Section 2.6. A Numerical Method

There are some differential equations that are very hard to find an exact solution. For example, the equation

$$y' = 0.1\sqrt{y} + 0.4x^2, \quad y(2)=4$$

is a nonliear and nonseparable equation. Our methods in previous seetions do not work in this case. However we can still find approximate numerical values of the unknown $y(x)$.

We can use the direction field with lineal passing the grid of integer coordinates.
As the solution curve passes the point $(2,4)$, the lineal element at this point is the tangent with slope given by

$$f(2,4) = 0.1\sqrt{4} + 0.4(2)^2 = 1.8.$$

The equation of the tangent line is

$$y = L(x) = 1.8 \cdot x + 0.4.$$

This line is call the **linearization** of $y(x)$ at $x=2$. When x_1 is close to 2, then

$$y(x_1) \approx L(x_1)$$

For example

$$y(2.1) \approx L(2.1) = 1.8\cdot(2.1) + 0.4$$
$$= 4.18.$$

We will generalize this process in the following method.

EULER's METHOD

Consider the equation

$$y' = f(x,y) \quad , \quad y(x_0) = x_0 \qquad (1)$$

We use the linearization of (1) at $x = x_0$ as

$$L(x) = y_0 + f(x_0,y_0)(x-x_0) \qquad (2)$$

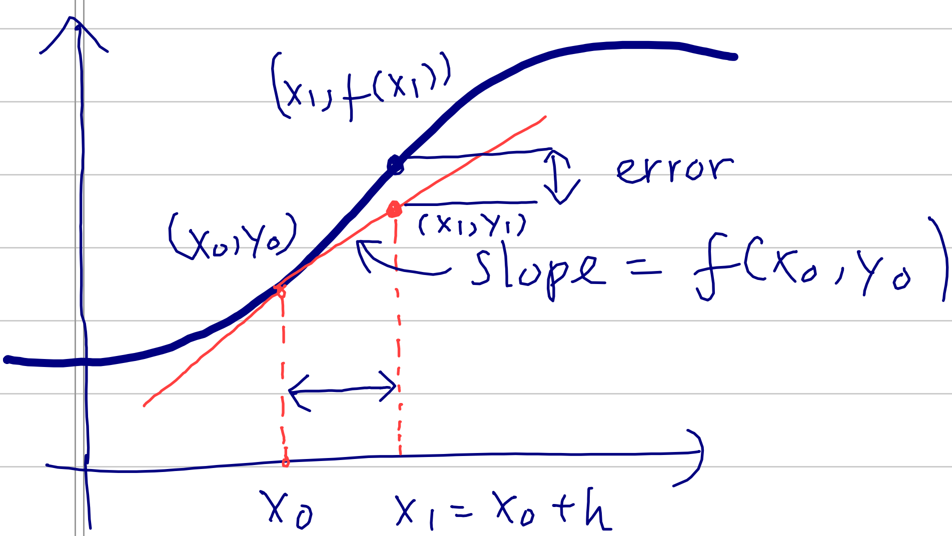
($L(x)$ passes (x_0, y_0) and have slope $= f(x_0,y_0)$ it means $L(x)$ is the tangent of the graph of $y(x)$ at (x_0,y_0))

Now let h be a positive 'increment' of the x-axis. Then by replacing x by $x_1 = x_0 + h$ in (2) we get

$$L(x_1) = y_0 + f(x_0,y_0)(x_0+h-x_0)$$

$\Leftrightarrow$

$$y_1 = y_0 + f(x_0,y_0)\, h, \text{ where } y_1 = L(x_1).$$



The point (x_1, y_1) on the tangent line $L(x)$ is an approx. of $(x_1, y(x_1))$ on the solution curve.

The accuracy of the appox. $L(x_1) \approx y(x_1)$ or $y_1 \approx y(x_1)$ depend on the size of h. We usually pick the "**step size**" h to be reasonably small.
We now repeat the process using a second "tangent line" at (x_1, y_1).
Let $x_2 = x_1 + h = x_0 + 2h$, and

$$y(x_2) = y(x_0 + 2h) = y(x_1 + h)$$
$$\approx y_2 = y_1 + h f(x_1, y_1).$$

Continuing in this manner, we see that $y_1, y_2, \ldots,$ can be define recursively by the general formular

$$y_{n+1} = y_n + h f(x_n, y_n),$$

where $x_n = x_0 + nh$.
This process using succesive "tangent lines" is called Euler's method.

<u>Example</u>: Consider the IVP $y' = 0.1\sqrt{y} + 0.4x^2$
$y(2) = 4$.
Use Euler's method to obtain an approx $y(2.5)$ using first $h = 0.1$, $h = 0.05$, $h = 0.01$.

<u>Solution</u>: $y_{n+1} = y_n + h(0.1\sqrt{y_n} + 0.4x_n^2)$.

(Show on slide the tables)

Example 2. Consider the IVP

$$y' = 0.2\,x\,y \;, \quad y(1)=1.$$

Use Euler's method to approximate $y=(1.5)$ using $h=0.1$, $h=0.05$, $h=0.01$.

"Use computer to give tables here".

The exact solution is $y = e^{0.1(x^2-1)}$

$\Rightarrow$ compare the values found above with the actual values.

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