Enumeration of lozenge tilings of a hexagon with holes on boundary

Tri Lai

Institute for Mathematics and its Applications
Minneapolis, MN 55455

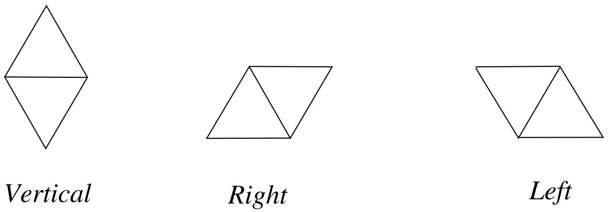
AMS sectional meeting, University of Nevada, Las Vegas
April 19, 2015

http://arxiv.org/abs/1502.05780

Outline of the talk

1. Definition and classical results
2. Main result
3. Proof of the main result
4. A q-analog
5. Plane partitions with constraints

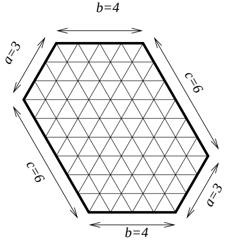
Lozenge tilings of a semi-regular hexagon



- A unit rhombus (or lozenge) is the union of two adjacent unit equilateral triangles.
- A lozenge tiling (or rhombus tiling) of a region R on the triangular lattice is a covering of the region by unit rhombi (or lozenges) so that there are no gaps or overlaps.
- Denote by $\mathbf{M}(R)$ the number of lozenge tilings of the region R.

Semi-regular hexagons

A semi-regular hexagon on the triangular lattice is a centrally symmetric hexagon of side-lengths a, b, c, a, b, c (in clockwise order, starting from the northwest side). Denote by $Hex(a, b, c)$ the hexagon.

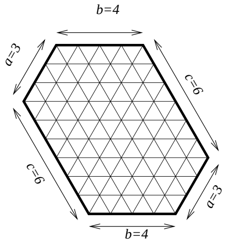


Semi-regular hexagons

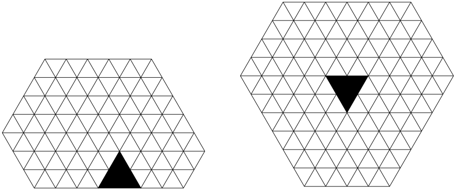
Theorem (MacMahon)

$$\mathbf{M}(Hex(a,b,c)) = \frac{\mathbf{H}(a)\,\mathbf{H}(b)\,\mathbf{H}(c)\,\mathbf{H}(a+b+c)}{\mathbf{H}(a+b)\,\mathbf{H}(b+c)\,\mathbf{H}(c+a)},$$

where the hyperfactorial $\mathbf{H}(n) := 0! \cdot 1! \cdot 2! \dots (n-1)!$.

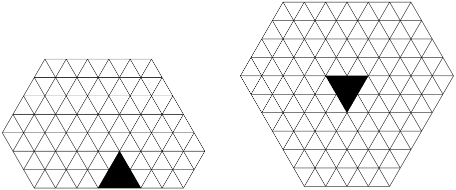


Enumeration of lozenge tilings of a hexagon with holes



- Cohn-Larsen-Propp (1998): Hexagon with triangular a hole on the base.

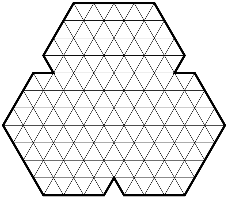
Enumeration of lozenge tilings of a hexagon with holes



- Cohn-Larsen-Propp (1998): Hexagon with triangular a hole on the base.
- Ciucu-Eisenkölbl-Krattenthaler-Zare (2001): Hexagon with a triangular hole at the "center".

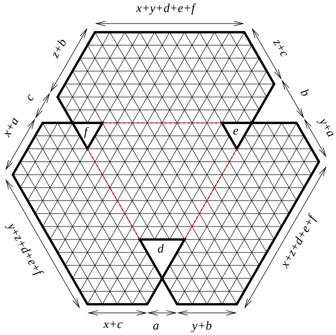
Propp's problem

Problem 3: Find the number of lozenge tilings in a hexagon of side-lengths $2n+3, 2n, 2n+3, 2n, 2n+3, 2n$, where the central unit triangles are removed from the long sides.



Eisenkölb solved (and generalized) the problem (1999).

Generalize Propp's Problem



$$F := F\begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix}$$

Main result

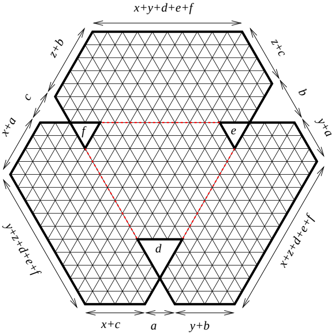
Theorem (L. 2015)

For nonnegative integers $x, y, z, a, b, c, d, e, f$

$$\mathbf{M}(F) = \frac{\mathbf{H}(x)\,\mathbf{H}(y)\,\mathbf{H}(z)\,\mathbf{H}(a)^2\,\mathbf{H}(b)^2\,\mathbf{H}(c)^2\,\mathbf{H}(d)\,\mathbf{H}(e)\,\mathbf{H}(f)\,\mathbf{H}(d+e+f+x+y+z)^4}{\mathbf{H}(a+d)\,\mathbf{H}(b+e)\,\mathbf{H}(c+f)\,\mathbf{H}(d+e+x+y+z)\,\mathbf{H}(e+f+x+y+z)\,\mathbf{H}(f+d+x+y+z)}$$
$$\times \frac{\mathbf{H}(A+2x+2y+2z)\,\mathbf{H}(A+x+y+z)^2}{\mathbf{H}(A+2x+y+z)\,\mathbf{H}(A+x+2y+z)\,\mathbf{H}(A+x+y+2z)}$$
$$\times \frac{\mathbf{H}(a+b+d+e+x+y+z)\,\mathbf{H}(a+c+d+f+x+y+z)\,\mathbf{H}(b+c+e+f+x+y+z)}{\mathbf{H}(a+d+e+f+x+y+z)^2\,\mathbf{H}(b+d+e+f+x+y+z)^2\,\mathbf{H}(c+d+e+f+x+y+z)^2}$$
$$\times \frac{\mathbf{H}(a+d+x+y)\,\mathbf{H}(b+e+y+z)\,\mathbf{H}(c+f+z+x)}{\mathbf{H}(a+b+y)\,\mathbf{H}(b+c+z)\,\mathbf{H}(c+a+x)}$$
$$\times \frac{\mathbf{H}(A-a+x+y+2z)\,\mathbf{H}(A-b+2x+y+z)\,\mathbf{H}(A-c+x+2y+z)}{\mathbf{H}(b+c+e+f+x+y+2z)\,\mathbf{H}(c+a+d+f+2x+y+z)\,\mathbf{H}(a+b+d+e+x+2y+z)},$$

where $A = a+b+c+d+e+f$.

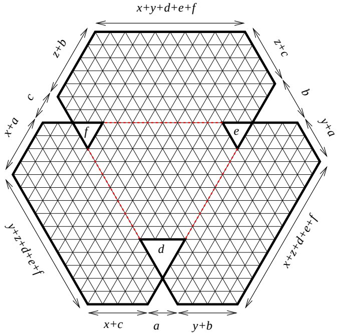
Example



$$\mathbf{M}\left(F\begin{pmatrix}2 & 1 & 2\\ 2 & 3 & 2\\ 3 & 2 & 2\end{pmatrix}\right) = 391,703,752,601,434,880,582,976,000,000,000$$

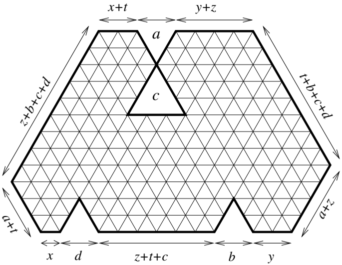
$$\sim 4 \times 10^{32}$$

Example



$$\mathbf{M}\left(F\begin{pmatrix}2 & 1 & 2\\ 2 & 3 & 2\\ 3 & 2 & 2\end{pmatrix}\right) = 2^{15}\cdot 3^3\cdot 5^9\cdot 7^9\cdot 11^3\cdot 13^3\cdot 17^4\cdot 23$$

M-shaped regions



$$B := B\begin{pmatrix} x & y & z & t \\ a & b & c & d \end{pmatrix}$$

M-shaped regions

M-shaped regions

Lemma

For non-negative integers a, b, c, d, x, y, z, t

$$\mathbf{M}\left(B\begin{pmatrix} x & y & z & t \\ a & b & c & d \end{pmatrix}\right) =$$

$$\frac{\mathbf{H}(x)\,\mathbf{H}(y)\,\mathbf{H}(z)\,\mathbf{H}(t)\,\mathbf{H}(a)^2\,\mathbf{H}(b)\,\mathbf{H}(c)\,\mathbf{H}(d)}{\mathbf{H}(a+c)\,\mathbf{H}(b+y)\,\mathbf{H}(d+x)}\,\frac{\mathbf{H}(a+b+c+d+x+z+2t)\,\mathbf{H}(a+b+c+d+y+2z+t)}{\mathbf{H}(a+b+c+d+z+t)^2}$$

$$\times\frac{\mathbf{H}(a+b+c+d+x+y+2z+2t)\,\mathbf{H}(a+b+c+d+x+y+z+t)}{\mathbf{H}(a+b+c+d+x+y+z+2t)\,\mathbf{H}(a+b+c+d+x+y+2z+t)}$$

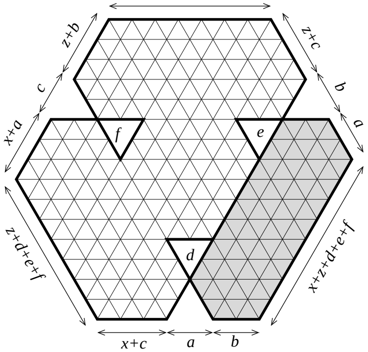
$$\times\frac{\mathbf{H}(a+b+c+y+z+t)\,\mathbf{H}(a+c+d+x+z+t)\,\mathbf{H}(b+c+d+z+t)^3}{\mathbf{H}(a+c+d+x+z+2t)\,\mathbf{H}(a+b+c+y+2z+t)\,\mathbf{H}(b+c+d+y+z+t)\,\mathbf{H}(b+c+d+x+z+t)}$$

$$\times\frac{\mathbf{H}(d+x+t)\,\mathbf{H}(b+y+z)\,\mathbf{H}(a+c+z+t)}{\mathbf{H}(b+c+z+t)\,\mathbf{H}(c+d+z+t)\,\mathbf{H}(a+y+z)\,\mathbf{H}(a+x+t)\,\mathbf{H}(b+d+z+t)}.$$

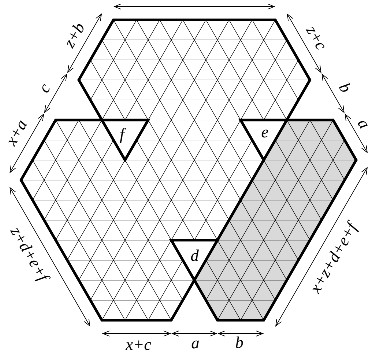
Proof of the main result

We prove by induction on $y+z$. The base cases: $y=0$ and $z=0$.

Case $y = 0$

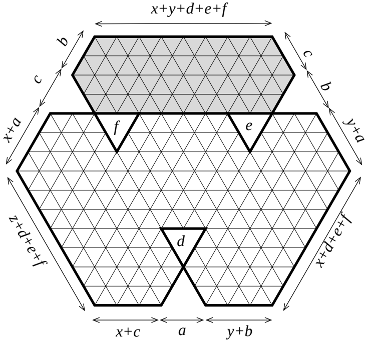


Case $y = 0$



$\mathbf{M}$(F-type Region) $=$ $\mathbf{M}$(Hexagon)$\mathbf{M}$(M-shaped region)

Case $z = 0$



Induction step.

- Assume that $y, z \geq 1$ and that the statement holds for all F-type regions with the sum of their y- and z-parameters less than $y + z$.

Induction step.

- Assume that $y, z \geq 1$ and that the statement holds for all F-type regions with the sum of their y- and z-parameters less than $y + z$.
- We use Kuo's condensation to create a recurrence.

Kuo's Graphical Condensation

A perfect matching of a graph G is a collection of disjoint edges covering all vertices of G.

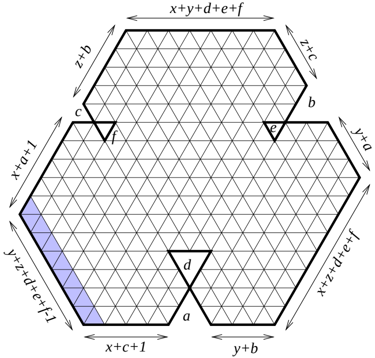
Theorem (Kuo 2004)

$G = (V_1, V_2, E)$ bipartite planar graph with $|V_1| = |V_2| + 1$. Assume that u, v, w, s are four vertices appearing in a cyclic order on a face of G so that $u, v, w \in V_1$ and $s \in V_2$. Then

$$\mathbf{M}(G - \{v\})\,\mathbf{M}(G - \{u, w, s\}) = \mathbf{M}(G - \{u\})\,\mathbf{M}(G - \{v, w, s\}) + \mathbf{M}(G - \{w\})\,\mathbf{M}(G - \{u, v, s\}).$$

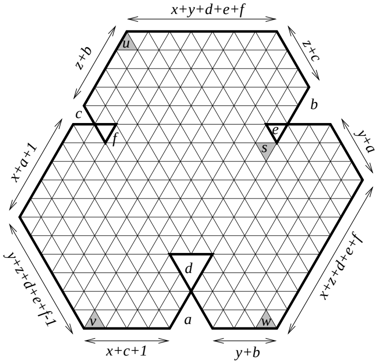
Apply Kuo Condensation

Let R be a region obtained from F by adding a band of unit triangles along the southwest side.

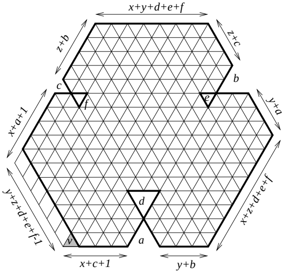


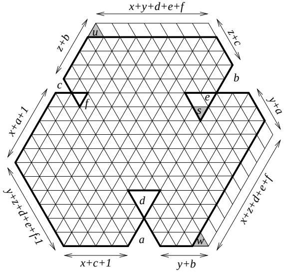
Let R be a region obtained from F by adding a band of unit triangles along the southwest side. Let G is the dual graph of the region R:

1. Vertices of G are unit triangles in R;
2. Edges of G are rhombi in R.



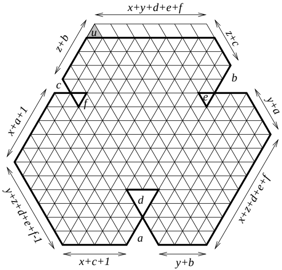
Apply Kuo Condensation

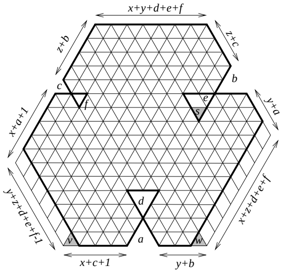




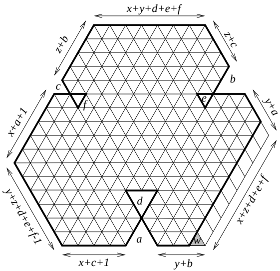
$$\mathbf{M}(G-\{v\}) = \mathbf{M}\left(F\begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix}\right) \text{ and } \mathbf{M}(G-\{u,w,s\}) = \mathbf{M}\left(F\begin{pmatrix} x+1 & y-1 & z-1 \\ a & b & c \\ d & e+1 & f \end{pmatrix}\right)$$

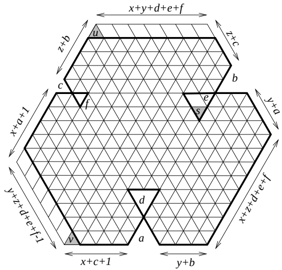
Apply Kuo Condensation





$$\mathbf{M}(G - \{u\}) = \mathbf{M}\left(F\begin{pmatrix} x+1 & y & z-1 \\ a & b & c \\ d & e & f \end{pmatrix}\right) \text{ and } \mathbf{M}(G - \{v, w, s\}) = \mathbf{M}\left(F\begin{pmatrix} x & y-1 & z \\ a & b & c \\ d & e+1 & f \end{pmatrix}\right)$$





$$\mathbf{M}(G - \{w\}) = \mathbf{M}\left(F\begin{pmatrix} x+1 & y-1 & z \\ a & b & c \\ d & e & f \end{pmatrix}\right) \text{ and } \mathbf{M}(G - \{u, v, s\}) = \mathbf{M}\left(F\begin{pmatrix} x & y & z-1 \\ a & b & c \\ d & e+1 & f \end{pmatrix}\right)$$

Recurrence

$$
\begin{aligned}
\mathbf{M}\left(F\begin{pmatrix} x & y & z\\ a & b & c\\ d & e & f\end{pmatrix}\right)\mathbf{M}\left(F\begin{pmatrix} x+1 & y-1 & z-1\\ a & b & c\\ d & e+1 & f\end{pmatrix}\right) = \\
\mathbf{M}\left(F\begin{pmatrix} x+1 & y & z-1\\ a & b & c\\ d & e & f\end{pmatrix}\right)\mathbf{M}\left(F\begin{pmatrix} x & y-1 & z\\ a & b & c\\ d & e+1 & f\end{pmatrix}\right) \\
+\mathbf{M}\left(F\begin{pmatrix} x+1 & y-1 & z\\ a & b & c\\ d & e & f\end{pmatrix}\right)\mathbf{M}\left(F\begin{pmatrix} x & y & z-1\\ a & b & c\\ d & e+1 & f\end{pmatrix}\right).
\end{aligned}
$$

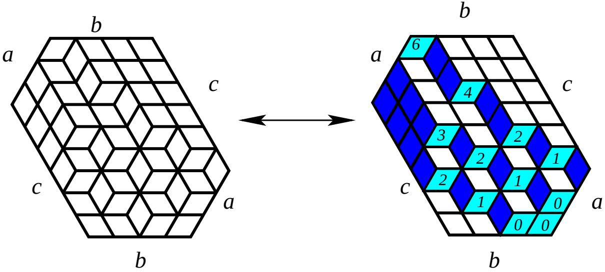
Recurrence

$$\mathbf{M}\left(F\begin{pmatrix}x & y & z\\ a & b & c\\ d & e & f\end{pmatrix}\right) = \frac{\mathbf{M}\left(F\begin{pmatrix}x+1 & y & z-1\\ a & b & c\\ d & e & f\end{pmatrix}\right)\mathbf{M}\left(F\begin{pmatrix}x & y-1 & z\\ a & b & c\\ d & e+1 & f\end{pmatrix}\right)}{\mathbf{M}\left(F\begin{pmatrix}x+1 & y-1 & z-1\\ a & b & c\\ d & e+1 & f\end{pmatrix}\right)}$$

$$+\frac{\mathbf{M}\left(F\begin{pmatrix}x+1 & y-1 & z\\ a & b & c\\ d & e & f\end{pmatrix}\right)\mathbf{M}\left(F\begin{pmatrix}x & y & z-1\\ a & b & c\\ d & e+1 & f\end{pmatrix}\right)}{\mathbf{M}\left(F\begin{pmatrix}x+1 & y-1 & z-1\\ a & b & c\\ d & e+1 & f\end{pmatrix}\right)}$$

□

6	4	2	1
3	2	1	0
2	1	0	0



Give three different colors to three types of rhombi: vertical, left, and right.

MacMahon's Theorem (again)

Theorem

$$\sum_{\pi} q^{|\pi|} = \frac{\mathbf{H}_q(a)\,\mathbf{H}_q(b)\,\mathbf{H}_q(c)\,\mathbf{H}_q(a+b+c)}{\mathbf{H}_q(a+b)\,\mathbf{H}_q(b+c)\,\mathbf{H}_q(c+a)},$$

where the sum is taken over all plane partitions π fitting in an $a \times b \times c$ box, and $|\pi|$ is the volume of π,

Definition:

- q-integer $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$
- q-factorial $[n]_q! = [1]_q[2]_q \ldots [n]_q$,
- q-hyperfactorial $\mathbf{H}_q(n) = [0]_q![1]_q! \ldots [n-1]_q!$.

Generalized plane partitions.

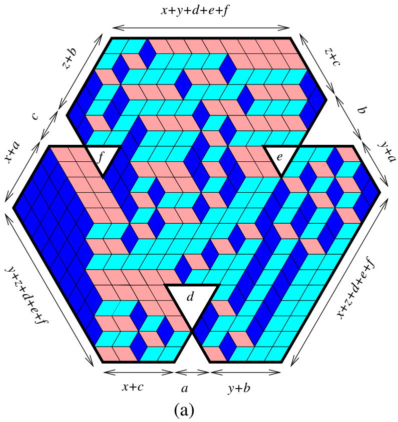
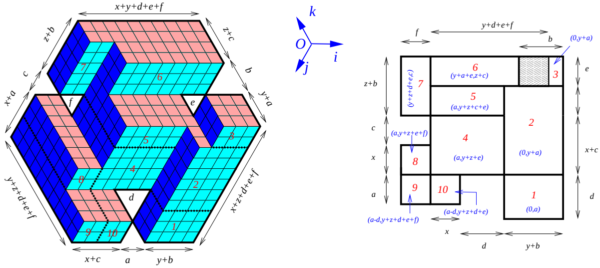


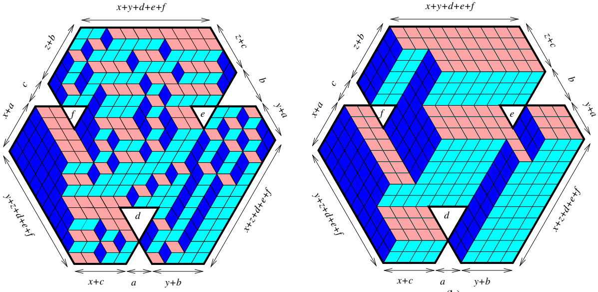
Figure: View a lozenge tiling as a stack of unit cubes.

The box $\mathcal{B}$



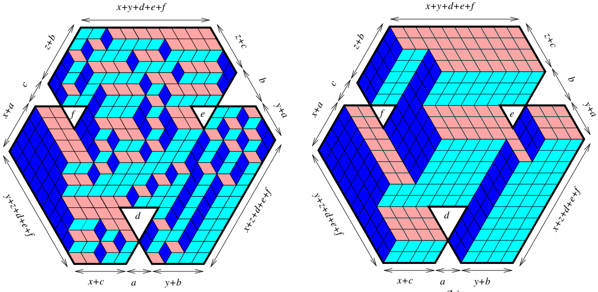
1. $\mathcal{B}$ consists of 11 non-overlapping smaller boxes, called rooms.
2. A room has four walls (left, right, back, front).
3. If two rooms share a portion of their walls, we remove it to make them connected.

Generalized plane partitions.



(a) (b)

- *Property of the stack:* The tops of columns are weakly decreasing along Oi and Oj.



(a) (b)

- *Property of the stack:* The tops of columns are weakly decreasing along Oi and Oj.
- We call the above stacks fitting in box $\mathcal{B}$ a generalized plane partitions.

A question inspired by MacMahon q-formula

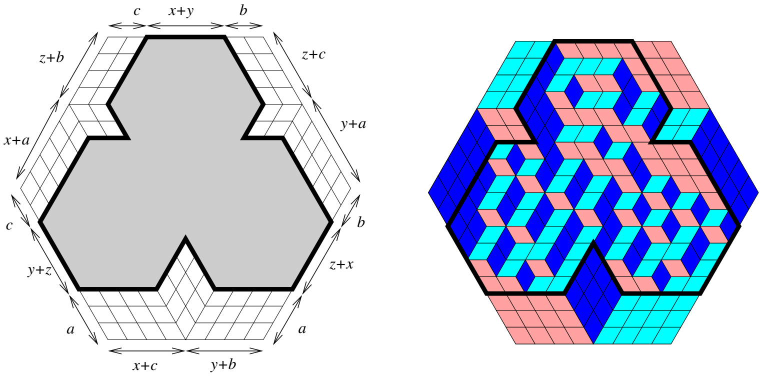
What is the generating function of the volume of generalized plane partitions π fitting in the compound box $\mathcal{B}$?

Theorem (L. 2015)

For nonnegative integers $x, y, z, a, b, c, d, e, f$

$$\sum_{\pi} q^{|\pi|} = \frac{\mathbf{H}_q(x)\,\mathbf{H}_q(y)\,\mathbf{H}_q(z)\,\mathbf{H}_q(a)^2\,\mathbf{H}_q(b)^2\,\mathbf{H}_q(c)^2\,\mathbf{H}_q(d)\,\mathbf{H}_q(e)\,\mathbf{H}_q(f)\,\mathbf{H}_q(d+e+f+x+y+z)^4}{\mathbf{H}_q(a+d)\,\mathbf{H}_q(b+e)\,\mathbf{H}_q(c+f)\,\mathbf{H}_q(d+e+x+y+z)\,\mathbf{H}_q(e+f+x+y+z)\,\mathbf{H}_q(f+d+x+y+z)}$$
$$\times \frac{\mathbf{H}_q(A+2x+2y+2z)\,\mathbf{H}_q(A+x+y+z)^2}{\mathbf{H}_q(A+2x+y+z)\,\mathbf{H}_q(A+x+2y+z)\,\mathbf{H}_q(A+x+y+2z)}$$
$$\times \frac{\mathbf{H}_q(a+b+d+e+x+y+z)\,\mathbf{H}_q(a+c+d+f+x+y+z)\,\mathbf{H}_q(b+c+e+f+x+y+z)}{\mathbf{H}_q(a+d+e+f+x+y+z)^2\,\mathbf{H}_q(b+d+e+f+x+y+z)^2\,\mathbf{H}_q(c+d+e+f+x+y+z)^2}$$
$$\times \frac{\mathbf{H}_q(a+d+x+y)\,\mathbf{H}_q(b+e+y+z)\,\mathbf{H}_q(c+f+z+x)}{\mathbf{H}_q(a+b+y)\,\mathbf{H}_q(b+c+z)\,\mathbf{H}_q(c+a+z)}$$
$$\times \frac{\mathbf{H}_q(A-a+x+y+2z)\,\mathbf{H}_q(A-b+2x+y+z)\,\mathbf{H}_q(A-c+x+2y+z)}{\mathbf{H}_q(b+c+e+f+x+y+2z)\,\mathbf{H}_q(c+a+d+f+2x+y+z)\,\mathbf{H}_q(a+b+d+e+x+2y+z)},$$

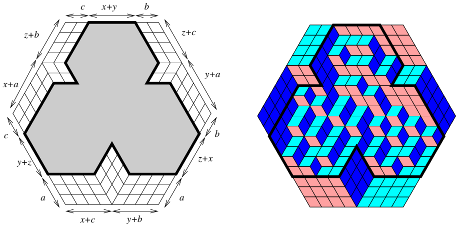
Plane partitions with constraints



(a) (b)

$$(z + x + a + b) \times (x + y + b + c) \times (y + z + c + a) - box$$

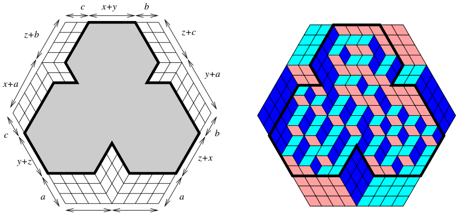
Plane partitions with constraints



(a) (b)

- The first $z + b$ entries of the columns $1, 2, \ldots, c$ are all $y + z + c + a$. Moreover, the remaining entries of these columns are at most $y + z + a$.

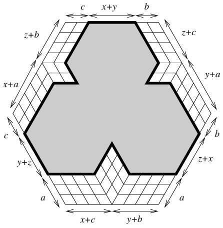
Plane partitions with constraints



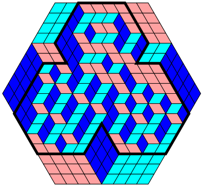
(a) (b)

- The first $z+b$ entries of the columns $1, 2, \ldots, c$ are all $y+z+c+a$. Moreover, the remaining entries of these columns are at most $y+z+a$.
- The last b entries of the columns $x+y+b+1, x+y+b+2, \ldots, x+y+b+c$ are all $y+a$.

Plane partitions with constraints

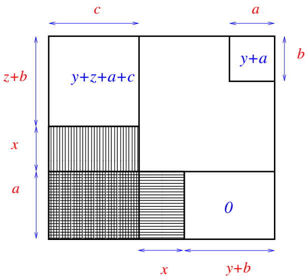


(a)


(b)

- The first $z + b$ entries of the columns $1, 2, \ldots, c$ are all $y + z + c + a$. Moreover, the remaining entries of these columns are at most $y + z + a$.
- The last b entries of the columns $x + y + b + 1, x + y + b + 2, \ldots, x + y + b + c$ are all $y + a$.
- The last $y + b$ entries of the rows $z + x + b + 1, z + x + b + 2, \ldots, z + x + b + a$ are all 0; and the remaining entries of these rows are at least a.

Plane partitions with constraints



- The first $z+b$ entries of the columns $1,2,\ldots,c$ are all $y+z+c+a$. Moreover, the remaining entries of these columns are at most $y+z+a$.
- The last a entries of the rows $1,2,\ldots,b$ are all $y+a$.
- The last $y+b$ entries of the rows $z+x+b+1, z+x+b+2, \ldots, z+x+b+a$ are all 0; and the remaining entries of these rows are at least a.

Plane partitions with constraints

Theorem

Let a, b, c, x, y, z be non-negative integers. The number of plane partitions having $z+x+a+b$ rows and $x+y+b+c$ columns with entries at most $y+z+c+a$, where all three constraints hold, is equal to

$$\mathbf{H}(x)\,\mathbf{H}(y)\,\mathbf{H}(z)\,\mathbf{H}(a)\,\mathbf{H}(b)\,\mathbf{H}(c)\,\mathbf{H}(x+y+z)$$
$$\times \frac{\mathbf{H}(a+b+c+2x+2y+2z)\,\mathbf{H}(a+b+c+x+y+z)^2}{\mathbf{H}(a+b+c+2x+y+z)\,\mathbf{H}(a+b+c+x+2y+z)\,\mathbf{H}(a+b+c+x+y+2z)}$$
$$\times \frac{\mathbf{H}(a+b+x+y+z)\,\mathbf{H}(a+c+x+y+z)\,\mathbf{H}(b+c+x+y+z)}{\mathbf{H}(a+x+y+z)^2\,\mathbf{H}(b+x+y+z)^2\,\mathbf{H}(c+x+y+z)^2}$$
$$\times \frac{\mathbf{H}(a+x+y)\,\mathbf{H}(b+y+z)\,\mathbf{H}(c+z+x)}{\mathbf{H}(a+b+y)\,\mathbf{H}(b+c+z)\,\mathbf{H}(c+a+x)}.$$

Next Projects

1. Find more examples of generalized plane partitions enumerated by simple product formulas.
2. Characterize the structure of the compound boxes, which give simple product formulas.
3. Symmetric generalized plane partitions.
4. Investigate (-1)-enumeration of generalize plane partitions.