Chapter 5: Partitions

1. Compositions.

**Defn.** A sequence \((a_1, a_2, \ldots, a_k)\) of nonnegative integers s.t. \(a_1 + a_2 + \ldots + a_k = n\) is called a **weak composition** of \(n\).

If, in addition, the \(a_i\) are positive, then the sequence \((a_1, a_2, \ldots, a_k)\) is called a **composition** of \(n\).

**Example 5.1.** The number of weak compositions of \(n\) into \(k\) parts is

\[
\binom{n+k-1}{k-1} = \binom{n+k-1}{n}
\]

**Proof.**

By adding 1 to each \(a_i\), we have a bijection between weak compositions of \(n\) and compositions of \(n+k\).

We will count the latter. The number of composition of \(n+k\) is exactly the way to divide a row of \(n+k\) identical balls into \(k\) nonempty parts.

We can divide this row of balls by using \(k-1\) sticks. These stick can be placed at \(n+k-1\)
gap between two consecutive balls. Finally, there are \( \binom{n+k-1}{k-1} \) ways to place \( k-1 \) sticks in \( n+k-1 \) gaps.

**Corollary:** The number of compositions of \( n \) into \( k \) parts is \( \binom{n-1}{k-1} \).

**Corollary:** The number of all compositions of \( n \) is \( 2^{n-1} \).

**Solution:** \[ \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1} \]

2. **Set Partitions**

**Defn.** A "partition" of the set \([n]\) is a collection of non-empty subsets so that each element of \([n]\) belongs to exactly one of the subsets.

The number of partitions of \([n]\) into \( k \) non-empty subsets is denoted by \( S(n,k) \). The number \( S(n,k) \) is called the **Stirling number of the second kind**.

**Example 5.2.** \( S(n,k) = S(n-1, k-1) + k \cdot S(n-1, k) \)

**Proof.** We classify the set of partitions of
n into K subsets.

1) One subset is \$\{n\}\$.
2) There no subset \$\{n\}\$.

Type one has \$S(n-1, k-1)\$ partitions.

For each partition of type 2, \$n\$ must be in one of \$k\$ subsets of a partition of \$[n-1]\$ into \$k\$ parts.

\[ \Rightarrow \text{Type 2 has } K \times S(n-1, k) \text{ partitions.} \]

**Example 5.3.** Find the number of surjection \( f: [n] \rightarrow [k] \).

**Solution.** Let \( f \) be a surjective \( [n] \rightarrow [k] \)

\[ \Rightarrow (f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(k)) \] form a ordered partition of \$[n]\$ into \$k\$ parts.

Since each non-ordered partition of \$n\$ into \$k\$ parts can generate \( k! \) ordered partitions (by labelling the \$k\$ subsets), the number of ordered partitions is \( k! \times S(n, k) \).

**Example 5.4.** Prove that for all real numbers \( x \)

and all non-negative integers \( n \)

\[ x^n = \sum_{k=0}^{n} S(n, k) \cdot (x)_k \]

where \( (x)_k = x(x-1) \ldots (x-k+1) \).
Proof. Both sides are polynomials of \( x \) with degree \( n \). We will show that the two polynomials agree for all positive integer \( x \).

\( \text{LHS is the number of all function } [n] \to [x] \) 
\( \text{We NTS that the RHS is the same:} \)

We consider the set \( f([n]) = I \).

If \( |I| = k \), then there are \( \binom{x}{k} \) choices for the image \( I \), then by the previous example there are \( k! \cdot S(n, k) \) choices for the surjective function \( f \).

As \( \binom{x}{k} = k! \cdot \binom{x}{k} \), we have the number of all function \( f : [n] \to [x] \) is \( \sum_{k=1}^{n} \binom{x}{k} S(n, k) \). \( \square \)

**Defn.** The number of all partitions of \([n]\) into non-empty subsets is denoted by \( B(n) \), and called the \( n \)th Bell number. We also set \( B(0) = 1 \).

Thus

\[ B(n) = \sum_{i=0}^{n} S(n, i) \]

**Example 5.5.** Prove that

\[ B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i) \]

**Proof.** We need to show that RHS counts all partitions of \([n+1]\).
Assume that the element \( a_{i+1} \) is in a subset of size \( n-i+1 \). Then there are \( i \) elements that are not in the same subset as \( a_{i+1} \). Therefore, there are \( (^{n-i}_{i}) \) ways to choose such \( i \) elements, and there are \( B(i) \) ways to partition them.

Summing over all possible values of \( i \), we are done.

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3. **Integer Partitions.**

**Defn.** Let \( a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_k \geq 1 \) be integers such that \( a_1 + a_2 + \ldots + a_k = n \).

Then the sequence \( (a_1, a_2, \ldots, a_k) \) is called a **partition** of the integer \( n \).

The number of all partitions of \( n \) is denoted by \( p(n) \).

The number of all partitions of \( n \) into \( k \) parts is denoted by \( p_k(n) \).

**Example:** \( p(5) = 7 \):

\( (5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1) \), and \( (1,1,1,1,1) \).

We have

\[
p(n) \sim \frac{1}{4 \sqrt{3}} e^{\frac{\pi \sqrt{2n}}{3}}.
\]
Defn. A ‘Ferrers diagram’ (shape) of a partition \( p = (x_1, x_2, \ldots, x_k) \) is a set of \( n \) square boxes forming \( k \) rows, such that all rows start at the same vertical line and row \( i \) has \( x_i \) boxes.

Example:

\[
(3,2) = \begin{array}{ccc}
1 & 1 & 1 \\
\end{array}
\]

\[
(2,2,1) = \begin{array}{ccc}
1 & 1 \\
1 & 1 \\
\end{array}
\]

Defn. If we reflect a Ferrers diagram of a partition \( p \) with respect to its main diagonal, we obtain a Ferrers diagram of another partition \( p' \). We call \( p' \) the \textbf{conjugate partition} of \( p \).

Example:

\[
p = (3,2) \quad \rightarrow \quad \begin{array}{ccc}
1 & 1 & 1 \\
\end{array} = p' = (2,2,1)
\]

Defn. A partition of \( n \) is called \textbf{self-conjugate} if it equals its conjugate.

\[
(3,2,1) = \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]
Example 5.6. The number of partitions of $n$ into at most $K$ parts is equal to that of partitions of $n$ into parts not larger than $K$.

Proof.

Let $\mu$ be a partition of $n$ into at most $K$ parts. The first column in a Ferrers diagram of $\mu$ is at most $K$.

Consider the conjugate $\mu'$ of $\mu$. The Ferrers diagram of $\mu'$ has the first row at most $K$. This means the conjugation gives a bijection between two sets of partitions.

Example 5.7. The number of partitions of $n$ into $m$ distinct odd parts is equal to that of all self-conjugate partitions of $n$.

Proof.

Consider the following bijection:

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  self-conjugate  distinct odd parts
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□
Example 5.8. Let \( q(n) \) be the number of partitions of \( n \) in which each part is at least 2. Prove that \( q(n) = p(n) - p(n-1) \), for all \( n \geq 2 \).

Proof: We need to show

\[
p(n) - q(n) = p(n-1)
\]

# partitions of \( n \)  
in which there is a part 1

# partitions of \( n-1 \)

bijection: remove a part 1

Example 5.4. Denote by \( p_{k,l}(n) \) the number of partitions of \( n \) into at most \( K \) parts, and all parts are at most \( L \). Prove that

\[
p_{k,l}(n) = p_{k-1,l}(n) + p_{k,l-1}(n-K).
\]

Hint:

+ \( \leq p \)  
+ \( \leq l-1 \)

\[
\begin{array}{c}
\text{Hint:} \\
\end{array}
\]
Example 5.10. Prove that

\[ P_{k, \ell}(n) = \binom{k+\ell}{\ell} \]

Proof: We prove by induction on \( k+1 \).

\[ P_{k, \ell}(n) = P_{k-1, \ell}(n) + P_{k, \ell-1}(n-k) \]

By induction hypothesis the R.H.S is equal to

\[ \binom{k+\ell-1}{\ell} + \binom{k+\ell-1}{\ell-1} = \binom{k+\ell}{\ell} \]

Example 5.11. Prove that the number of partitions of \( n \) into \( K \) distinct parts is equal to the number of partitions of \( n - \frac{K(K+1)}{2} \) into at most \( K \) parts.

Hint:

\[ \begin{array}{c}
\includegraphics{example.png}
\end{array} \]

Connection between two types of partitions

Let \( \Pi = \{ \pi_1, \pi_2, \ldots, \pi_K \} \) be a set partition of \( \{n\} \). If we arrange the sequence of subset sizes \( |\pi_1|, |\pi_2|, \ldots, |\pi_K| \) in nonincreasing
order to get a seq. $\lambda_1, \lambda_2, \cdots, \lambda_k$. Then $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ is an integer partition of $n$. We say $\lambda$ is the type of the set partition $\pi$.

**Theorem.** Let $\lambda = (\lambda_1, \cdots, \lambda_k)$ be an integer partition of $n$, in which part $i$ appear $m_i$ times. Then the number of set partitions of type $\lambda$ is

$$\frac{n!}{(\lambda_1, \lambda_2, \cdots, \lambda_k)}$$

$$m_1! \cdot m_2! \cdots m_j!$$

**Proof:** Take $\lambda_i$ balls of color $i$ ($i \in [k]$). Arrange them in a row by $n!$ different ways. Consider the following map between these linear orderings and set partitions of $[n]$ with type $\lambda$.

For each ordering $\pi$ we define a set partition $\Pi = f(\pi)$ as follows: Two numbers $i$ and $j$ are in the same subset if the balls at positions $i$ and $j$ in the ordering $\pi$ have the same color.

It is easy to see that $f$ is onto. However, it is not 1-1. Each set partition of type $\lambda$ corresponds to exactly $\Pi_{m_1!}$.

(?)
Indeed, for example if $\lambda_1 = \lambda_2$, then having $\lambda_1$ balls of color 1 in a subset $A$ of positions, and having $\lambda_1$ balls of color 2 in a subset $B$ of positions will result in the same partition as having $\lambda_1$ balls of color 2 in $A$ and $\lambda_1$ balls of color 2 in $B$.

In general, if $m_i$ is the multiplicity of $i$ in $\lambda$, then there are $m_i!$ ways the $m_i$ color classes having $i$ balls each can be permuted any each other.

Thus, \# of set partitions of type $\lambda$ is

$$\frac{1}{\prod m_i!} \times \# \text{ of linear orderings.} \left( \begin{array}{c} n \\ a_1, a_2, \ldots, a_{\lambda_\kappa} \end{array} \right) \frac{m_1! m_2! \cdots m_{\lambda_\kappa}!}{m_1! m_2! \cdots m_{\lambda_\kappa}!} \square$$

**Theorem (Euler)**

The number of partitions of $n$ into odd parts is equal to the number of partitions of $n$ into distinct parts.

**Proof (Glaisher)**

Let $\lambda = (1^{m_1} 3^{m_2} 5^{m_3} \ldots)$ be a partition of $n$ into odd parts. For every $i$, let $P_i(\lambda)$ contain part $i$ if $m_i$ written in binary has 1 at the $r$-th position.
Define \( \varphi: D_n \to O_n \) as follows. Start with \( m = (\mu_1, \mu_2, \ldots) \in D_n \). Substitute every even part \( \mu_i \) by two parts \( \mu_i/2 \). Repeat until the resulting partition \( \lambda \) has no even part. Set \( \varphi(m) = \lambda \).

Exercise: check that \( \varphi \) is well defined, and \( \varphi^{-1} = \varphi \).

**Example 5.12.** Euler used the following recurrence to calculate \( p(n) \):

- Pentagonal number is an integer of the form \( \frac{k(3k+1)}{2} \). Example: 0, 1, 2, 5, 7, 12, 15, 22, 26, 35, ...

\[
p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - p(n-22) - p(n-26) + \ldots
\]

\[
= \sum_{m=1}^{\infty} (-1)^{m-1} \left[ p(n - \frac{m(3m-1)}{2}) + p(n - \frac{m(3m+1)}{2}) \right].
\]

This is called "Euler's Pentagonal Theorem".

\[
\frac{3n^2-n}{2}
\]
Example 5.13. (Rojers-Ramanujam)

The number of partitions of \( n \) into parts \( \equiv \pm 1 \pmod{5} \) is equal to the number of partitions of \( n \) into parts which differ by at least 2.

\[
\left( \sum_{k=1}^{\infty} \frac{q^k}{(1-q)(1-q^2)\cdots(1-q^k)} \right)^{\infty} = \prod_{i=0}^{\infty} \frac{1}{(1-q^{5i+1})(1-q^{5i+4})}
\]

People are still seeking for a direct bijective proof of the identity.

Note that the theorem should be named as Rojers-Ramanujan-Schur Theorem. Only Rojers and Schur gave proof for this beautiful formula. Ramanujan came up with the identity but couldn't give a proof. However, Hardy (a Ramanujan's close friend) named the theorem after Rojers and Ramanujan.