Section 1.2. Initial-Value Problems

We are interested in problems in which we seek a solution $y(x)$ of a diff. eq. so that $y(x)$ also satisfies certain prescirbed side conditions, that is, conditions that are imposed on the unknown function $y(x)$ and its derivatives at a number x_0.

The general form of an nth-order diff. eq. subject to n side conditions specified at x_0:

Solve: $$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \qquad (1)$$

Subject to: $$y(x_0) = y_0,\ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrary constant. This is called an **nth-order initial-value problem (IVP)**. The values of y and its first $n-1$ derivatives at x_0, $y_0, y_1, \ldots, y_{n-1}$, are called **initial condition (IC)**.

GEOMETRIC INTERPRETATION.

The case $n=1$ and $n=2$ in (1),

Solve: $$\frac{dy}{dx} = f(x, y)$$

Subject to: $$y(x_0) = y_0 \qquad (2)$$

and

Solve: $$d^2y/dx^2 = f(x, y, y')$$

Subject to: $$y(x_0) = y_0,\ y'(x_0) = y_1. \qquad (3)$$

are examples of first- and second-order initial-value problems.

For (2) we are finding a solution of the eq $y' = f(x, y)$ on an interval I containing x_0 so that its graph passes through the specific point (x_0, y_0).

For (3) we would like to find a solution $y(x)$ of $y'' = f(x, y, y')$ s.t. it graph not only passes through (x_0, y_0) but the slope of the curve at this point is also the number y_1.

Example 1.

(a) $y = ce^x$ is a one-parameter family of solutions of the equation $y' = y$. If we impose an initial condition $y(0) = 4$, then substituting $x = 0$, $y = 4$ in the family determines the constant $4 = ce^0 = c$. Thus $y = 4e^x$ is a solution of the IVP

$$y' = y, \quad y(0) = 4.$$

(b) Now if we demand that a solution curve pass through the point $(2, 5)$ rather than $(0, 4)$, then $y(2) = 5$ yields $5 = ce^2$, or $c = 5e^{-2}$. Thus $y = -2e^{x-2}$ is a solution of the IVP:

$$y' = y, \quad y(2) = 5.$$

□

The interval I of definition of the solution depends on the initial condition.

Example 2. The first-order diff equation

$$y' + 2xy^2 = 0$$

has a solution $y = \dfrac{1}{x^2+c}$.

If we impose the initial condition $y(0) = -1$, then $c = -1$. Thus $y = 1/(x^2-1)$.

The function $y = \dfrac{1}{x^2-1}$ has domain $\mathbb{R} - \{\pm 1\}$. So the interval I of definition of the solution $y = \dfrac{1}{x^2-1}$ is an interval containing 0 in which $y(x)$ is defined and differentiable.

$\Rightarrow$ The largest such interval is $(-1, 1)$. $\square$

Example 3. The second-order diff. eq.

$$y'' + 16y = 0$$

has a two-parameter family of solutions

$$y = c_1 \cos 4x + c_2 \sin 4x.$$

If we demand the initial condition

$$y\left(\tfrac{\pi}{2}\right) = -3, \quad y'\left(\tfrac{\pi}{2}\right) = 4,$$

then we have

$$c_1 \cos 2\pi + c_2 \sin 2\pi = -3$$

and

$$-4c_1 \sin 2\pi + 4c_2 \cos 2\pi = 4$$

Since $\cos 2\pi = 1$ and $\sin 2\pi = 0$, $c_1 = -3$ and $c_2 = 1$. Hence $y = -3\cos 4x + \sin 4x$ is a sol of the IVP

$\square$

Existence and Uniqueness.

Two fundamental questions in the study of IVP are

① Does a solution of the problem exist?

② If a solution exists, is it unique?

Example 4. Each of the functions $y=0$ and $y=\frac{1}{16}x^4$ satisfies the diff. eq. $\frac{dy}{dx} = xy^{1/2}$ and the initial condition $y(0)=0$.

However, if we change the initial condition, say $y(0)=1$, then the above functions are not solutions of the IVP anymore.

Theorem. Existence of a Unique Solution

Let R be a rectangular region defined by $a \le x \le b$, $c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x,y)$ and $\partial f/\partial y$ are continuous on R, then there exists an interval $I_0: (x_0-h, x_0+h)$, $h>0$, contained in $[a,b]$, and a unique function $y(x)$, defined on I_0, that is a solution of the IVP:

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

Example 5. Consider the IVP in Example 4:

$$\frac{dy}{dx} = f(x,y) = x y^{1/2}, \quad y(0) = 0.$$

We have $\frac{\partial f}{\partial y} = \frac{x}{2y^{1/2}}$ and $f(x,y) = x y^{1/2}$
are continuous on the upper half-plane defined by $y > 0$. Thus, the theorem tells us that through any point (x_0, y_0), $y_0 > 0$, there is an interval centered at x_0 on which the given diff. eq. has a unique solution.

Thus, for example, even without