Enumeration of domino tilings of a double Aztec rectangle

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Our main result is a such connection.
A lozenge (or unit rhombus) is the union of two adjacent unit equilateral triangles.

A lozenge tiling of a region $R$ on the triangular lattice is a covering of $R$ by lozenges so that there are no gaps or overlaps.
Semi-regular hexagons

**Theorem (MacMahon ∼ 1900)**

\[
\mathbb{T}(Hex(a, b, c)) = \frac{H(a) H(b) H(c) H(a + b + c)}{H(a + b) H(b + c) H(c + a)},
\]

where the “hyperfactorial” is defined as

\[
H(n) := 0! \cdot 1! \cdot 2! \ldots (n - 1)!. 
\]
Theorem (MacMahon’s $q$-Theorem)

\[
\sum_{\pi} q^{\text{vol}(\pi)} = \frac{\mathcal{H}_q(a) \mathcal{H}_q(b) \mathcal{H}_q(c) \mathcal{H}_q(a + b + c)}{\mathcal{H}_q(a + b) \mathcal{H}_q(b + c) \mathcal{H}_q(c + a)},
\]

where the sum is taken over all monotonic stacks of unit cubes (plane partitions) $\pi$ fitting in an $a \times b \times c$ box.

Definition:

- **$q$-integer** $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$
- **$q$-factorial** $[n]_q! = [1]_q[2]_q \ldots [n]_q$,
- **$q$-hyperfactorial** $\mathcal{H}_q(n) = [0]_q![1]_q! \ldots [n - 1]_q!$.
Study of domino tilings came from statistical mechanics with the work of Kasteleyn, and Temperley and Fisher in 1961.

**Theorem (Elkies, Kuperberg, Larsen and Propp 1992)**

The *Aztec diamond* of order $n$ has $2^{n(n+1)/2}$ domino tilings.

![Figure: The Aztec diamond of order 5 and one of its tilings.](image-url)
The all-horizontal domino $T_0$ has rank 0.

The rank $r(T)$ of a tiling $T$ is the smallest number of elementary moves to obtain the tiling $T$ from $T_0$. 
The Weighted Aztec Diamond Theorem

**Theorem (Weighted Aztec Diamond Theorem)**

For any positive integer $n$ and indeterminates $t$ and $q$

$$
\sum_{T} t^{v(T)} q^{r(T)} = \prod_{k=0}^{n-1} (1 + tq^{2k+1})^{n-k},
$$

where $v(T)$ is half number of vertical dominoes in $T$.
The Aztec Rectangle

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(a) (b)
The Double Aztec Rectangle

Enumeration of domino tilings of a double Aztec rectangle

\( \mathcal{DR}^{m_2,n_2}_{m_1,n_1,k} \)
In general, \( \mathcal{DR}^{m_2,n_2}_{m_1,n_1,k} \) does not have all-horizontal domino tiling.
Main Theorem

Theorem (L. 2016)

Assume that $m_1 \leq n_1$, $m_2 \leq n_2$, $k \leq \min(m_2, n_2 - 1)$, and $n_1 - m_1 = n_2 - m_2$. Then

$$\sum_{T \in \mathcal{T}(\mathcal{D}R_{m_1}^{m_2}, n_1, n_2, k)} t^{v(T)} q^{r(T)} = t^{(m_1^2 + 1)} + (m_2^2 + 1) + (n_1 - m_1)(m_1 + k)/2 q^E$$

$$\times \prod_{i=0}^{m_1-1} (1 + t^{-1} q^{2i+1} )^{m_1-i} \prod_{i=0}^{m_2-1} (1 + t^{-1} q^{-2i-1} )^{m_2-i}$$

$$\times P_q(n_1 - m_1, m_2 - k + 1, m_1 + k),$$

where

$$P_q(a, b, c) = \frac{H_q(a) H_q(b) H_q(c) H_q(a + b + c)}{H_q(a + b) H_q(b + c) H_q(c + a)}.$$
A consequence

Corollary

Assume that $m_1 \leq n_1$, $m_2 \leq n_2$, $k \leq \min(m_2, n_2 - 1)$, and $n_1 - m_1 = n_2 - m_2$. Then

$$T \left( DR_{m_1, n_1, k}^{m_2, n_2} \right) = 2^{m_1+1} + 2^{m_2+1}$$

$$\times T \left( \text{Hex}(n_1 - m_1, m_2 - k + 1, m_1 + k) \right).$$

Remark: The conditions $m_1 \leq n_1$, $m_2 \leq n_2$, $k \leq \min(m_2, n_2 - 1)$, and $n_1 - m_1 = n_2 - m_2$ to make sure the region has tilings.
A bijection between tilings and perfect matchings

Figure: Bijection between tilings of the Aztec diamond of order 5 and perfect matchings of its dual graph.

- The **dual graph** of a region $R$ is the graph whose vertices are the "cells" in $R$ and whose edges connect precisely two adjacent cells.
- A **perfect matching** of a graph $G$ is a collection of disjoint edges covering all vertices of $G$.
Idea: “Transform” the dual graph of a double Aztec rectangle into the dual graph of a hexagon.
\[ M(G) = 2^\# \text{ rows of diamonds in } K M(G') \]
Figure: The compound replacement

(a) \rightarrow (b) \rightarrow (c) \rightarrow (d)
Applying the compound replacement

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Applying the subgraph replacement

\[ T(\mathcal{D}R_{m_1,n_1,k}^{m_2,n_2}) = 2^{\binom{m_1+1}{2} + \binom{m_2+1}{2}} \times T(\text{Hex}(n_1 - m_1, m_2 - k + 1, m_1 + k)). \]
Thank You!