FLATNESS AND TORSION IN TENSOR POWERS OF MODULES OVER LOCAL RINGS

SRIKANTH B. IYENGAR

This is a fairly literal transcript of my lectures at the 7th Japan-Vietnam Joint Seminar on Commutative Algebra, held in Quy Nhon, Vietnam, from 12th–16th December 2011. The aim of those lectures was to describe my recent work with Avramov [2] on a conjecture of Vasconcelos [18] concerning the relationship between flatness and torsion in tensor product of modules over regular rings, that in turn is inspired by results of Auslander's, contained in his landmark paper [5]. My intention was to make the lectures completely comprehensible for anyone familiar with the foundations of commutative algebra; for example, the core chapters in Matsumura's book [16], but not much else. This meant, for example, that I assumed familiarity with basic properties of depth for modules over local rings, but spent considerable time stating and proving the results about smooth algebras used in [2].

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1. Finitely generated modules over regular rings

Let R be a commutative noetherian ring, and let $R \longrightarrow A$ be a homomorphism *essentially of finite type*. This means that A is a localization of a finitely generated R-algebra, and hence it is itself commutative and noetherian. Henceforth I abbreviate 'essentially of finite type' to 'e.f.t.'

Let M be a finitely generated A-module. The problem that these lectures address is to find *effective* criteria for detecting when M is flat when viewed as an R-module. For applications to algebraic geometry the most interesting case is perhaps $M = A$. To explain what the issue is, we recall a result due to André [1, Lemma II.58].

1.1. *Andr's criterion for flatness.* The following conditions are equivalent:

(1) M is flat over R;
(2) $\operatorname{Tor}_1^R(M, N) = 0$ for each finitely generated R-module N;
(3) $\operatorname{Tor}_1^R(M, R/\mathfrak{m}) = 0$ for each maximal ideal $\mathfrak{m}$ of R.

When R is local, this gives a reasonable test for the flatness of M, for it is not difficult to check that $\operatorname{Tor}_1^R(M, R/\mathfrak{m}) = 0$ holds: one needs only the first two steps in a free resolution of $R/\mathfrak{m}$ over R, in other words, the generators of $\mathfrak{m}$ and of its first syzygy module, and those are well-known; see [16, Section 21].

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This ceases to be an reasonable test when R has infinitely many maximal ideals, even if M is finitely generated over R.

When R is *regular*, that is to say, when the local ring $R_{\mathfrak{p}}$ is regular for all $\mathfrak{p}$ in $\operatorname{Spec} R$, Vasconcelos [18, Conjecture 6.2] proposed a rather different test for flatness:

1.2. *Vasconcelos' conjecture.* If R is regular and $M^{\otimes_R^n}$ is torsion-free over R for *some* $n \geq \dim R$, then M is flat over R.

If true, then the following conditions would be equivalent:

(1) M is flat over R;
(2) $M^{\otimes_R^n}$ is torsion-free over R for all $n \geq 1$.
(3) $M^{\otimes_R^n}$ is torsion-free over R for some $n \geq \dim R$.
(4) $M^{\otimes_R^n}$ is torsion-free over R for $n = \dim R$.

Indeed, the implications (1) $\implies$ (2) $\implies$ (4) $\implies$ (3) is clear, whilst (3) $\implies$ (1) is Vasconcelos' conjecture.

The point about Conjecture 1.2 is that checking for torsion-freeness is computationally easy, at least if A is a finitely generated R-algebra; see [18, Section 2].

1.3. *Status of the conjecture.* Affirmative answers to Vasconcelos' conjecture are known in the following situations:

$\dim R \leq 1$: Then R is a Dedekind domain, and for *any* R-module, flatness and torsion-freeness are equivalent conditions; see [7, Ch. VII, Sections 4 & 5].

M is finitely generated as an R-module: This was proved by Auslander [5, Theorem 3.2]—see Theorem 1.9 below—and motivates Conjecture 1.2.

$\dim R \leq 2$: For $M = A$ and $n = 2$ this is proved in [18, Proposition 6.1], and for any R-module M, still for $n = 2$, this is proved in [2, Proposition 5.1].

R is smooth over some field: This is the main result of [2], and the argument is explained in some detail in these lectures. When the characteristic of the field is 0, a different proof, applicable also to analytic algebras, was given by Adamus, Bierstone, and Milman [4, Theorem 1.3]; see also A. Galligo and M. Kwieciński [10] for $M = A$.

The rest of this lecture is devoted to proving Auslander's result, referred to above. We begin by introducing the relevant notions.

1.4. *Torsion-freeness.* Let R be a commutative noetherian ring and M an R-module, not necessarily finitely-generated. The *torsion submodule* of M is

$$\top_R M = \{m \in M \mid rm = 0 \text{ for some non-zero divisor } r \in R.\}$$

Thus, $\top_R M$ is the kernel of the natural map $M \to M \otimes_R Q(R)$, where $Q(R)$ is the ring of fractions of R. There is then an exact sequence of R-modules

$$0 \longrightarrow \top_R M \longrightarrow M \longrightarrow \perp_R M \longrightarrow 0 \tag{1.4.1}$$

The R-module M is said to be *torsion* if $\top_R M = M$, and *torsion-free* if $M \cong \perp_R M$. Note that M is torsion if and only if $M \otimes_R Q(R) = 0$, while it is torsion-free if and only if $M \subseteq M \otimes_R Q(R)$ holds.

The following facts concerning these notions are easy to verify:

(1) $\top_R M$ is torsion and $\perp_R M$ is torsion-free.
(2) $(\top_R M)_{\mathfrak{p}} \cong \top_R M_{\mathfrak{p}} \cong \top_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$, for each $\mathfrak{p} \in \operatorname{Spec} R$.

(3) M is torsion-free if and only if $M_{\mathfrak{p}}$ is torsion-free over $R_{\mathfrak{p}}$ for all $\mathfrak{p}$ not in $\operatorname{Ass} R$, the associated primes of R.
(4) M is torsion-free if $\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \geq 1$ for $\mathfrak{p} \notin \operatorname{Ass} R$; equivalently, if $\operatorname{Ass}_R M \subseteq \operatorname{Ass} R$. The converse holds R has no embedded associated primes, that is to say, when $\operatorname{Ass} R$ equals $\operatorname{Min} R$, the minimal primes of R.
(5) When R has no embedded associated primes, and M is finitely-generated, it is torsion-free if and only if there is an embedding $M \subseteq R^n$.

1.5. *Regular rings.* In the remainder of this lecture, we assume R is regular local ring. The "Rigidity Theorem" below was proved by Auslander [5, Theorem 2.1] for unramified rings, and for all regular rings by Lichtenbaum [15, Corollary 1].

Theorem 1.6. *Let R be a regular local ring and M, N finitely generated R-modules. If $\operatorname{Tor}_i^R(M, N) = 0$ for some $i \geq 0$, then $\operatorname{Tor}_j^R(M, N) = 0$ for all $j \geq i$.* □

A proof of this result for smooth algebras is discussed in Section 2.

Theorem 1.7. *Let R be a regular local ring and M, N non-zero finitely generated R-modules. If $M \otimes_R N$ is torsion-free, then*
(1) $\operatorname{Tor}_i^R(M, N) = 0$ *for* $i \geq 1$*;*
(2) M *and* N *are torsion-free.*

Proof. For the moment, assume M is torsion-free, so $\top_R M = 0$. Since M is finitely generated, there is an exact sequence of R-modules $0 \to M \to R^n \to M' \to 0$, and this gives rise to the exact sequence and the isomorphisms below:

$$0 \longrightarrow \operatorname{Tor}_1^R(M', N) \longrightarrow M \otimes_R N \longrightarrow N^n \longrightarrow M' \otimes_R N \longrightarrow 0$$
$$\operatorname{Tor}_{i+1}^R(M', N) \cong \operatorname{Tor}_i^R(M, N) \quad \text{for all } i \geq 1$$

Since $\operatorname{Tor}_1^R(M', N)$ is torsion whilst $M \otimes_R N$ is torsion-free, by hypothesis, the exact sequence implies $\operatorname{Tor}_1^R(M', N) = 0$, and hence that $\operatorname{Tor}_j^R(M', N) = 0$ for all $j \geq 1$, by the Rigidity Theorem 1.6. The isomorphisms then yield $\operatorname{Tor}_j^R(M, N) = 0$ for all $j \geq 1$.

It thus suffices to prove that M is torsion-free; by symmetry it would follow that so in N. Applying $- \otimes_R N$ to (1.4.1) yields an exact sequence

$$\operatorname{Tor}_1^R(\perp_R M, N) \longrightarrow (\top_R M) \otimes_R N \xrightarrow{0} M \otimes_R N \xrightarrow{\cong} (\perp_R M) \otimes_R N \longrightarrow 0$$

The second map (from the left) has to be zero because $(\top_R M) \otimes_R N$ is torsion whilst $M \otimes_R N$ is torsion-free, by hypothesis. This then explains also the isomorphism above. In particular, $(\perp_R M) \otimes_R N$ is torsion-free and since $\perp_R M$ is torsion-free, the argument in the preceding paragraph yields $\operatorname{Tor}_1^R(\perp_R M, N) = 0$. Another appeal to the exact sequence above gives $(\top_R M) \otimes_R N = 0$, and since $N \neq 0$ and R is local, it follows that $\top_R M = 0$; this is by an application of Nakayama's Lemma, which applies since M, hence also $\top_R M$, and N are finitely generated. □

Recall that a local ring R is regular if and only if $\operatorname{pd}_R M$, the *projective dimension* of M, is finite for each finitely generated R-module M; see [16, Theorem 19.2].

Lemma 1.8. *Let R be a regular local ring and M, N finitely generated R-modules.*
(1) *If M is torsion-free, then* $\operatorname{pd}_R M \leq \dim R - 1$.
(2) *If $\operatorname{Tor}_i^R(M, N) = 0$ for $i \geq 1$, then* $\operatorname{pd}_R(M \otimes_R N) = \operatorname{pd}_R M + \operatorname{pd}_R N$.

Proof. For (1), one can use either the Auslander-Buchsbaum Equality [16, Theorem 19.1], or note that there is an exact sequence $0 \to M \to R^n \to M' \to 0$, as M is torsion-free, and track depth along that.

(2): Let F and G be minimal free resolutions of M and N, respectively. The hypothesis translates to the statement that the complex $F \otimes_R G$ is a free resolution of $M \otimes_R N$, since $\mathrm{H}_i(F \otimes_R G) \cong \mathrm{Tor}_i^R(M, N)$. It remains to observe that $F \otimes_R G$ is minimal, because F and G are. $\square$

The result below is a minor extension of [5, Theorem 3.2], which focuses on the case $d \geq \dim R$. The non-trivial ingredient in its proof is the Rigidity Theorem 1.6.

Theorem 1.9. *Let R be a regular local ring. If M is finitely generated R-module and $M^{\otimes_R^d}$ is torsion-free for some $d \geq 1$, then there is an inequality*

$$\mathrm{pd}_R M \leq \left\lfloor \frac{\dim R - 1}{d} \right\rfloor .$$

In particular, if $d \geq \dim R$, then M is free.

Proof. The first and the second implications below are by Theorem 1.7, while the third one is by Lemma 1.8.

$$\begin{aligned} M^{\otimes_R^d} \text{ torsion-free} &\implies M^{\otimes_R^p} \text{ torsion-free for each } p = 1,..,d \\ &\implies \mathrm{Tor}_i^R(M, M^{\otimes_R^{p-1}}) = 0 \text{ for each } p = 1,..,d \text{ and } i \geq 1 \\ &\implies \mathrm{pd}_R(M^{\otimes_R^p}) = \mathrm{pd}_R M + \mathrm{pd}_R(M^{\otimes_R^{p-1}}) \text{ for each } p = 1,..,d \\ &\implies \mathrm{pd}_R(M^{\otimes_R^d}) = d \,\mathrm{pd}_R M \end{aligned}$$

It remains to note that $\mathrm{pd}_R(M^{\otimes_R^d}) \leq \dim R - 1$, again by Lemma 1.8. $\square$

The result above is sharp, as the following example from [5, pp. 638] shows.

Example 1.10. Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension d and pick a minimal generating set $r_1, \dots, r_d$ for the maximal ideal $\mathfrak{m}$. Consider the R-module W with presentation

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} r_1 \\ \vdots \\ r_d \end{bmatrix}} R^d \longrightarrow W \longrightarrow 0 .$$

In particular, W is not free. It is not difficult to verify that $W^{\otimes_R^n}$ is torsion-free for each $n = 1,..,d-1$, and that $\top_R(W^{\otimes_R^d}) \cong k$; see [14].

2. Rigidity of Tor over smooth algebras

Let K be a field and R a K-algebra e.f.t. The *enveloping algebra* $R^e = R \otimes_K R$ is again a K-algebra e.f.t. and the map

$$\mu \colon R^e \longrightarrow R \quad \text{where } \mu(r \otimes s) = rs$$

is a homomorphism of K-algebras. Its kernel is the ideal $(r \otimes 1 - 1 \otimes r \mid r \in R)$.

Let M and N be R-modules. There is a natural R^e-module structure on $M\otimes_K N$, defined by $(r\otimes s)\cdot(m\otimes n) = (rm\otimes sn)$, and the natural map $(M\otimes_K N) \to (M\otimes_R N)$ induces an isomorphism

$$R \otimes_{R^e} (M \otimes_K N) \cong (M \otimes_R N)\,.$$

The derived version of this is the "reduction to the diagonal" isomorphisms:

Lemma 2.1. *There are isomorphisms* $\operatorname{Tor}_i^{R^e}(R, M\otimes_K N) \cong \operatorname{Tor}_i^R(M,N)$ *for* $i \in \mathbb{Z}$.

Proof. Let $F \simeq M$ and $G \simeq N$ be free resolutions, over R, of M and N, respectively. Observe that $F \otimes_K G$ is a complex of free R^e-modules with

$$\mathrm{H}_i(F \otimes_K G) = \begin{cases} M \otimes_K N & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

Thus, $F \otimes_K G$ is a resolution of $M \otimes_K N$ over R^e. Thus

$$\operatorname{Tor}_i^{R^e}(R, M \otimes_K N) \cong \mathrm{H}_i(R \otimes_{R^e} (F \otimes_K G)) \cong \mathrm{H}_i(F \otimes_R G) \cong \operatorname{Tor}_i^R(M,N)\,. \qquad \square$$

The point of the preceding lemma is that (homological) properties of R as an R^e-module have an impact on corresponding properties of all R-modules. Theorem 2.5 below is a perfect archetype of this phenomenon.

2.2. *Smooth algebras.* Let K be a field. We say that a K-algebra R e.f.t. is *smooth* if for all (equivalently, finite) extensions $K \subseteq L$ of fields, the ring $L \otimes_K R$ is regular.

A smooth K-algebra is evidently regular, but the converse does not hold, as the second example below shows. However, when K has characteristic 0, or is a perfect field of positive characteristic, these two notions coincide; this follows from results of Zariski [19, Section 7.2 and and 8.3].

Example 2.3. A polynomial ring $K[t_1,\dots,t_n]$ in indeterminates $t_1,\dots,t_n$ is smooth over K, because for any extension $K \subseteq L$ of field, $L\otimes_K K[\boldsymbol{t}] \cong L[\boldsymbol{t}]$, which is regular.

Example 2.4. Suppose K is a field of characteristic $p > 0$ and there exists an $a \in K$ that does not have a pth root in K, then the K-algebra $R = K[t]/(t^p - a)$, which is a field obtained by adjoining a pth root of a, is *not* smooth, for if L is any extension field of R containing a pth root of a (for example, R itself), then

$$L \otimes_K R \cong L[t]/(t^p - a) \cong L[u]/(u^p)$$

where $u = t - a^{1/p}$, and this ring is not regular.

The next result relates smoothness of the K-algebra R to properties of R viewed as an R^e-module. The equivalence of conditions (1)–(3) is implicit in the work of Hochschild, Kostant, and Rosenberg [11], at least when K is perfect. The implication (4) $\Longrightarrow$ (1) goes further back than that; see [7, Ch. IX, Proposition 7.5].

Theorem 2.5. *The following conditions are equivalent:*

(1) *R is a smooth K-algebra.*
(2) *R^e is a regular ring.*
(3) $\operatorname{Ker}(\mu)_{\mathfrak{q}}$ *generated by a regular sequence in* $S_{\mathfrak{q}}$ *for each* $\mathfrak{q} \in \operatorname{Spec} S$.
(4) $\operatorname{pd}_{R^e} R$ *is finite.*

Proof. (1) $\implies$ (2). The homomorphism of rings $\iota\colon R \to R^e$ where $i(r) = r \otimes 1$ is flat, because it is obtained from the structure homomorphism $K \to R$ by base-change. For each $\mathfrak{p} \in \operatorname{Spec} R$, the fiber over $\mathfrak{p}$ is the ring

$$k(\mathfrak{p}) \otimes_R (R \otimes_K R) \cong k(\mathfrak{p}) \otimes_K R\,,$$

where $k(\mathfrak{p})$ is the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ at $\mathfrak{p}$. This is regular, because R is smooth. Since R and the fibers of ι are regular, R^e is regular; see [16, Theorem 23.7].

(2) $\implies$ (3). For each $\mathfrak{q} \in \operatorname{Spec} R^e$ containing $\operatorname{Ker}(\mu)$, the map $\mu_{\mathfrak{q}}\colon (R^e)_{\mathfrak{q}} \to R_{\mathfrak{q}}$ is a surjective homomorphism between regular local rings, and hence it kernel, that is to say, $\operatorname{Ker}(\mu)_{\mathfrak{q}}$, is generated by a regular sequence, by a theorem of Chevalley [8, Proposition 9].

(3) $\implies$ (4). Fix a $\mathfrak{q} \in \operatorname{Spec} R^e$, and let $\boldsymbol{x} = x_1, \dots, x_c$ be regular sequence in $(R^e)_{\mathfrak{q}}$ such that $(\boldsymbol{x}) = \operatorname{Ker}(\mu)_{\mathfrak{q}}$. The Koszul complex on $\boldsymbol{x}$ is then a free resolution of $R_{\mathfrak{q}}$ over $(R^e)_{\mathfrak{q}}$, so that there are (in)equalities

$$\operatorname{pd}_{(R^e)_{\mathfrak{q}}} R_{\mathfrak{q}} = c \le d$$

where d is the minimal number of generators of $\operatorname{Ker}(\mu)$. It follows that $\operatorname{pd}_{R^e} R \le d$.

(4) $\implies$ (1). The reduction to the diagonal isomorphisms, Lemma 2.1, yield

$$\operatorname{Tor}_i^R(M,N) \cong \operatorname{Tor}_i^{R^e}(R, M \otimes_K N) = 0 \quad \text{for all } i > \operatorname{pd}_{R^e} R\,.$$

This proves that R is regular. A more direct proof is to note that if F is a projective resolution of R over R^e, then for any R-module M, the complex $F \otimes_R M$, with the R action from the left, is a projective resolution of M over R.

Consider any extension of fields $K \to L$. One has a natural isomorphism

$$(R \otimes_K L) \otimes_L (R \otimes_K L) \cong R^e \otimes_K L$$

and the projective dimension of the L-algebra $R \otimes_K L$ as a module over its enveloping algebra $(R \otimes_K L)^e$ is at most $\operatorname{pd}_{R^e} R$, and hence it is also finite. Therefore $R \otimes_K L$ is also regular, by the argument in the preceding paragraph. □

One of the consequences of the theorem above is that over smooth algebras the computation of Tor reduces to one about Koszul homology, over (a localization of) the enveloping algebra, as explained in the remark below.

In what follows, given a sequence $\boldsymbol{x} = x_1, \dots, x_c$ of elements in a commutative ring S and an S-module L, we write $\mathrm{H}_i(\boldsymbol{x}; L)$ for the ith Koszul homology with respect to $\boldsymbol{x}$, with coefficients in L.

Remark 2.6. Let R be a smooth K-algebra, and let M, N be R-modules.

Fix a prime ideal $\mathfrak{q} \supseteq \operatorname{Ker}(\mu)$ in R^e and a regular sequence $\boldsymbol{x} = x_1, \dots, x_c$ in $(R^e)_{\mathfrak{q}}$ that generates $\operatorname{Ker}(\mu)_{\mathfrak{q}}$; such a sequence exists, by Theorem 2.5. For each $i \in \mathbb{Z}$, there are then isomorphisms

$$\begin{aligned}
\operatorname{Tor}_i^R(M,N)_{\mathfrak{q}} &\cong \operatorname{Tor}_i^{R^e}(R, M \otimes_K N)_{\mathfrak{q}} \\
&\cong \operatorname{Tor}_i^{(R^e)_{\mathfrak{q}}}(R_{\mathfrak{q}}, (M \otimes_K N)_{\mathfrak{q}}) \\
&\cong \mathrm{H}_i(\boldsymbol{x}; (M \otimes_K N)_{\mathfrak{q}})\,.
\end{aligned}$$

Indeed, the first isomorphism is obtained from Lemma 2.1 and the third holds because the Koszul complex in $\boldsymbol{x}$ is a free resolution of $R_{\mathfrak{q}}$ over $(R^e)_{\mathfrak{q}}$.

2.7. *Modules essentially of finite type.* Let R be a commutative noetherian ring. Following [2], an R-module M is said to be *essentially of finite type*, again abbreviated to e.f.t., if there exists an R-algebra A e.f.t. such that the R-action on M structure extends to A, and for that A-module structure M is finitely generated.

For example, any finitely generated R-module is e.f.t., but there are many more.

Remark 2.8. Suppose R is a K-algebra e.f.t., where K is any commutative noetherian ring. If M and N are e.f.t. R-modules, with respect to R-algebra A and B, respectively, then $M \otimes_K N$ is an e.f.t. R^e-module, with respect to $A \otimes_K B$, and hence also an e.f.t. R-module, since the homomorphism $R \to R^e$ is e.f.t.

In particular, taking $K = R$, one gets that $M \otimes_R N$ is an an e.f.t. R-module.

The rigidity property of Koszul homology for finitely generated modules extends to e.f.t. modules.

Lemma 2.9. *Let $\boldsymbol{x} = x_1, \dots, x_c$ be a sequence of elements in commutative noetherian ring S, and let L be an e.f.t. S-module. If $\mathrm{H}_i(\boldsymbol{x}, L) = 0$ for some $i \geq 0$, then $\mathrm{H}_j(\boldsymbol{x}, L) = 0$ for all $j \geq i$.*

Proof. Suppose L is finitely generated over a e.f.t. S-algebra A. With $\alpha\colon S \to A$ the structure morphism, the invariance of base property of Koszul homology implies that $\mathrm{H}_i(\boldsymbol{x}; L)$ coincides the Koszul homology of the sequence $\alpha(x_1), \dots, \alpha(x_c)$, with coefficients in the A-module L. Thus the desired result follows from the rigidity of Koszul homology of finitely generated modules, see [6, Proposition 2.6]. □

For smooth algebras, the result below contains Theorem 1.6.

Theorem 2.10. *Suppose that R is smooth over some field, and let M, N be R-modules e.f.t. If $\mathrm{Tor}_i^R(M, N) = 0$ for some $i \geq 0$, then*

$$\mathrm{Tor}_j^R(M, N) = 0 \quad \textit{for all} \quad j \geq i.$$

Proof. It suffices to prove that for each $\mathfrak{q}$ in $\mathrm{Spec}(R^e)$ containing $\mathrm{Ker}(\mu)$, one has

$$\mathrm{Tor}_j^R(M, N)_{\mathfrak{q}} = 0 \quad \text{for all} \quad j \geq i.$$

In view of the discussion in Remark 2.6, this follows from the rigidity of Koszul homology, Lemma 2.9, since $(M \otimes_K N)_{\mathfrak{q}}$ is an e.f.t. module over $(R^e)_{\mathfrak{q}}$. □

3. Torsion in tensor products

In this lecture, the goal is extend Theorem 1.7 to modules e.f.t., albeit only over smooth algebras. We begin with some elementary remarks concerning e.f.t. modules.

Let M be an R-module e.f.t., with respect to an e.f.t. R-algebra A.

3.1. We may assume A is of the localization of a polynomial ring over R, and in particular, a flat R-algebra.

3.2. The R-modules $\top_R M$ and $\perp_R M$ are e.f.t. with respect to A.

Indeed, it is not hard to check that $\top_R M$ is an A-submodule of M, and hence also finitely generated as an A-module. It then follows that the quotient, $\perp_R M$, is also a finitely generated A-module.

3.3. If N is an R-module e.f.t. with respect to B, then the R-modules $\operatorname{Tor}_i^R(M,N)$ are e.f.t. with respect to $A\otimes_R B$, for all $i\in\mathbb{Z}$.

Indeed, let M and N be finitely generated over R-algebras A and B, respectively. We may assume also that A is flat over R, by 3.1. Let $F\simeq M$ be a resolution of M over A, with each F_i finitely generated and free; this is possible because A is noetherian and M is a finitely generated A-module. Then F is a flat resolution of M over R, because A is flat as an R-module, so $\operatorname{Tor}_i^R(M,N)$ can be computed as homology of the complex $F\otimes_R N$. It remains to observe that $F\otimes_R N$ is a complex over the noetherian ring $A\otimes_R B$, consisting of finitely generated modules.

3.4. If $L\subseteq M$ is an A-submodule and $L\otimes_R M=0$, then $L=0$; here L is viewed as an R-module by restriction of scalars. In particular, $M\otimes_R M=0$ implies $M=0$.

Indeed, there is surjection $L\otimes_R M\to L\otimes_A M$, so if $L\otimes_R M=0$ holds, then it follows that $L\otimes_A M=0$. Fix a prime ideal $\mathfrak{q}$ of A. Since $L_\mathfrak{q}\otimes_{A_\mathfrak{q}} M_\mathfrak{q}=0$, and $L_\mathfrak{q}$ is an $A_\mathfrak{q}$-submodule of $M_\mathfrak{q}$ and the latter is finitely generated, it follows that $L_\mathfrak{q}=0$; this is by an application of Nakayama's Lemma [16, Theorem 4.8]. Since $\mathfrak{q}$ was arbitrary, one deduces that $L=0$.

Over smooth algebras, the following result is an extension of Theorem 1.7.

Theorem 3.5. *Let R be a smooth algebra over some field. If M and N are e.f.t. R-modules such that $M\otimes_R N$ is torsion-free, then*

(1) $\operatorname{Tor}_i^R(M,N)=0$ *for* $i\geq 1$*;*
(2) $\operatorname{Tor}_i^R(\top_R M,N)=0=\operatorname{Tor}_i^R(M,\top_R N)$ *for all* $i\in\mathbb{Z}$.

Observe that (2) contains the assertion: $(\top_R M)\otimes_R N=0=M\otimes_R(\top_R N)$.

The key new idea in the proof of the theorem is the following result, due to Huneke and R. Wiegand [12, Theorem 6.3]; see also Remark 3.7.

3.6. Let $\boldsymbol{x}$ be a sequence of elements in local ring S and L a finitely generated S-module. If $\operatorname{length}_S \mathrm{H}_1(\boldsymbol{x};L)$ is non-zero and finite, then $\operatorname{depth}_S \mathrm{H}_0(\boldsymbol{x};L)=0$.

We can now present the proof of Theorem 3.5. Part of the argument is reminiscent of the proof of Theorem 1.7 dealing with finitely generated modules, but the crux of proof (the Claim below) uses a different idea.

Proof of Theorem 3.5. The difficult step is proving the

Claim: $\operatorname{Tor}_1^R(M,N)=0$ whenever the R-module $M\otimes_R N$ is torsion-free.

Indeed, the Rigidity Theorem 2.10 then gives $\operatorname{Tor}_i^R(M,N)=0$ for all $i\geq 1$, and that is the first of the desired statements. Applying $-\otimes_R N$ to the exact sequence (1.4.1) then yields an exact sequence

$$0\longrightarrow \operatorname{Tor}_1^R(\perp_R M,N)\longrightarrow \top_R M\otimes_R N\xrightarrow{0} M\otimes_R N\xrightarrow{\cong}\perp_R M\otimes_R N\longrightarrow 0$$

Now $\top_R M\otimes_R N$ is always torsion, whilst $M\otimes_R N$ is torsion-free, by hypothesis; this explains the 0 in the diagram above, and hence also the isomorphism. This implies $\perp_R M\otimes_R N$ is torsion-free and so $\operatorname{Tor}_1^R(\perp_R M,N)=0$, by another application of the claim; note that $\perp_R M$ is also e.f.t. over R, by 3.2. Hence $\top_R M\otimes_R N=0$, by the exact sequence above, and that implies $\operatorname{Tor}_i^R(\top_R M,N)=0$ for all i, by the Rigidity Theorem 2.10. This justifies the first part of (2), in the statement of the theorem; the second part holds by symmetry.

It thus remains to prove the claim. To this end, suppose R is smooth over K, and let A and B, be e.f.t. R-algebras over which M and N, respectively, are finitely generated. Set

$$Q = R \otimes_K R \quad \text{and} \quad C = A \otimes_K B$$

By 3.3, each $\operatorname{Tor}_i^R(M,N)$ is finitely generated over $A \otimes_R B$, and hence over C. Assume $\operatorname{Tor}_1^R(M,N) \neq 0$, and pick an $\mathfrak{m} \in \operatorname{Spec} C$ minimal in its support, so that $\operatorname{Tor}_1^R(M,N)_\mathfrak{m}$ is non-zero but of finite length. Set $\mathfrak{q} = \mathfrak{m} \cap Q$ and $\mathfrak{p} = \mathfrak{q}R$, and consider the local homomorphisms

$$R_\mathfrak{p} \twoheadleftarrow Q_\mathfrak{q} \to C_\mathfrak{m} \tag{3.6.1}$$

induced by the surjection $R \twoheadleftarrow Q$ and the homomorphism $R \otimes_K R \to A \otimes_K B$. Since R is smooth over K, there exists a $Q_\mathfrak{q}$-regular sequence $\boldsymbol{x}$ such that $R_\mathfrak{p} \cong Q_\mathfrak{q}/(\boldsymbol{x})$ and isomorphisms

$$\operatorname{Tor}_i^R(M,N)_\mathfrak{m} \cong \mathrm{H}_i(\boldsymbol{x}; (M \otimes_K N)_\mathfrak{m})\,;$$

see Remark 2.6. Since of $\operatorname{length}_{C_\mathfrak{m}} \operatorname{Tor}_1^R(M,N)_\mathfrak{m}$ is non-zero and finite, it follows from the isomorphisms above and 3.6 that $\operatorname{depth}_{C_\mathfrak{m}}(M \otimes_R N)_\mathfrak{m} = 0$, that is to say, there is a non-zero $C_\mathfrak{m}$-linear map on the right-hand side:

$$\frac{Q_\mathfrak{q}}{\mathfrak{q}Q_\mathfrak{q}} \hookrightarrow \frac{C_\mathfrak{m}}{\mathfrak{m}C_\mathfrak{m}} \hookrightarrow (M \otimes_R N)_\mathfrak{m}$$

The one on the left is induced by the local homomorphism in (3.6.1); it is $Q_\mathfrak{q}$-linear. It follows that the depth of $(M \otimes_R N)_\mathfrak{m}$ over $Q_\mathfrak{q}$ is zero. It follows that the same holds for $(M \otimes_R N)_\mathfrak{q}$, since $(M \otimes_R N)_\mathfrak{m} \cong ((M \otimes_R N)_\mathfrak{q})_\mathfrak{m}$.

The action of Q on $M \otimes_R N$ factors through R, so $\operatorname{depth}_{R_\mathfrak{p}}(M \otimes_R N)_\mathfrak{p} = 0$, that is to say, $\mathfrak{p} \in \operatorname{Ass}_R(M \otimes_R N)$. The hypothesis that $M \otimes_R N$ is torsion-free then implies that $\mathfrak{p} \in \operatorname{Ass} R$ as well, and hence that $R_\mathfrak{p}$ is a field. Therefore one has

$$\operatorname{Tor}_1^R(M,N)_\mathfrak{p} \cong \operatorname{Tor}_1^{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$$

from which it follows that $\operatorname{Tor}_1^R(M,N)_\mathfrak{m}$ is 0 as well, since it is obtained from $\operatorname{Tor}_1^R(M,N)_\mathfrak{p}$ by further localization. This contradicts the choice of $\mathfrak{m}$. □

Remark 3.7. With notation as in 3.6, the main result of [17] by K. Takahashi, H. Terakawa, K.-I. Kawasaki, Y. Hinohara is: for *any* $i \geq 0$, if $\operatorname{length}_S \mathrm{H}_{i+1}(\boldsymbol{x}; L)$ is non-zero and finite, then $\operatorname{depth}_S \mathrm{H}_i(\boldsymbol{x}; L) = 0$.

As explained in [17], this gives the following refinement of the rigidity of Koszul homology: Let S be any commutative noetherian ring, $\boldsymbol{x}$ a finite sequence of elements in S, and L a finitely generated S-module. The result stated in the preceding paragraph translates to the claim that there is an inclusion

$$\operatorname{Min}_S \mathrm{H}_{i+1}(\boldsymbol{x}; L) \subseteq \operatorname{Ass}_S \mathrm{H}_i(\boldsymbol{x}; L) \quad \text{for all } i \geq 0.$$

From this one gets inclusion $\operatorname{Supp}_S \mathrm{H}_{i+1}(\boldsymbol{x}; L) \subseteq \operatorname{Supp}_S \mathrm{H}_i(\boldsymbol{x}; L)$, which is equivalent to the rigidity statement in **??**.

4. Torsion in tensor powers

It is easy to construct examples of a pair of e.f.t. modules, neither of which is torsion-free but whose tensor product is, even over a local ring. This is in contract to the case of finitely generated modules; see Theorem 1.7. The following is all one has in this generality.

Theorem 4.1. *Let R be a smooth algebra over some field and M an e.f.t. R-module. If $M^{\otimes_R^d}$ is torsion-free for some $d \geq 1$, then for each $p = 1, \dots, d$, the following statements hold:*

1. *$M^{\otimes_R^p}$ is torsion-free;*
2. *$\operatorname{Tor}_i^R(M, M^{\otimes_R^{p-1}}) = 0$ for all $i \geq 1$.*

Proof. Noting that $M^{\otimes_R^d} \cong M \otimes_R M^{\otimes_R^{d-1}}$, when $M^{\otimes_R^d}$ is torsion-free, Theorem 3.5 yields that $\operatorname{Tor}_i^R(M, M^{\otimes_R^{d-1}}) = 0$ for all $i \geq 1$, and that $M \otimes_R \top_R(M^{\otimes_R^{d-1}}) = 0$. Applying $M^{\otimes_R^{d-2}} \otimes_R -$ to the last equality then gives

$$M^{\otimes_R^{d-1}} \otimes_R \top_R(M^{\otimes_R^{d-1}}) = 0$$

and this entails $\top_R(M^{\otimes_R^{d-1}}) = 0$, by 3.2 and 3.4. Thus the R-module $M^{\otimes_R^{d-1}}$ is torsion-free; now iterate. □

4.2. *Codepth.* Let $(S, \mathfrak{n}, k)$ be a local ring. The *codepth* of an S-module M is

$$\operatorname{codepth}_S M = \sup\{i \geq 0 \mid \operatorname{Tor}_i^S(k, M) \neq 0\}.$$

When M is finitely generated, this number equals the projective dimension of M.

Compare the result below with Lemma 1.8.

Lemma 4.3. *Let $(S, \mathfrak{n}, k)$ be a regular local ring, and let M, N be S-modules.*

1. $\operatorname{codepth}_S M = \dim S - \operatorname{depth}_S M$ *;*
2. $\operatorname{codepth}_S M \leq \dim S - 1$ *when M is torsion-free;*
3. *If $\operatorname{Tor}_i^S(M, N) = 0$ for all $i \geq 1$, then*

$$\operatorname{codepth}_S (M \otimes_S N) = \operatorname{codepth}_S M + \operatorname{codepth}_S N\,.$$

Proof. Let $s_1, \dots, s_e$ be a minimal generating set for the ideal $\mathfrak{n}$; thus, $e = \dim S$, as S is regular. The Koszul complex on $\boldsymbol{s}$ is then a free resolution of k over S, so

$$\operatorname{Tor}_i^S(k, M) \cong \mathrm{H}_i(\boldsymbol{s}; M) \quad \text{for all } i.$$

The equality in (1) is by depth sensitivity of Koszul homology, [16, Theorem 16.8].

(2) This follows from (1) because $\operatorname{depth}_R M \geq 1$ when M is torsion-free.

(3) Let $F \simeq M$ and $G \simeq N$ be free resolutions over S. When $\operatorname{Tor}_i^S(M, N) = 0$ for all $i \geq 1$, the complex $F \otimes_S G$ is a free resolution of $M \otimes_S N$, and this gives the first isomorphism below:

$$\begin{aligned} \operatorname{Tor}_*^S(k, M \otimes_S N) &\cong \mathrm{H}_*(k \otimes_S (F \otimes_S G)) \\ &\cong \mathrm{H}_*((k \otimes_S F) \otimes_k (k \otimes_S G)) \\ &\cong \mathrm{H}_*(k \otimes_S F) \otimes_k \mathrm{H}_*(k \otimes_S G) \\ &\cong \operatorname{Tor}_*^R(k, M) \otimes_k \operatorname{Tor}_*^R(k, N)\,. \end{aligned}$$

The second one is standard, while the third is by the Künneth isomorphism [7, Ch. IV, Theorem 7.2]. Computing the suprema gives the desired equality. □

As noted earlier, when M is finitely generated, $\operatorname{codepth}_S M = \operatorname{pd}_S M$. A special case of [9, Corollary 1.2] due to Chouinard provides an analogue that applies to any R-module. Here $\operatorname{fd}_R M$ denotes the *flat dimension* of M.

4.4. Let R be a regular ring and M any R-module. Then $\operatorname{fd}_R M \leq c$ if and only if

$$\operatorname{codepth}_{R_\mathfrak{p}} M_\mathfrak{p} \leq c \quad \text{for all } \mathfrak{p} \in \operatorname{Spec} R.$$

The result below contains [2, Theorem 4.1], which deals with the case $d \geq \dim R$; the the argument is the same, and mimics the proof Theorem 1.9, with Theorem 4.1 and Lemma 4.3 substituting for Theorem 1.7 and Lemma 1.8, respectively.

Theorem 4.5. *Let R be a smooth algebra over some field and M an e.f.t. R-module. If $M^{\otimes_R^d}$ is torsion-free for some $d \geq 1$, then*

$$\operatorname{fd}_R M \leq \left\lfloor \frac{\dim R - 1}{d} \right\rfloor .$$

In particular, if $d \geq \dim R$, then M is flat.

Proof. In view of 4.4, it is enough to prove that there is an inequality

$$d \operatorname{codepth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \dim R - 1 \quad \text{for each } \mathfrak{p} \in \operatorname{Spec} R.$$

Noting that $R_{\mathfrak{p}}$ is again a smooth affine algebra, and that $(M_{\mathfrak{p}})^{\otimes_{R_{\mathfrak{p}}}^d}$ is a torsion-free $R_{\mathfrak{p}}$-modules, since it is isomorphic to $(M^{\otimes_R^d})_{\mathfrak{p}}$, one can localize at $\mathfrak{p}$ and assume R is a local ring. The task is then to prove that $d \operatorname{codepth}_R M \leq \dim R - 1$.

By Theorem 4.1(2) and Lemma 4.3(3) there are equalities

$$\operatorname{codepth}_R (M^{\otimes_R^p}) = \operatorname{codepth}_R M + \operatorname{codepth}_R (M^{\otimes_R^{p-1}}) \quad \text{for all } 1 \leq p \leq d.$$

From these one gets the equality below

$$d \operatorname{codepth}_R M = \operatorname{codepth}_R (M^{\otimes_R^d}) \leq \dim R - 1$$

The inequality is by Lemma 4.3(2), since $M^{\otimes_R^d}$ is torsion-free. ☐

Remark 4.6. The statement of the theorem above holds, with minor modifications in the proof, also when R is smooth over a Dedekind ring; see [14].

4.7. *Geometric formulations.* Let $\beta\colon R \to B$ be a homomorphism of commutative rings, and let N be a B-module. With ${}^a\beta\colon \operatorname{Spec} B \to \operatorname{Spec} R$ the induced map there is an equality

$${}^a\beta(\operatorname{Ass}_B N) = \operatorname{Ass}_R N\,;$$

in particular, N is torsion-free as an R-module if and only if ${}^a\beta(\operatorname{Ass}_B N) \subseteq \operatorname{Ass} R$. For a proof, do [16, Exercise 6.7].

With this remark, Theorem 4.5 yields: *Let R be a smooth algebra over some field, $R \to A$ a homomorphism e.f.t., and M a finitely generated A-module. If there exists an integer $d \geq \dim R$ such that*

$${}^a\beta(\operatorname{Ass}_B(M^{\otimes_R^d})) \subseteq \operatorname{Min} R$$

where $\beta\colon R \to B = A^{\otimes_R^d}$ is the natural map, then M is flat over R.

There is an obvious extension of this statement to a scheme-theoretic one: In the rest of this article, we consider a morphism of noetherian schemes $f\colon X \to Y$ essentially of finite type, and for each $d \geq 1$ let

$$f^{\{d\}}\colon X^{\{d\}} \to Y\,,$$

be the natural morphism, where $X^{\{d\}} = X \times_Y \cdots \times_Y X$ is the d-fold fiber product. For each $i = 1, \ldots, d$, let

$$\pi_i\colon X^{\{d\}} \to X$$

be the canonical projections. Since flatness is, by definition, a local property, see Hartshorne [13, Ch. III, Section 9], the discussion above gives the following result; cf. [13, Ch. III, Proposition 9.7].

Theorem 4.8. *Assume that Y is essentially smooth over some field, and let $\mathcal{F}$ be a coherent sheaf on X. If $f^{\{d\}}$ maps the associated points of $\bigotimes_{i=1}^{d}\pi_i^*\mathcal{F}$ to generic points of Y for some $d \geq \dim Y$, then $\mathcal{F}$ is flat over Y.* □

A natural question is to investigate what consequences flow from the torsion-freeness of tensor powers when the base (R in the affine case, Theorem 4.5, and Y in the one above) is not smooth (or regular). It is clear that the statements do not extend verbatim; for instance, when R is one dimensional and singular, torsion-freeness does not imply flatness, even for finitely generated modules. As far as I know, the best result in this direction is the following beautiful theorem from [3]. Note that flat maps are open, but the converse does not hold.

Theorem 4.9. *Assume Y is of finite type over a field, normal, and of dimension d. Then f is an open mapping if and only if $f^{\{d\}}$ takes generic points of $\mathcal{O}_X^{\{d\}}$ to generic points of Y.* □

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Department of Mathematics, University of Nebraska, Lincoln, NE 68588, U.S.A.
E-mail address: siyengar2@unl.edu