Enumeration of lozenge tilings of a hexagon with holes on boundary

Tri Lai

Institute for Mathematics and its Applications
University of Minnesota
Minneapolis, MN 55455

February 3, 2015

Outline of talk

1. Definition and classical results
2. Main result
3. Proof of the main result
4. A q-analog

Plane partitions

Definition

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ be positive integers. A plane partition of shape $(\lambda_1, \lambda_2, \dots, \lambda_k)$ is an array of non-negative integers

$$\begin{array}{ccccccc} n_{1,1} & n_{1,2} & n_{1,3} & \dots & \dots & \dots & n_{1,\lambda_1} \\ n_{2,1} & n_{2,2} & n_{2,3} & \dots & \dots & n_{2,\lambda_2} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ n_{k,1} & n_{k,2} & n_{k,3} & \dots & n_{k,\lambda_k} & & \end{array}$$

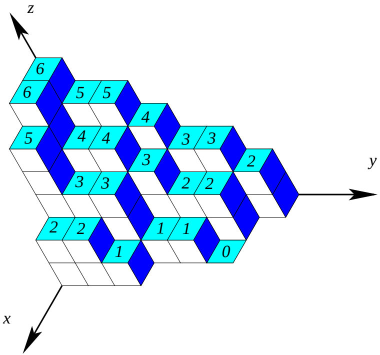
so that $n_{i,j} \geq n_{i,j+1}$ and $n_{i,j} \geq n_{i+1,j}$ (i.e. all rows and columns are weakly decreasing).

Plane partitions

Example: A plane partition of shape $(7, 6, 6, 3)$.

6	5	5	4	3	3	2
6	4	4	3	2	2	
5	3	3	1	1	0	
2	2	1				

3-D interpretation of plane partitions



The heights of columns are weakly decreasing along Ox and Oy.

MacMahon's Theorem

Theorem (MacMahon)

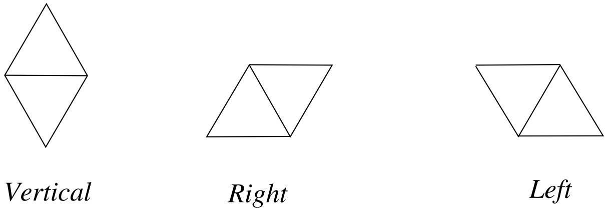
The number of plane partitions of rectangular shape $(\underbrace{b, b, \dots, b}_{a})$ *with entries at most* c *is equal to*

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}=\frac{\mathbf{H}(a)\,\mathbf{H}(b)\,\mathbf{H}(c)\,\mathbf{H}(a+b+c)}{\mathbf{H}(a+b)\,\mathbf{H}(b+c)\,\mathbf{H}(c+a)},$$

where hyperfactorial $\mathbf{H}(n) := 0!1!2!\dots(n-1)!$.

The plane partition is usually identified with its 3-D interpretation. So the plane partitions of rectangular shape $a \times b$ with the entries at most c is called "the plane partitions fitting in an $a \times b \times c$ box".

Lozenge tilings of a semi-regular hexagon

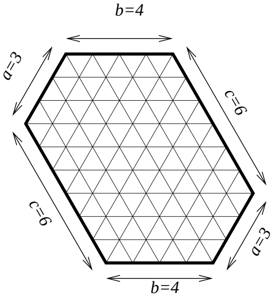


- A unit rhombus (or lozenge) is the union of two adjacent unit equilateral triangles.
- A lozenge tiling of a region R on the triangular lattice is a covering of the region by unit rhombi (or lozenge) so that there are no gaps or overlaps.
- Denote by $\mathbf{M}(R)$ the number of lozenge tilings of the region R.

Semi-regular hexagons

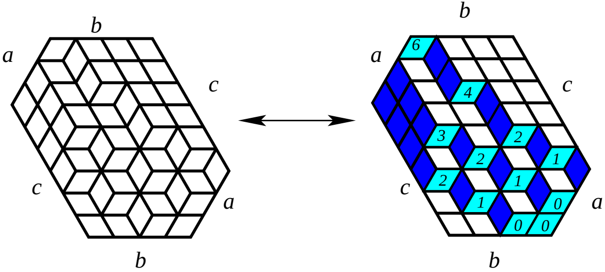
A semi-regular hexagon on the triangular lattice is a centrally symmetric hexagon of side-lengths a, b, c, a, b, c (in cyclic order, starting from the northwest side). Denote by $Hex(a, b, c)$ the hexagon.

Q: How many different lozenge tilings does the hexagon $Hex(a, b, c)$ **have?**



Lozenge tilings and plane partition

6	4	2	1
3	2	1	0
2	1	0	0



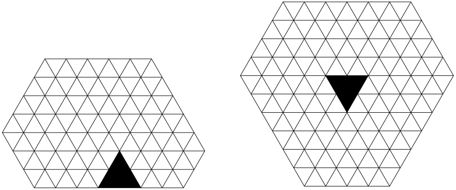
Give three different colors to three types of rhombi: vertical, left, and right.

Lozenge tilings and plane partition

Theorem (MacMahon)

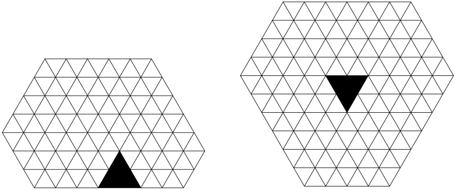
$$\mathbf{M}(Hex(a,b,c)) = \frac{\mathbf{H}(a)\,\mathbf{H}(b)\,\mathbf{H}(c)\,\mathbf{H}(a+b+c)}{\mathbf{H}(a+b)\,\mathbf{H}(b+c)\,\mathbf{H}(c+a)}.$$

Enumeration of lozenge tilings of a hexagon with holes



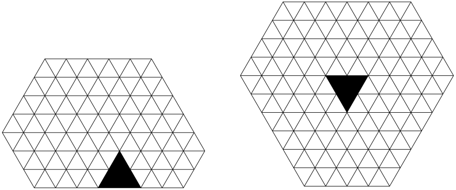
- Cohn-Larsen-Propp (1998): Hexagon with triangular a hole on the base.

Enumeration of lozenge tilings of a hexagon with holes



- Cohn-Larsen-Propp (1998): Hexagon with triangular a hole on the base.
- Ciucu-Eisenkölbl-Krattenthaler-Zare (2001): Hexagon with a triangular hole at the "center".

Enumeration of lozenge tilings of a hexagon with holes



- Cohn-Larsen-Propp (1998): Hexagon with triangular a hole on the base.
- Ciucu-Eisenkölbl-Krattenthaler-Zare (2001): Hexagon with a triangular hole at the "center".
- Ciucu-Krattenthaler (2013): Hexagon with "shamrock" hole at the "center".

A shamrock.

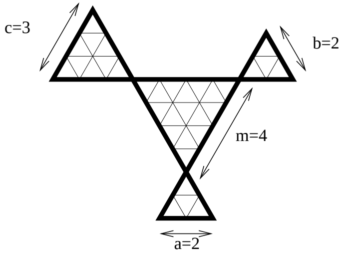


Figure : The shamrock $S(m, a, b, c)$.

Shamrock removed at center

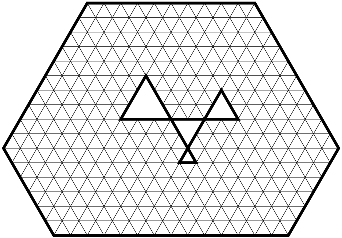


Figure : A hexagon with a shamrock $S(2, 1, 2, 3)$ removed at the center.

A question

What is the number of the lozenge tilings of a hexagon when a shamrock hole appears on boundary?

Shamrock removed along the boundary

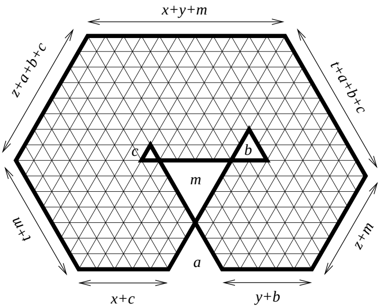


Figure : A hexagon with shamrock removed along the boundary.

$$Q\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$$

Main result

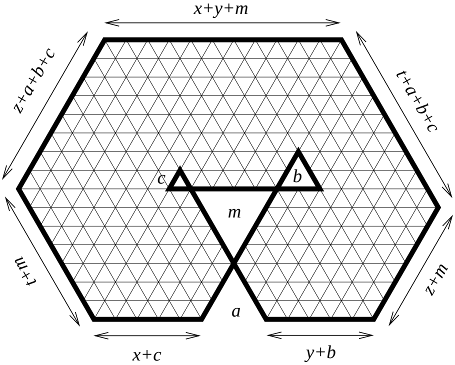
Theorem (L. 2015+)

For nonnegative integers x, y, z, t, m, a, b, c

$$\mathbf{M}\left(Q\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}\right) = \frac{\mathbf{H}(\Delta+x+y+z+t)}{\mathbf{H}(\Delta+x+y+t)\,\mathbf{H}(\Delta+x+y+z)}$$
$$\times \frac{\mathbf{H}(\Delta+x+t)\,\mathbf{H}(\Delta+x+y)\,\mathbf{H}(\Delta+y+z)\,\mathbf{H}(\Delta)}{\mathbf{H}(\Delta+z+t)\,\mathbf{H}(\Delta+x)\,\mathbf{H}(\Delta+y)}$$
$$\times \frac{\mathbf{H}(m+b+c+z+t)\,\mathbf{H}(m+a+c+x)\,\mathbf{H}(m+a+b+y)}{\mathbf{H}(m+b+y+z)\,\mathbf{H}(m+c+x+t)}$$
$$\times \frac{\mathbf{H}(c+x+t)\,\mathbf{H}(b+y+z)}{\mathbf{H}(a+c+x)\,\mathbf{H}(a+b+y)\,\mathbf{H}(b+c+z+t)}$$
$$\times \frac{\mathbf{H}(m)^3\,\mathbf{H}(a)^2\,\mathbf{H}(b)\,\mathbf{H}(c)\,\mathbf{H}(x)\,\mathbf{H}(y)\,\mathbf{H}(z)\,\mathbf{H}(t)}{\mathbf{H}(m+a)^2\,\mathbf{H}(m+b)\,\mathbf{H}(m+c)\,\mathbf{H}(x+t)\,\mathbf{H}(y+z)}, \qquad (1)$$

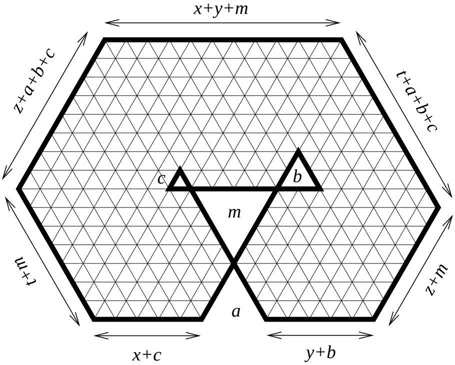
where $\Delta = m+a+b+c$.

Example



$$M\left(Q\begin{pmatrix}4 & 3 & 2 & 3\\ 4 & 3 & 2 & 1\end{pmatrix}\right) = 236,606,074,710,198,953,760,000.$$

Example



$$M\left(Q\begin{pmatrix}4 & 3 & 2 & 3\\ 4 & 3 & 2 & 1\end{pmatrix}\right) = 2^8 \cdot 3^5 \cdot 5^4 \cdot 7^8 \cdot 11^3 \cdot 13^3 \cdot 19^2.$$

"Magnet bar" region

When $b = c = 0$, the region is called a "magnet bar" by Ciucu and Krattenthaler.

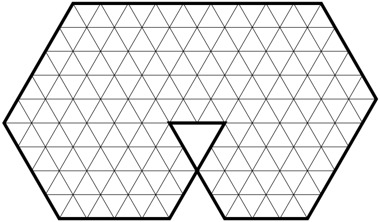
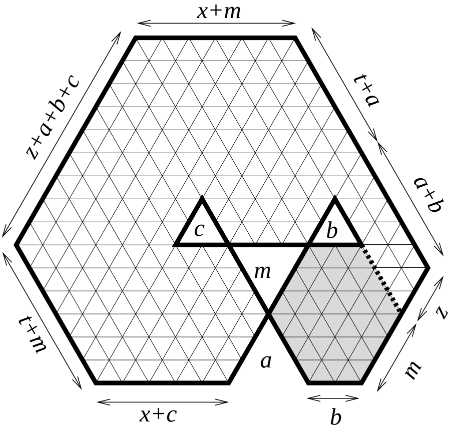


Figure : A magnet bar.

Proof of the main result

We prove by induction on $y+z+t$. The base cases: $y=0$, $z=0$, and $t=0$.

Case $y = 0$



Region-Splitting Lemma

If the numbers of up-pointing and down-pointing triangles in a region R are equal, then we say that R is balanced.

Lemma (L. 2014)

Let R be balanced region on the triangular lattice. Assume that a subregion Q of R satisfies following two conditions:

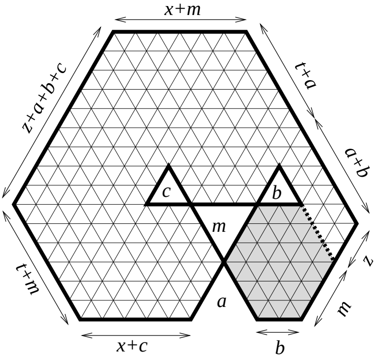
(i) (Separating Condition) *The border between Q and $R - Q$ separates two types of unit triangles.*

(ii) (Balancing Condition) *Q is balanced.*

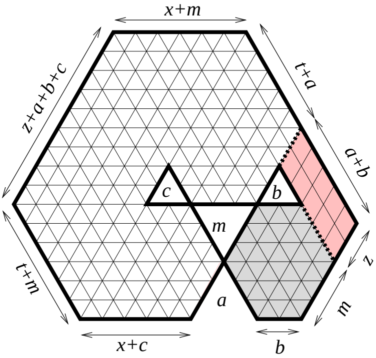
Then

$$\mathbf{M}(R) = \mathbf{M}(Q)\,\mathbf{M}(R - Q). \tag{2}$$

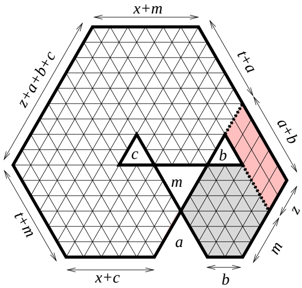
Case $y = 0$



Case $y = 0$

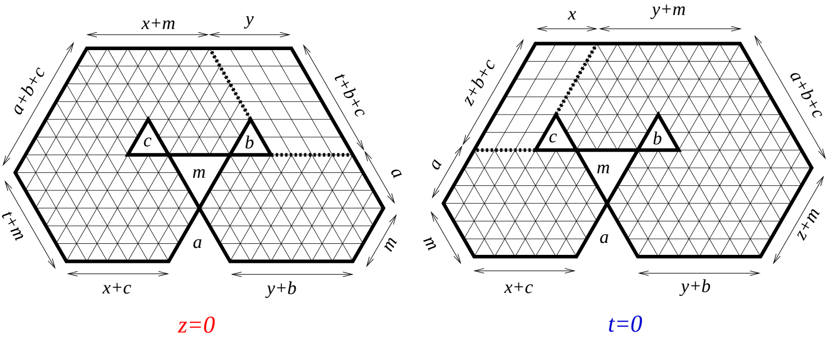


Case $y = 0$



$$\mathbf{M}(\mathit{Region}) = \mathbf{M}(\text{Hexagon})\,\mathbf{M}(\text{Magnet bar})$$

Case $z = 0$ and the case $t = 0$



Induction step.

- Assume that theorem holds for all regions with the sum of their y-, z- and t- parameters less than $y + z + t$.

Induction step.

- Assume that theorem holds for all regions with the sum of their y-, z- and t- parameters less than $y + z + t$.
- We use Kuo's condensation to create a recurrence.

Kuo's Graphical Condensation

A perfect matching of a graph G is a collection of disjoint edges covering all vertices of G.

Theorem (Kuo 2004)

$G = (V_1, V_2, E)$ bipartite planar graph with $|V_1| = |V_2|$. Assume that u, v, w, t are four vertices appearing in a cyclic order on a face of G so that $u, w \in V_1$ and $v, t \in V_2$. Then

$$\mathbf{M}(G)\,\mathbf{M}(G - \{u, v, w, t\}) = \mathbf{M}(G - \{u, v\})\,\mathbf{M}(G - \{w, t\}) + \mathbf{M}(G - \{u, t\})\,\mathbf{M}(G - \{v, w\}).$$

Apply Kuo Condensation

Let G is the dual graph of the region $Q := Q\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$:

Vertices of G are unit triangles in Q; edges of G are rhombi in Q.

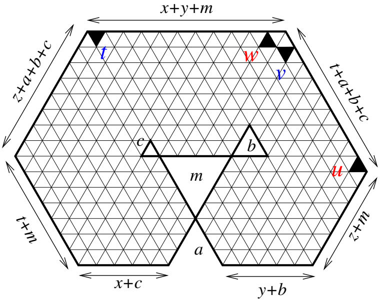


Figure : Applying Kuo condensation. The four black triangles correspond to the four vertices u, v, w, t.

Apply Kuo Condensation

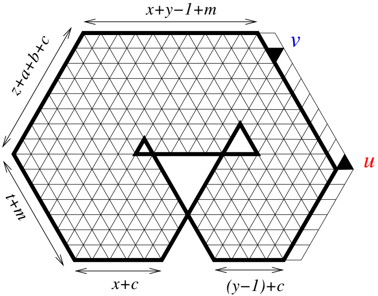


Figure : The region corresponding to $G - \{u, v\}$.

$$\mathbf{M}(G - \{u, v\}) = \mathbf{M}\left(Q\begin{pmatrix} x & y-1 & z & t \\ m & a & b & c \end{pmatrix}\right)$$

Apply Kuo Condensation

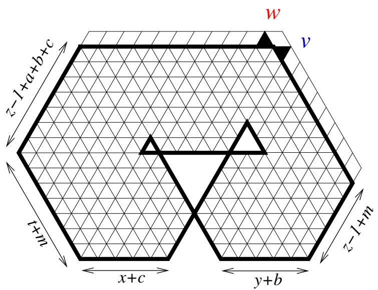


Figure : The region corresponding to $G - \{v, w\}$.

$$\mathbf{M}(G - \{v, w\}) = \mathbf{M}\left(Q\begin{pmatrix} x & y & z-1 & t \\ m & a & b & c \end{pmatrix}\right)$$

Apply Kuo Condensation

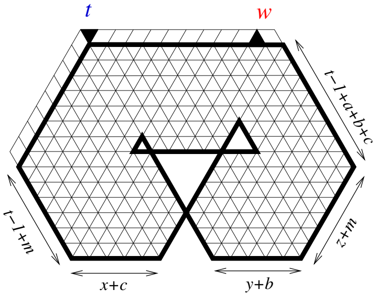


Figure : The region corresponding to $G - \{w, t\}$.

$$\mathbf{M}(G - \{w, t\}) = \mathbf{M}\left(Q\begin{pmatrix} x & y & z & t-1 \\ m & a & b & c \end{pmatrix}\right)$$

Apply Kuo Condensation

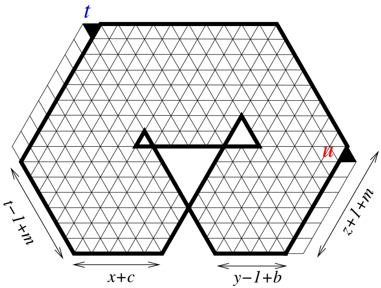


Figure : The region corresponding to $G - \{t, u\}$.

$$\mathbf{M}(G - \{t,u\}) = \mathbf{M}\left(Q\begin{pmatrix} x & y-1 & z+1 & t-1 \\ m & a & b & c \end{pmatrix}\right)$$

Apply Kuo Condensation

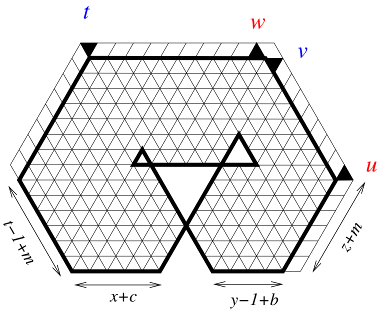


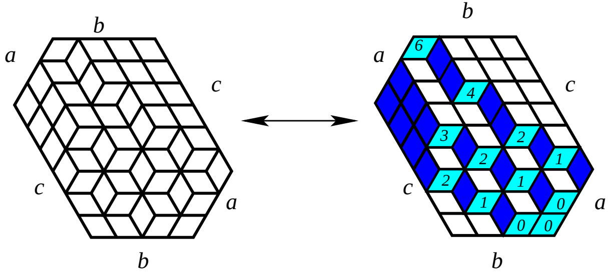
Figure : The region corresponding to $G - \{u, v, w, t\}$.

$$\mathbf{M}(G - \{u, v, w, t\}) = \mathbf{M}\left(Q\begin{pmatrix} x & y-1 & z & t-1 \\ m & a & b & c \end{pmatrix}\right)$$

Recurrence

$$\mathbf{M}\left(Q\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}\right)\mathbf{M}\left(Q\begin{pmatrix} x & y-1 & z & t-1 \\ m & a & b & c \end{pmatrix}\right) =$$
$$\mathbf{M}\left(Q\begin{pmatrix} x & y-1 & z & t \\ m & a & b & c \end{pmatrix}\right)\mathbf{M}\left(Q\begin{pmatrix} x & y & z & t-1 \\ m & a & b & c \end{pmatrix}\right)$$
$$+\mathbf{M}\left(Q\begin{pmatrix} x & y-1 & z+1 & t-1 \\ m & a & b & c \end{pmatrix}\right)\mathbf{M}\left(Q\begin{pmatrix} x & y & z-1 & t \\ m & a & b & c \end{pmatrix}\right).$$

Lozenge tilings and plane partition



A plane partition is usually identified with its 3-D interpretation.

MacMahon's Theorem (again)

Theorem

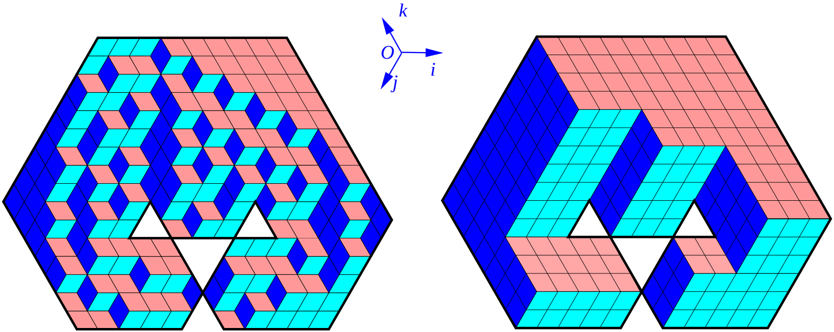
$$\sum_{\pi} q^{|\pi|} = \frac{\mathbf{H}_q(a)\,\mathbf{H}_q(b)\,\mathbf{H}_q(c)\,\mathbf{H}_q(a+b+c)}{\mathbf{H}_q(a+b)\,\mathbf{H}_q(b+c)\,\mathbf{H}_q(c+a)},$$

where the sum is taken over all plane partitions π *fitting in an* $a \times b \times c$ *box, and* $|\pi|$ *is the number of cubes in* π*,*

Definition:

- q-integer $[n]_q := 1 + q + q^2 + \dots + q^{n-1}$
- q-factorial $[n]_q! = [1]_q[2]_q \dots [n]_q$,
- q-hyperfactorial $\mathbf{H}_q(n) = [0]_q![1]_q! \dots [n-1]_q!$.

Generalized plane partitions.



(a) (b)

Figure : View a lozenge tiling as a stack of unit cubes.

"Floor plan" of the box $\mathcal{B}\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$

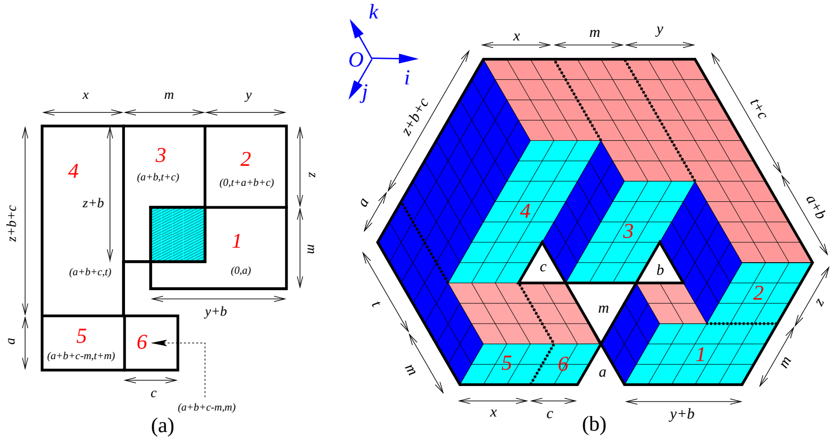
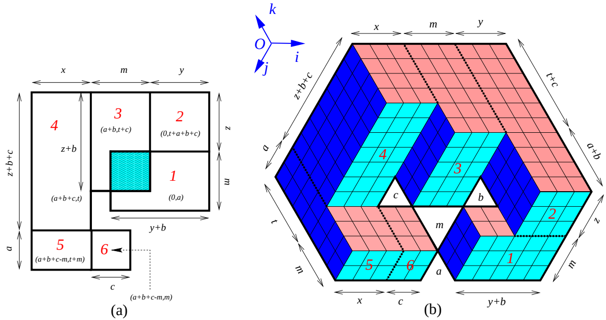


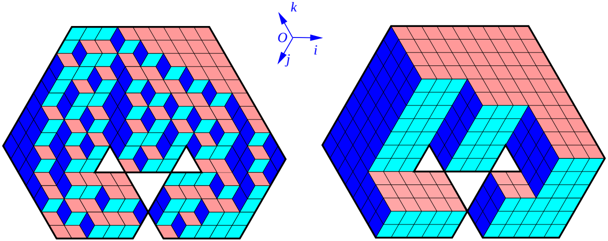
Figure : Projective diagram of the box.

"Floor plan" of the box $\mathcal{B}\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$



1. $\mathcal{B}$ consists of 6 non-overlapping smaller boxes, called rooms.
2. A room has four walls (left, right, back, front).
3. If two rooms share a portion of their walls, we remove it to make them connected.

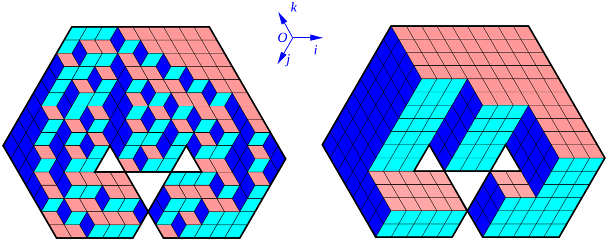
Generalized plane partitions.



(a) (b)

- *Property of the stack:* The tops of columns are weakly decreasing along Oi and Oj.

Generalized plane partitions.



(a) (b)

- *Property of the stack:* The tops of columns are weakly decreasing along Oi and Oj.
- We call the above stacks fitting in box $\mathcal{B}$ a generalized plane partitions.

A question inspired by MacMahon q-formula

What is the q-sum of the generalized plane partitions π fitting in the box $\mathcal{B}\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$?

q-analog

Theorem (L. 2015+)

For nonnegative integers x, y, z, t, m, a, b, c

$$\sum_{\pi} q^{|\pi|} = \frac{\mathbf{H}_q(\Delta+x+y+z+t)}{\mathbf{H}_q(\Delta+x+y+t)\,\mathbf{H}_q(\Delta+x+y+z)}$$
$$\times \frac{\mathbf{H}_q(\Delta+x+t)\,\mathbf{H}_q(\Delta+x+y)\,\mathbf{H}_q(\Delta+y+z)\,\mathbf{H}_q(\Delta)}{\mathbf{H}_q(\Delta+z+t)\,\mathbf{H}_q(\Delta+x)\,\mathbf{H}_q(\Delta+y)}$$
$$\times \frac{\mathbf{H}_q(m+b+c+z+t)\,\mathbf{H}_q(m+a+c+x)\,\mathbf{H}_q(m+a+b+y)}{\mathbf{H}_q(m+b+y+z)\,\mathbf{H}_q(m+c+x+t)}$$
$$\times \frac{\mathbf{H}_q(c+x+t)\,\mathbf{H}_q(b+y+z)}{\mathbf{H}_q(a+c+x)\,\mathbf{H}_q(a+b+y)\,\mathbf{H}_q(b+c+z+t)}$$
$$\times \frac{\mathbf{H}_q(m)^3\,\mathbf{H}_q(a)^2\,\mathbf{H}_q(b)\,\mathbf{H}_q(c)\,\mathbf{H}_q(x)\,\mathbf{H}_q(y)\,\mathbf{H}_q(z)\,\mathbf{H}_q(t)}{\mathbf{H}_q(m+a)^2\,\mathbf{H}_q(m+b)\,\mathbf{H}_q(m+c)\,\mathbf{H}_q(x+t)\,\mathbf{H}_q(y+z)}, \quad (3)$$

where $\Delta = m+a+b+c$.

Weighted case

- Kuo Condensation works well for the weighted case.

Weighted case

- Kuo Condensation works well for the weighted case.
- $\mathbf{M}(G)$ is the sum of weights of all perfect matchings, the weight of a matching is the product of weights of its constituent edges.

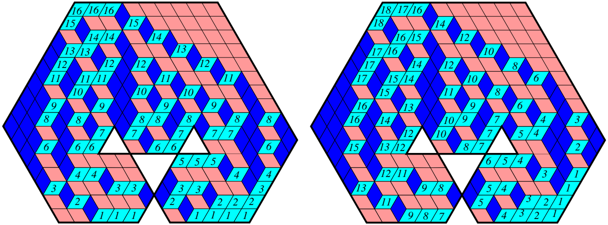
Weighted case

- Kuo Condensation works well for the weighted case.
- $\mathbf{M}(G)$ is the sum of weights of all perfect matchings, the weight of a matching is the product of weights of its constituent edges.
- $\mathbf{M}(R)$ is the sum of weights of all tilings, the weight of a tiling is the product of weights of its constituent rhombi.

Weighted case

- Kuo Condensation works well for the weighted case.
- $\mathbf{M}(G)$ is the sum of weights of all perfect matchings, the weight of a matching is the product of weights of its constituent edges.
- $\mathbf{M}(R)$ is the sum of weights of all tilings, the weight of a tiling is the product of weights of its constituent rhombi.
- We want to find some simple weight assignment(s) on the rhombi so that
  1. $\mathbf{M}(R)$ is not "too far " from $\sum_{\pi} q^{|\pi|}$.
  2. Kuo Condensation is applicable.

Two weight assignments



(a) (b)

- Only weight right lozenge.
- (a) Weight=q^x, where x is the distance from the top of the rhombus to the bottom of the region.
- (b) Weight=q^y, where y is the distance from the left side of the rhombus to the southeast side of the region.

Relation of the weight assignments

Lemma

For any tiling T of the region $Q\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$, the ratio

$$wt_1(T) : wt_2(T) : q^{|\pi_T|} = q^A : q^B : 1,$$

A and B do not depend on the choice of T.

Compare weights

$$A\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix} = (y+b)\binom{m+1}{2} + myz + y\binom{z+1}{2} + m(z+b)(m+a)$$
$$+ m\binom{z+b+1}{2} + x(m+a)(z+b+c) + x\binom{z+b+c+1}{2} + x\binom{a+1}{2} + c\binom{a+1}{2}.$$

$$B\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix} = m\binom{y+b+1}{2} + z\binom{y+1}{2} + m(z+b)(y+a+b) + (b+z)\binom{m+1}{2}$$
$$+ x(z+b+c)(y+m+a+b+c) + (z+b+c)\binom{x+1}{2} + a(x+c)(y+a+b) + a\binom{x+c+1}{2}.$$

Compare weight

Denote

$$\mathbf{M}^q\left(Q\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}\right) := \sum_T wt_1(T)$$

$$\mathbf{M}_q\left(Q\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}\right) := \sum_T wt_2(T)$$

We have

$$\mathbf{M}^q(Q) : \mathbf{M}_q(Q) : \sum_\pi q^{|\pi|} = q^A : q^B : 1$$

Refinement

Need to show

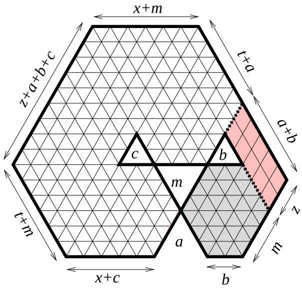
$$\mathbf{M}^q(Q) = q^A \frac{\mathbf{H}_q(\Delta+x+y+z+t)}{\mathbf{H}_q(\Delta+x+y+t)\,\mathbf{H}_q(\Delta+x+y+z)}$$
$$\times \frac{\mathbf{H}_q(\Delta+x+t)\,\mathbf{H}_q(\Delta+x+y)\,\mathbf{H}_q(\Delta+y+z)\,\mathbf{H}_q(\Delta)}{\mathbf{H}_q(\Delta+z+t)\,\mathbf{H}_q(\Delta+x)\,\mathbf{H}_q(\Delta+y)}$$
$$\times \frac{\mathbf{H}_q(m+b+c+z+t)\,\mathbf{H}_q(m+a+c+x)\,\mathbf{H}_q(m+a+b+y)}{\mathbf{H}_q(m+b+y+z)\,\mathbf{H}_q(m+c+x+t)}$$
$$\times \frac{\mathbf{H}_q(c+x+t)\,\mathbf{H}_q(b+y+z)}{\mathbf{H}_q(a+c+x)\,\mathbf{H}_q(a+b+y)\,\mathbf{H}_q(b+c+z+t)}$$
$$\times \frac{\mathbf{H}_q(m)^3\,\mathbf{H}_q(a)^2\,\mathbf{H}_q(b)\,\mathbf{H}_q(c)\,\mathbf{H}_q(x)\,\mathbf{H}_q(y)\,\mathbf{H}_q(z)\,\mathbf{H}_q(t)}{\mathbf{H}_q(m+a)^2\,\mathbf{H}_q(m+b)\,\mathbf{H}_q(m+c)\,\mathbf{H}_q(x+t)\,\mathbf{H}_q(y+z)} \quad (4)$$

Special cases

1. Case $m = a = b = c = 0$: Hexagon.
2. Case $m = b = c = 0$: Hexagon with one triangular hole.
3. Case $b = c = 0$: Magnet bar.

Proof of q-analog

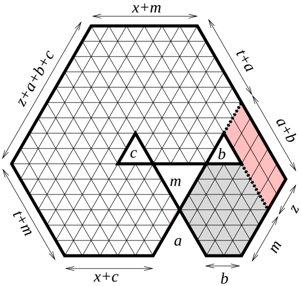
Base case $y = 0$.



$\mathbf{M}^q(Region) =$ (Weight of forced rhombi)$\cdot\mathbf{M}^q$(Hexagon)$\mathbf{M}_q$(Magnet bar)

Proof of q-analog

Base case $y = 0$.



$$\mathbf{M}^q(Region) = 1 \cdot \mathbf{M}^q(\text{Hexagon}) q^{B-A}\, \mathbf{M}^q(\text{Magnet bar}).$$

Apply Kuo Condensation

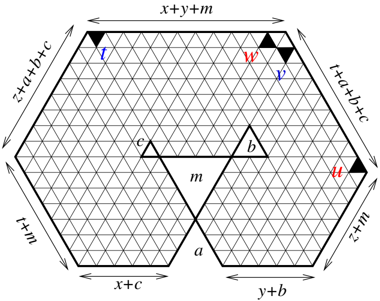


Figure : Applying Kuo condensation. The four black triangles correspond to the four vertices u, v, w, t.

Apply Kuo Condensation

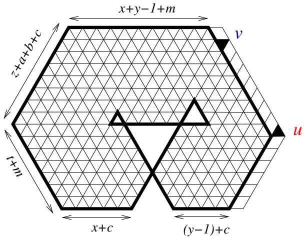


Figure : The region corresponding to $G - \{u, v\}$.

$$\mathbf{M}(G - \{u, v\}) = q^{\binom{z+m+1}{2}} \mathbf{M}^q \left(Q \begin{pmatrix} x & y-1 & z & t \\ m & a & b & c \end{pmatrix} \right)$$

Apply Kuo Condensation

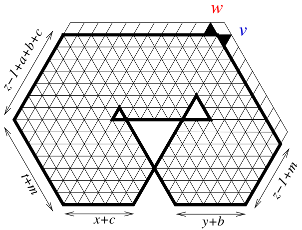


Figure : The region corresponding to $G - \{v, w\}$.

$$\mathbf{M}(G-\{v,w\}) = q^{(x+y+m-1)(z+t+\Delta)}\,\mathbf{M}^q\left(Q\begin{pmatrix} x & y & z-1 & t \\ m & a & b & c \end{pmatrix}\right)$$

Apply Kuo Condensation

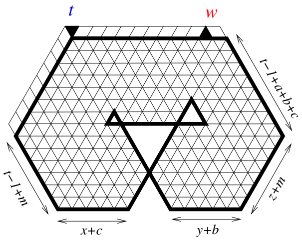


Figure : The region corresponding to $G - \{w, t\}$.

$$\mathbf{M}(G-\{w,t\}) = q^{(x+y+m-2)(z+t+\Delta)}\,\mathbf{M}^q\left(Q\begin{pmatrix} x & y & z & t-1 \\ m & a & b & c \end{pmatrix}\right)$$

Apply Kuo Condensation

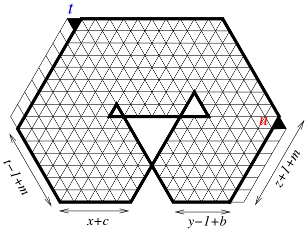


Figure : The region corresponding to $G - \{t, u\}$.

$$\mathbf{M}(G - \{t, u\}) = q^{\binom{z+m+1}{2}} \mathbf{M}^q \left(Q \begin{pmatrix} x & y-1 & z+1 & t-1 \\ m & a & b & c \end{pmatrix} \right)$$

Apply Kuo Condensation

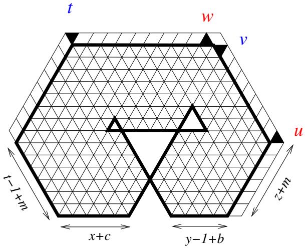


Figure : The region corresponding to $G - \{u, v, w, t\}$.

$$\mathbf{M}(G - \{u,v,w,t\}) = q^{\binom{z+m+1}{2}+(x+y+m-2)(z+t+\Delta)} \mathbf{M}^q\left(Q\begin{pmatrix} x & y-1 & z & t-1 \\ m & a & b & c \end{pmatrix}\right)$$

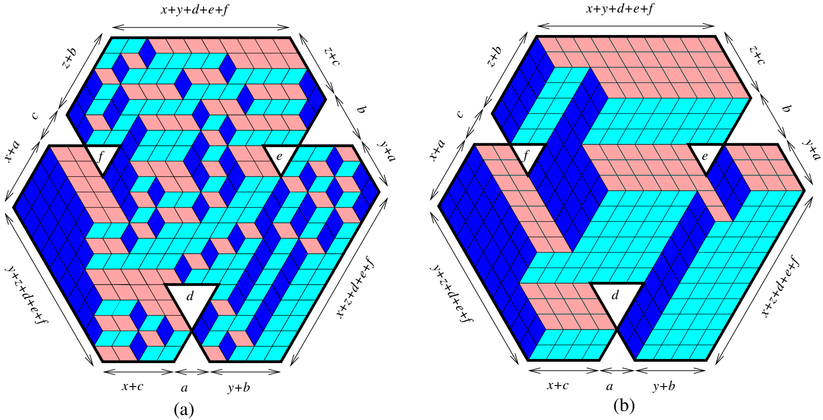
Recurrence

$$\mathbf{M}^q\left(Q\begin{pmatrix} x & y & z & t\\ m & a & b & c\end{pmatrix}\right)\mathbf{M}^q\left(Q\begin{pmatrix} x & y-1 & z & t-1\\ m & a & b & c\end{pmatrix}\right)=$$

$$\mathbf{M}^q\left(Q\begin{pmatrix} x & y-1 & z & t\\ m & a & b & c\end{pmatrix}\right)\mathbf{M}^q\left(Q\begin{pmatrix} x & y & z & t-1\\ m & a & b & c\end{pmatrix}\right)$$

$$+\,q^{z+t+\Delta}\,\mathbf{M}^q\left(Q\begin{pmatrix} x & y-1 & z+1 & t-1\\ m & a & b & c\end{pmatrix}\right)\mathbf{M}^q\left(Q\begin{pmatrix} x & y & z-1 & t\\ m & a & b & c\end{pmatrix}\right)$$

Future project



Questions?

Thank you !