Cyclically Symmetric Tilings of Hexagons with Four Holes

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Introduction

- A plane partition is a rectangular array of non-negative integers with weakly decreasing rows and columns.
- One can view a plane partition with a rows, b columns, and entries at most c as a stack of unit cubes fitting in an $a \times b \times c$ box.

6 4 2 1
3 2 1 0
2 1 0 0

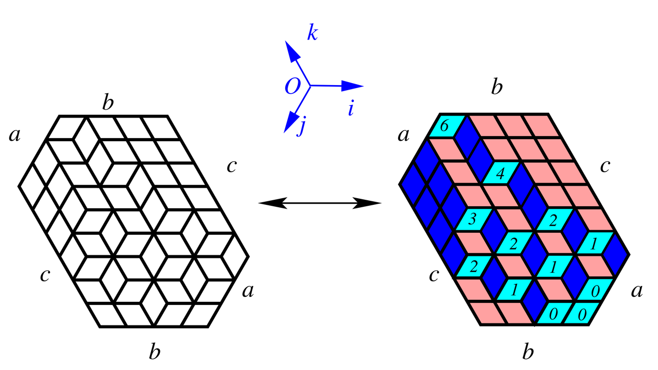

Fig. 1: Viewing a plane partition (top) as a stack of unit cubes (lower-right), and as a lozenge tiling of a hexagon (lower-left).

MacMahon's Theorem (1911)

$$\sum_{\pi} q^{|\pi|} = PP_q(a,b,c) = \frac{\mathrm{H}_q(a)\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(a+b+c)}{\mathrm{H}_q(a+b)\,\mathrm{H}_q(b+c)\,\mathrm{H}_q(c+a)}, \quad (1)$$

where the sum is taken over all plane partitions π fitting in an $a \times b \times c$ box, and $|\pi|$ is the volume of π.

- q-integer $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$
- q-factorial $[n]_q! := [1]_q[2]_q \ldots [n]_q$,
- q-hyperfactorial $\mathrm{H}_q(n) := [0]_q![1]_q! \ldots [n-1]_q!$.

Mills–Robbins–Rumsey Theorem (1982)

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{a} \frac{1-q^{3i-1}}{1-q^{3i-2}} \prod_{1\le i<j\le a} \frac{1-q^{3(2i+j-1)}}{1-q^{3(2i+j-2)}} \times \prod_{1\le i<j,k\le a} \frac{1-q^{3(i+j+k-1)}}{1-q^{3(i+j+k-2)}} \quad (2)$$

summing over all cyclically symmetric plane partitions π fitting in an $a \times a \times a$ box.

By letting $q = 1$:

- MacMahon's Theorem implies a product formula for the number of tilings of a hexagon
- Mills–Robbins–Rumsey's Theorem implies a product formula for cyclically symmetric tilings of a regular hexagon

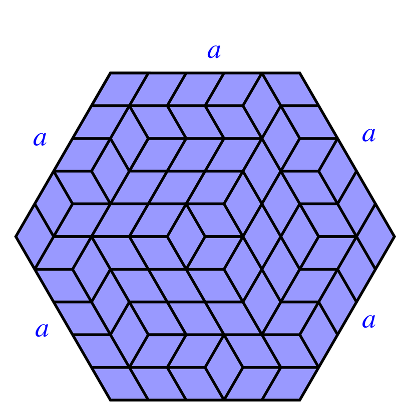

Fig. 2: A cyclically symmetric tiling

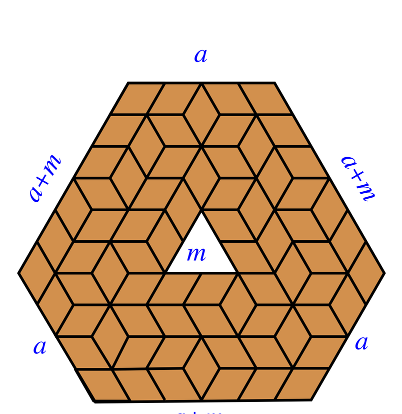

Fig. 3: A cyclically symmetric tilings of a 'cored-hexagon'.

- Ciucu–Krattenthaler (2000) found a formula for cyclically symmetric tilings of a cored-hexagon
- Krattenthaler (2006) present a bijection between cyclically symmetric tilings of a cored-hexagon and descending plane partitions.

Hexagons with Four Holes

Hexagons with Four Holes of Type I:

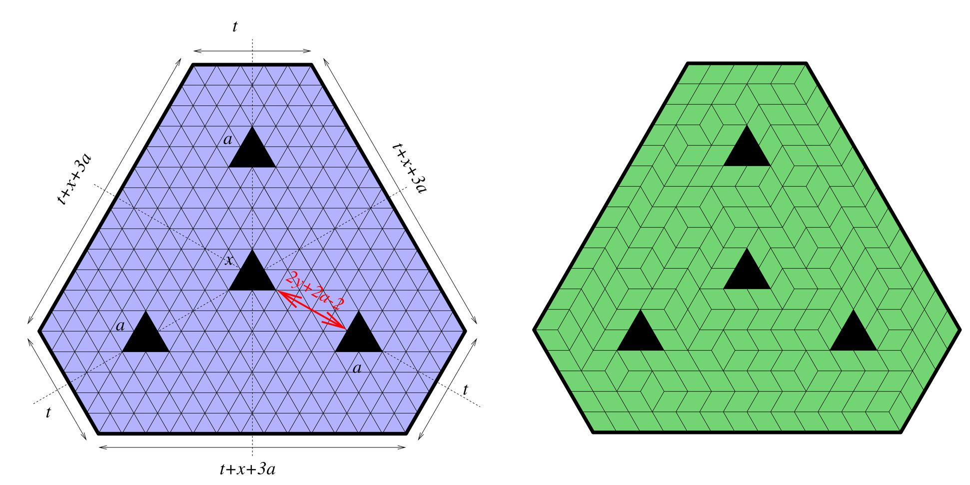

Fig. 4: The region $\mathcal{H}_{t,y}(a,x)$ and a cyclically symmetric tiling.

Main Theorem 1 (L. – Rohatgi 2017)

For non-negative integers a, t, x, y

$$\mathrm{CS}(\mathcal{H}_{t,y}(2a,2x)) = 2^{4a+t} P_1\left(x+1, y, \left\lceil \frac{t}{2} \right\rceil - y - 1, a\right) P_2\left(x+1, y, \left\lfloor \frac{t}{2} \right\rfloor - y, a\right), \quad (3)$$

$$\mathrm{CS}(\mathcal{H}_{t,y}(2a+1,2x)) = 2^{4a+t+2} F_1\left(x+1, y, \left\lceil \frac{t}{2} \right\rceil - y - 1, a+1\right) F_2\left(x+1, y, \left\lfloor \frac{t}{2} \right\rfloor - y, a\right). \quad (4)$$

Remark: When the central hole has an odd side, $\mathrm{CS}(\mathcal{H}_{t,y}(a,x))$ is not 'nice'.

Hexagons with Four Holes of Type II:

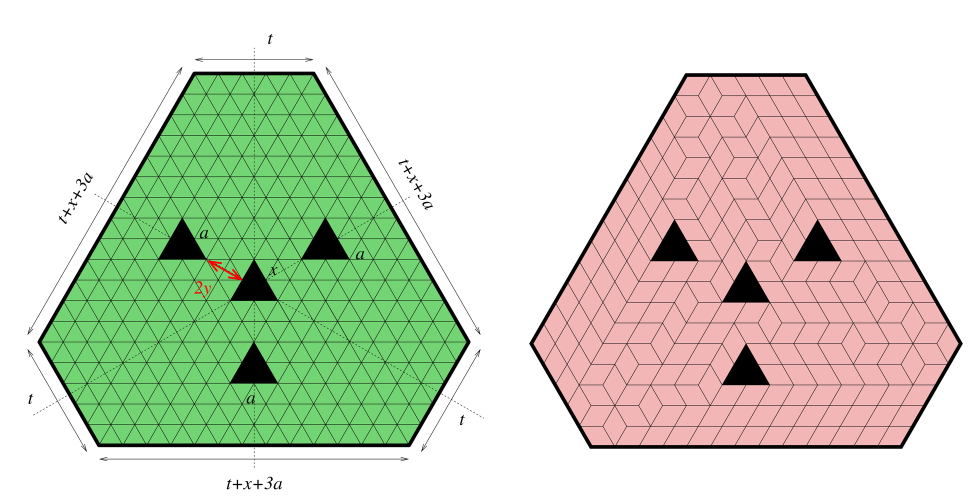

Fig. 5: The region $\overline{\mathcal{H}}_{t,y}(a,x)$ and a cyclically symmetric tiling.

Main Theorem 2 (L. – Rohatgi 2017)

Assume that a, t, x, y are non-negative integers. Then

$$\mathrm{CS}(\overline{\mathcal{H}}_{t,y}(2a,2x)) = 2^{4a+t} E_1\left(x+1, y-1, \left\lceil \frac{t}{2} \right\rceil - y + 1, a\right) E_2\left(x+1, y, \left\lfloor \frac{t}{2} \right\rfloor - y + 1, a\right), \quad (5)$$

$$\mathrm{CS}(\overline{\mathcal{H}}_{t,y}(2a,2x+1)) = 2^{4a+t} E_3\left(x+2, y-1, \left\lceil \frac{t}{2} \right\rceil - y + 1, a\right) E_4\left(x+1, y, \left\lfloor \frac{t}{2} \right\rfloor - y + 1, a\right). \quad (6)$$

Remark:

- When the satellite holes have an odd side, $\mathrm{CS}(\overline{\mathcal{H}}_{t,y}(a,x))$ is not 'nice'.
- We omit the explicit formulae of E_i's here (see details in `arXiv:1705.01122`)

Pochhammer symbol $(x)_n := x(x+1)\ldots(x+n-1)$
'Skipping' Pochhammer symbol $[x]_n := x(x+2)(x+4)\ldots(x+2(n-1))$

$$P_1(x,y,z,a) := \frac{1}{2^{y+z}} \prod_{i=1}^{y+z} \frac{(2x+6a+2i)_i[2x+6a+4i+1]_{i-1}}{(i)_i[2x+6a+2i+1]_{i-1}} \times \prod_{i=1}^{a} \frac{(z+i)_{y+a-2i+1}(x+y+2z+2a+2i)_{2y+2a-4i+2}}{(i)_y(y+2z+2i-1)_{y+2a-4i+3}} \times \prod_{i=1}^{a} \frac{(x+3i-2)_{y-i+1}(x+3y+2i-1)_{i-1}}{(2z+2i)_{y+2a-4i+1}(x+y+z+2a+i)_{y+a-2i+1}}.$$

$$P_2(x,y,z,a) := \frac{[x+3y]_a(x+2y+z+2a)_a}{2^{2(ay+z)}[x+3y+2z+2a+1]_a} \times \prod_{i=1}^{y+z} \frac{(2x+6a+2i-2)_{i-1}[2x+6a+4i-1]_i}{(i)_i[2x+6a+2i-1]_{i-1}} \times \prod_{i=1}^{a} \frac{(z+i)_{y+a-2i+1}(x+y+2z+2a+2i-1)_{2y+2a-4i+3}}{(i)_y(y+2z+2i-1)_{y+2a-4i+3}} \times \prod_{i=1}^{a} \frac{(x+3i-2)_{y-i}(x+3y+2i-1)_{i-1}}{(2z+2i)_{y+2a-4i+1}(x+y+z+2a+i-1)_{y+a-2i+2}}.$$

Explicit Formulas for $\mathrm{CS}(\mathcal{H}_{t,y}(a,x))$

$$F_1(x,y,z,a) = \frac{1}{2^{ya+z}} \frac{[x+y+2z+2a+1]_y}{[x+y+2a-1]_y} \frac{\prod_{i=1}^{\lfloor \frac{a}{3} \rfloor}(x+3y+6i-3)_{3a-9i+1}}{\prod_{i=1}^{\lfloor \frac{a-1}{3} \rfloor}(x+3y+6i-2)} \times \prod_{i=1}^{y+z} \frac{i!(x+3a+i-3)!(2x+6a+2i-4)_i(x+3a+2i-2)_i(2x+6a+3i-4)}{(x+3a+2i-2)!(2i)!} \times \prod_{i=1}^{a-1}(x+3i-2)_{y-i+1}(x+y+2z+2a+2i)_{2y+2a-4i} \times \prod_{i=1}^{y} \frac{[2i+3]_{z-1}(x+3a+3i-5)_{2y+z-a-4i+5}}{(a+i+1)_{z-1}(i)_{a+1}[2i+3]_{a-2}[2x+6a+6i-7]_{z+2y-4i+3}}.$$

$$F_2(x,y,z,a) = \frac{1}{2^{y(a+2)+2a+z+1}} \frac{\prod_{i=1}^{\lfloor \frac{y+1}{3} \rfloor}(x+3i-2)_{3y-9i+4}}{\prod_{i=1}^{\lfloor \frac{y}{3} \rfloor}(x+3y-6i)} \times \prod_{i=1}^{y+z} \frac{i!(x+3a+i-1)!(2x+6a+2i)_i(x+3a+2i)_i(x+3a+3i)}{(x+3a+2i)!(2i)!} \times \prod_{i=1}^{y} \frac{[2i+3]_{z-1}(x+y+2a+2i-1)_{y+z-3i+2}(x+y+2z+2a+2i)_{2y+2a-4i+3}}{(a+i+2)_{z-1}(i)_{a+2}[2i+3]_{a-1}[2x+6a+6i-1]_{2y+z-4i+2}}.$$

Outlines of the proof of Main Theorem 1:

1. Convert symmetric tilings of $\mathcal{H}_{t,y}(a,x)$ into symmetric matchings of its dual graph (see Figure 6)
2. Convert symmetric matchings of the dual graph to matchings of the orbit graph G (see Figure 7)
$$\mathrm{CS}(\mathcal{H}_{t,y}(a,x)) = \mathrm{M}(G)$$
3. Use Ciucu's factorization to break the orbit graph into two components (see Figure 8)
$$\mathrm{CS}(\mathcal{H}_{t,y}(a,x)) = \mathrm{M}(G) = 2^k\,\mathrm{M}(G^+)\,\mathrm{M}(G^-)$$
4. Use tiling-matching duality to write $\mathrm{CS}(\mathcal{H}_{t,y}(a,x))$ as product of tiling numbers of two regions (see Figure 9)
$$\mathrm{CS}(\mathcal{H}_{t,y}(a,x)) = \mathrm{M}(G) = 2^k\,\mathrm{M}(G^+)\,\mathrm{M}(G^-) = 2^k\,\mathrm{M}(T^+)\,\mathrm{M}(T^-)$$
5. Use 'Kuo condensation' to prove that the regions T^+ and T^- are enumerated by P_i and F_i.

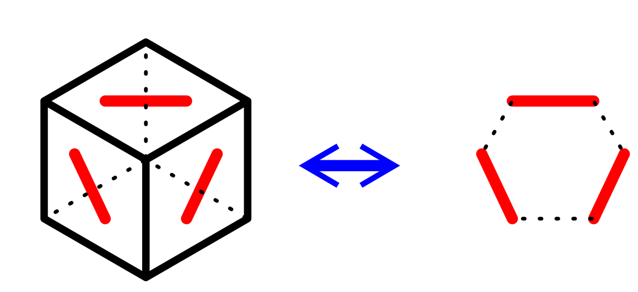

Fig. 6: Correspondence between tilings and perfect matchings

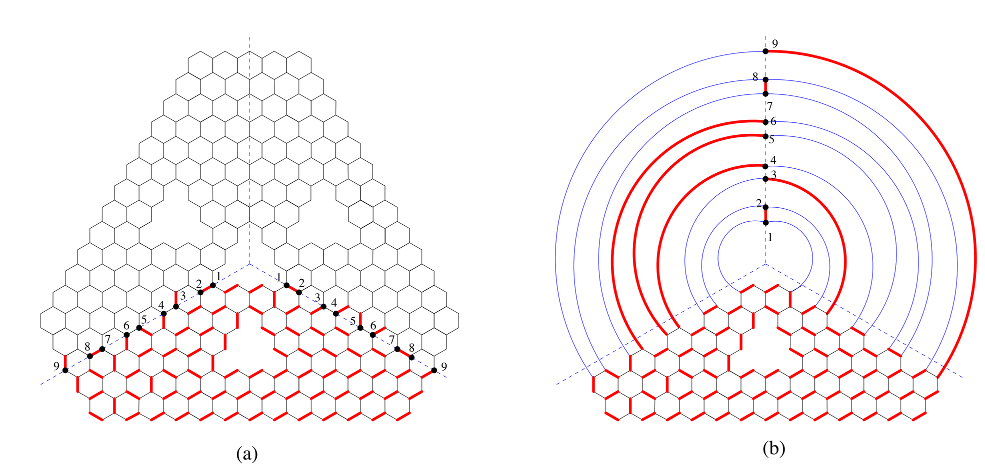

Fig. 7: Converting symmetric matchings to matchings of the orbit graph

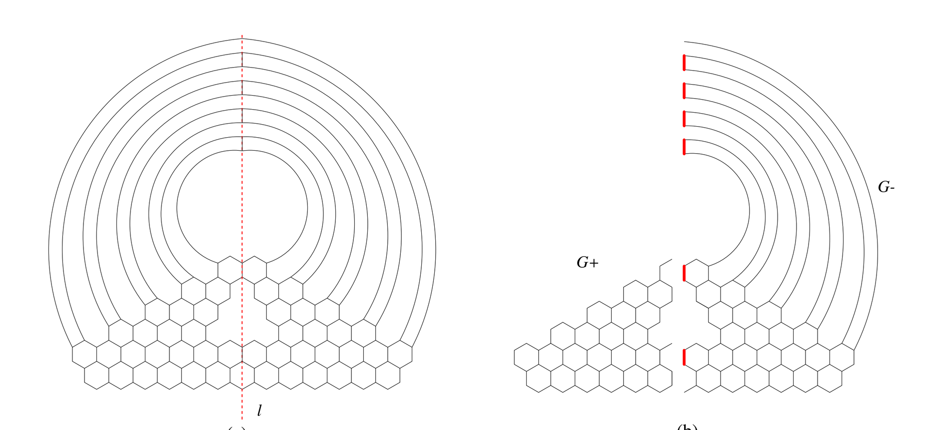

Fig. 8: Breaking the orbit graph into two components

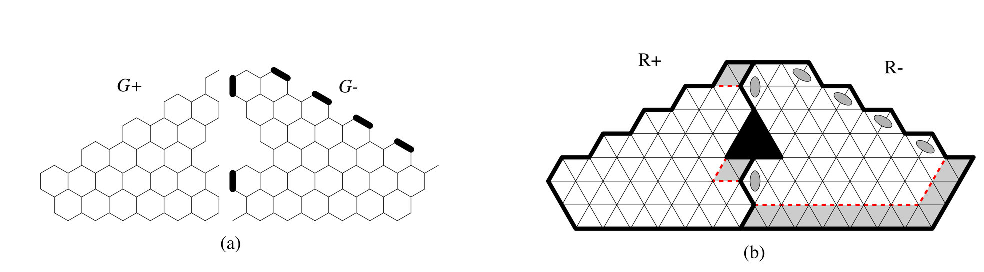

Fig. 9: Use the tiling-matching duality (inreverse)