NOTES ON DIFF. EQUATIONS

Section 1.1. Definitions and Terminology

**Definition.** An equation containing the derivatives of one or more functions (or dependent variables), with respect to one or more independent variables, is said to be a **differential equation**.

Example 0: \( \frac{dy}{dx} = 0.2xy \).

**Classification.** We classify differential equations according to **type**, **order**, and **linearity**.

**Classification by Type**

If a differential equation contains only ordinary derivatives of one or more unknown functions w.r.t. a single independent variable, it is called an **ordinary differential equation (ODE)**.

An eq. involving partial derivatives of one or more unknown functions of two or more independent variables is said to be a **partial differential equation (PDE)**.
Example 1.

(a) \( \frac{dy}{dx} + 8y^2 = e^{x+5} \), \( \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 4 \),

and

\( \frac{dy}{dx} + \frac{dz}{dx} = 2z \cdot y \)

are ODEs.

(b) \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \), \( \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y^2} \), and

\( \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} \)

are PDEs.

We can use **Leibniz notation** \( \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots \)

or **prime notation** \( y', y'', y''', \ldots \) for ordinary derivatives.

In general, the \( n \)-th derivative of \( y \) is written by

\( \frac{d^ny}{dx^n} \) OR \( y^{(n)} \).

In physics, people may use **Newton’s dot notation**, for example the second derivative of \( y \) is written as \( \ddot{y} \).

Partial derivatives are often denoted by a **subscript notation**. \( \partial u/\partial x \equiv u_x \), \( \partial^2 u/\partial x^2 \equiv u_{xx} \).
CLASSIFICATION BY ORDER.

The order of a differential equation (either ODE or PDE) is the order of the highest derivative in the equation. For example

1. \[ \frac{d^3 y}{dx^3} - 6 \left( \frac{dy}{dx} \right)^4 + 4 y x = \sin x \]

   second order     first order

is a third-order ODE.

2. \[ \frac{\partial^3 u}{\partial x \partial y^2} + \left( \frac{\partial u}{\partial z} \right)^5 = \frac{\partial^2 u}{\partial x \partial z} \]

is a third-order PDE.

A first-order ODE is sometimes written in the differential form

\[ M(x,y) \, dx + N(x,y) \, dy = 0. \]

Example 2. If we assume that \( y \) is the dependent variable in a first-order ODE, then we can write \( dy = y' \, dx \).

a) By dividing the differential \( dx \), we have
(y + 3x) \, dx + 4x^2 \, dy = 0 \, \text{ is equivalent to } \\
y + 3x + 4x^2 \frac{dy}{dx} = 0.

(b) By multiplying the equation 

\[ 7x^2 \, \gamma \frac{dy}{dx} + x^3 - \gamma = 0 \]

by \( dx \), we get the alternative differential form 

\[ (x^3 - \gamma) \, dx + 7x^2 \, \gamma \, dy = 0. \]

In symbols we can express an \( n \)-th-order ODE in one independent variable by the general form

\[ F(x, y, y', \ldots, y^{(n)}) = 0, \quad (*) \]

where \( F \) is a real-valued function of \( n+2 \) variables. From now on, we always assume that it is possible to solve the above equation uniquely for the highest derivatives \( y^{(n)} \) in terms of the remaining \( n+1 \) variables as

\[ \frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \quad (**) \]

The equation (**) is referred to as the normal form of (*)

We shall use the normal forms

\[ \frac{dy}{dx} = f(x, y) \quad \text{and} \quad \frac{d^2 y}{dx^2} = f(x, y, y') \]
to represent general first- and second-order ODEs.

**Example 3.**

(a) \[ 4x \frac{dy}{dx} + y^2 = 5x \]

has the normal form \[ \frac{dy}{dx} = \frac{5x - y^2}{4x} \]

(b) \[ y'' - y' + 8y = 0 \] has the normal form \[ y'' = y' - 8y. \]

**Classification by Linearity.**

An \( n \)-th order ODE \[ F(x, y, y', \ldots, y^{(n)}) = 0 \] is said to be **linear** if \( F \) is linear in \( y, y', \ldots, y^{(n)} \).

It means that an \( n \)-th order ODE is linear when it has form

\[ a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \ldots + a_0(x) y - g(x) = 0 \]

or

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_0(x) y = g(x) \]

**Obs**

- All variables \( y, y', \ldots, y^{(n)} \) are of the first degree.
- The coefficients \( a_0, a_1, \ldots, a_n \) depend at most on the variable \( x \).
A nonlinear ODE is simply the one that is not linear.

**Example 4:**
(a) The equations
\[(y-x)\,dx + 4x\,dy = 0\]
\[y'' - 2y' + y = 0\]
\[x^3 \frac{d^3y}{dx^3} + x \frac{dy}{dx} + 5y = \sin x\]
are linear. (Why?)

(b) The equations
\[(1-y)\,y' + 2y^2 = 4x\]
\[\frac{d^2y}{dx^2} + e^y = 0\]
\[\frac{d^5y}{dx^5} + y^3 = 0\]
are nonlinear. (Why?)

**Solutions.**

**Definition.** Any function \( \phi \), defined on an interval \( I \) and possessing at least \( n \) derivatives that are continuous on \( I \), which, when substituted into an \( n \)th-order ODE reduces the equation to an identity, is said
to be a solution of the equation on the interval $I$.

In other words, a solution of an nth-order ordinary diff. equation $F(x, y, y', \ldots, y^{(n)}) = 0$ is a function $\phi$ that possesses at least $n$ derivatives and for which

$$F(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0$$

for all $x$ in $I$. We say that $\phi$ "satisfies" the diff. equation on $I$. For our purposes, we always assume that a solution $\phi$ is a real-valued function.

**Interval of definition.**

We cannot think "solution" of an ODE without simultaneously thinking "interval". The interval $I$ in the definition of the solution is called the interval of defn, the interval of existence, the interval of validity, or the domain of the solution and can be an open interval $(a, b)$, a closed interval $[a, b]$, an infinite interval $(a, \infty)$, and so on.

**Example 5.** Verify that

(a) $\frac{dy}{dx} = x y^{\frac{1}{2}}$ has a solution $y = \frac{1}{16} x^4$ on $(-\infty, \infty)$

(b) $y'' - 2y' + y = 0$ has a solution $y = xe^x$ on $(-\infty, \infty)$. \(\square\)
Solution Curve.

The graph of a solution \( \phi \) of an ODE is called a solution curve.

Since \( \phi \) is differentiable on \( I \), \( \phi \) is continuous on \( I \). The domain of the function \( \phi \) may be different from \( I \).

Example 6:

(a) The function \( y = \frac{1}{x} \) is defined for any \( x \neq 0 \)
\( D = \mathbb{R} \setminus \{0\} \)

(b) \( y = \frac{1}{x} \) is a solution of the equation
\[ xy' + y = 0 \]
on any interval that does not contain 0, such as \((-\infty, -1), (-1, 0), (0, \infty)\). But it is not the whole domain \( \mathbb{R} \setminus \{0\} \).

Explicit and Implicit Solutions.

A solution in which the dependent variable is express solely in terms of the independent variable and constants is said to be an explicit solution.

A relation \( G(x, y) = 0 \) is said to be an implicit solution of an ODE on an interval \( I \), provided that there exists at least one function \( \phi \) that satisfies the relation as well as the ODE on \( I \).
Example 7.

The relation \( x^2 + y^2 = 25 \) is an implicit solution of the diff eq.

\[
\frac{dy}{dx} = -\frac{x}{y}
\]
on \((-5, 5)\).

And \( y = \phi_1(x) = \sqrt{25-x^2} \) and \( y = \phi_2(x) = -\sqrt{25-x^2} \) are explicit solutions of the diff eq on \((-5, 5)\). □

Families of Solutions.

Similar to the study of integral calculus, when solving a diff equation \( F(x, y, y') = 0 \), we usually obtain a solution containing a constant parameter \( c \). A solution of \( F(x, y, y') = 0 \) containing a constant \( c \) is actually a set or a family of solutions \( G(x, y, c) = 0 \) called a one-parameter family of solutions. Similarly, when solving an \( n \)-th-order diff eq. \( F(x, y, y', \ldots, y^{(n)}) = 0 \), we seek an \( n \)-parameter family of solutions \( G(x, y, c_1, c_2, \ldots, c_n) = 0 \).

A solution of a diff. eq. that are free of parameters is called a particular solution.

Example 8:

(a) For all real values of \( c \), the one-parameter family \( y = cx - x \cos x \) is an explicit solution of \( xy' - y^2 = x^2 \sin x \).
(b) The two-parameter family \( y = c_1 e^x + c_2 x e^x \) is an explicit solution of
\[
y'' - 2y' + y = 0. \quad \Box
\]
Sometimes a diff eq possesses a solution that is not a member of a family of solutions. Such an extra solution is called a singular solution.
For example, \( y = \frac{1}{16} x^4 \) and \( y = 0 \) are solutions of eq. \( \frac{dy}{dx} = xy^{1/2} \) on \((-\infty, \infty)\). The first one is not singular, since it is a member of the family of solutions \( y = (\frac{1}{4} x^2 + c)^2 \). However, \( y = 0 \) is a singular solution.

Example 9. Consider the diff eq.
\[
y' + 16y = 0.
\]
It has a family of solutions \( y = c_1 \cos 4x + c_2 \sin 4x \).

Example 10. \( y = c x^4 \) is a solution of
\[
xy' - 4y = 0.
\]
We can define a new solution
\[
y = \begin{cases} -x^4, & x < 0 \\ x^4, & x > 0 \end{cases}. \quad \Box
\]
We can similarly work on systems of diff eq's.
For example

\[
\frac{dx}{dt} = f (t, x, y)
\]

\[
\frac{dy}{dt} = g (t, x, y)
\]