{2x} \right] = 0
\]

\[
\Rightarrow w = C_1 e^{-2x} \quad \text{or} \quad u' = C_1 e^{-2x}
\]

\[
\Rightarrow u = -\frac{1}{2} C_1 e^{-2x} + C_2.
\]
Thus

\[
y = u(x) e^x = -\frac{C_1}{2} e^{-x} + C_2 e^x.
\]

By choosing \( C_2 = 0 \) and \( C_1 = -2 \), we obtain the desired second solution, \( y_2 = e^{-x} \).
Because \( W(e^x, e^{-x}) \neq 0 \), the solutions are linearly independent on \((-\infty, \infty)\).

**GENERAL CASE**

We divide (1) by \( a_2(x) \) to obtain

\[
y'' + p(x) y' + Q(x) y = 0 \quad \text{(3)}
\]
where \( p(x) \) and \( q(x) \) are continuous on some interval \( I \).

Suppose \( y_1(x) \) is a known solution of (3) on \( I \) and that \( y_1(x) \neq 0 \) for every \( x \in I \).

If we define \( y = y_1(x) \), then

\[
y' = u y_1' + y_1 u' \quad \text{and} \quad y'' = u y_1'' + 2 y_1' u' + y_1 u''
\]

Thus

\[
y'' + p y' + q y = u y_1'' + 2 y_1' u' + y_1 u'' + p (u y_1' + y_1 u') + q y
\]

\[
= u [y_1'' + p y_1' + q y_1] + y_1 u'' + (2 y_1' + p y_1) u'
\]

\[
= y_1 u'' + (2 y_1' + p y_1) u'.
\]

\( \Rightarrow \)

\[
y_1 u'' + (2 y_1' + p y_1) u' = 0 \quad \text{or}
\]

\[
y_1 w' + (2 y_1' + p y_1) w = 0 \quad (w = u'). \quad (4)
\]

The eq (4) are both linear and separable.

\( \Rightarrow \)

\[
\frac{d w}{w} + 2 \frac{y_1'}{y_1} \, dx + p \, dx = 0
\]

\[
\ln |w| + 2 \ln |y_1| = - \int p \, dx + C
\]

\( \Rightarrow \)

\[
\ln |w y_1^2| = - \int p \, dx + C
\]

or

\[
w y_1^2 = C_1 e^{- \int p \, dx}.
\]
Since \( w = u' \), we have
\[
U = c_1 \int \frac{-f p d x}{y_1^2} \, d x + c_2.
\]

By choosing \( c_1 = 1, \ c_2 = 0 \) we have
\[
y_2 = y_1(x) \int \frac{e^{-\int p(x) \, d x}}{y_1(x)} \, d x. \quad (5)
\]

**Example:** \( y_1 = x^2 \) is a solution of
\[
x^2 y'' - 3x y' + 4 y = 0 \quad (6)
\]
Find the general solution of the equation \((0, \infty)\).

**Solution:**
\[(6) \iff y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0\]

By (2), \( y_2 = x^2 \int \frac{e^{\int p(x) \, d x}}{x^4} \, d x \)
\[
= x^2 \int \frac{e^{\int p(x) \, d x}}{x^4} \, d x
= x^2 \int \frac{e^{\int 3 \, d x}}{x^4} \, d x
= x^2 \int \frac{e^{3 \ln x}}{x^4} \, d x
= x^2 \int \frac{x^3 \, d x}{x^4}
= x^2 \int \frac{d x}{x}
= x^2 \ln x.
\]
The general solution is
\[
y = c_1 y_1(x) + (x^2 \ln x) = c_1 x^2 + c_2 x^2 \ln x. \quad \square
\]