Let \( R \) be a commutative noetherian ring. Recall that the support of a finitely generated \( R \)-module \( M \) is the set of prime ideals \( p \) in \( R \) such that \( M_p \neq 0 \). For arbitrary modules and, more generally, for complexes of modules, there are various possible notions of support. Among them it is by now clear that the right definition, from a homological perspective, is the one introduced by Foxby in \([3]\), and recalled below. With this notion, Foxby \([3, 2.8, 2.9]\) proved that when \( X \) is a complex with \( H^n(X) = 0 \) for \( n \ll 0 \), a prime \( p \) is in the support of \( X \) if and only if the injective hull of \( R/p \) appears in the minimal semi-injective resolution of \( X \).

The purpose of this note is to describe examples that show that such a result does not extend to arbitrary complexes, contrary to expectation; see Remark 2.

**Support.** We write \( \text{Spec} \, R \) for the set of all the prime ideals in \( R \). For each \( p \) in \( \text{Spec} \, R \), the residue field \( R_p/R_p \) of the local ring \( R_p \) is denoted by \( k(p) \). The **support** of a complex \( X \) of \( R \)-modules is the subset

\[
\text{supp}_R X = \{ p \in \text{Spec} \, R \mid H(X \otimes_R k(p)) \neq 0 \}.
\]

This notion was introduced by Foxby \([3, p.157]\), under the name ‘small support’, to distinguish it from the ‘big support’, namely, the set \( \{ p \in \text{Spec} \, R \mid H(X)_p \neq 0 \} \). They coincide when the \( R \)-module \( H(X) \) is finitely generated—see \([3, 2.1]\)—but not in general. Also, \( \text{supp}_R X \) and \( \text{supp}_R H(X) \) need not coincide; see \([2, 9.4]\).

For each \( R \)-module \( M \) we write \( \text{ass}_R M \) for the set of its associated primes and \( E_R(M) \) for its injective hull; see Matsumura’s book \([9, \S\S 6,18]\).

**Injective modules.** Using \([9, 18.4]\), it is easy to verify that the support of \( E_R(R/p) \) equals \( \{ p \} \), which also equals \( \text{ass}_R E_R(R/p) \). By the structure theorem for injective \( R \)-modules \([9, 18.5]\) any injective \( R \)-module is of the form \( \bigoplus_{p \in \text{Spec} \, R} E(R/p)^{\mu(p)} \), where each \( \mu(p) \) is a non-negative integer (possibly \( \infty \)) which depends only on \( E \). It then follows that there are equalities:

\[
\text{supp}_R E = \{ p \in \text{Spec} \, R \mid \mu(p) \neq 0 \} = \text{ass}_R E.
\]

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It is this observation that suggests the possibility of reading the support of a complex from its injective resolutions.

**Injective resolutions.** We say that a complex $I$ of $R$-modules is *homotopically injective* if $\text{Hom}_R(-, I)$ preserves quasi-isomorphisms; it is *semi-injective* if in addition each $R$-module $I^n$ is injective. For example, a complex $I$ of injective $R$-modules with $I^n = 0$ for $n \ll 0$ is semi-injective. Each complex $X$ of $R$-modules admits a *semi-injective resolution*: a quasi-isomorphism $X \to I$, where $I$ is semi-injective. Moreover, one can choose an $I$ such that for each integer $n$ the extension $\text{Ker}(\partial^n) \subseteq I^n$ is essential; here $\partial$ is the differential on $I$. Such a *minimal* semi-injective resolution of $X$ is unique, up to isomorphism of complexes. For details see [1] and [6, Appendix B].

In the result below the additional hypotheses on $I$ hold if $R$ is regular, for then any complex of injectives is semi-injective, by [5, 2.4,2.8]. They hold also when $I$ is a minimal and $H_n(X) = 0$ for $n \ll 0$, for then $I^i = 0$ for $i \ll 0$ so $I$ and its localizations are semi-injective. Thus, it extends Foxby’s result mentioned earlier.

**Proposition 1.** Let $R$ be a commutative noetherian ring and $X$ a complex of $R$-modules. If a complex $I$ of injective modules is quasi-isomorphic to $X$, then $\text{supp}_R X \subseteq \bigcup_{n \in \mathbb{Z}} \text{ass}_R I^n$.

Equality holds if $I_p$ is minimal and homotopically injective for each $p \in \text{Spec } R$.

**Remark 2.** It is claimed in [7, 5.1] that the inclusion above is an equality whenever $I$ is a minimal semi-injective resolution of $X$. This is however not the case; see Proposition 5 for counter-examples. The proof of [7, 5.1] breaks down in the penultimate line, where it is asserted that a certain complex is homotopically injective; what can be salvaged from it is Proposition 1. The latter result is also implicit in [4], so we provide only a sketch. In the same vein, the last line of [2, 9.2] is incorrect: only conditions (2)–(4) in op. cit. are equivalent, and are implied by condition (1).

Given an ideal $a$ in $R$, we write $\Gamma_a(-)$ for the $a$-torsion functor on the category of $R$-modules, and $R\Gamma_a(-)$ for its right derived functor; see [3] or [8].

**Sketch of proof.** By localization, it suffices to prove the following statement: Suppose that the ring $R$ is local, with maximal ideal $m$ and residue field $k$. If $m$ is in $\text{supp}_R X$, then the complex $\Gamma_m(I)$ is non-zero; the converse holds if $I$ is minimal semi-injective. It follows from [4, 2.1.4.1] that the conditions below are equivalent:

(i) $H(X \otimes_R k) \neq 0$;
(ii) $H(R\text{Hom}_R(k, X)) \neq 0$;
(iii) $H(R\Gamma_m(X)) \neq 0$.

Since the complex $I$ consists of injective modules and is quasi-isomorphic to $X$, the complexes $R\Gamma_m(X)$ and $\Gamma_m(I)$ are quasi-isomorphic; see [8, 3.5.1]. Therefore, if $m$ is in $\text{supp}_R X$, the complex $\Gamma_m(I)$ must be non-zero.

Suppose $m \notin \text{supp}_R X$ holds, so that $H(R\text{Hom}_R(k, X)) = 0$. When $I$ is semi-injective $R\text{Hom}_R(k, X)$ is quasi-isomorphic to $\text{Hom}_R(k, I)$, which is isomorphic to $\text{Hom}_R(k, \Gamma_m(I))$; when $I$ is also minimal the differential on $\text{Hom}_R(k, I)$ is zero, and hence $H(\text{Hom}_R(k, I)) = 0$ implies $\Gamma_m(I) = 0$. □
Next we focus on our main task; namely, giving examples that show that Proposition 1 is probably the best possible. Their construction is motivated by an observation of Neeman [10, 6.5]; see also Iacob and Iyengar [5]. First, we record an elementary remark about associated primes of products.

Remark 3. Let $R$ be a commutative noetherian ring and let $\{M_\lambda\}$ be a family of $R$-modules. There are inclusions

$$\bigcup_{\lambda} \text{ass}_R M_\lambda \subseteq \text{ass}_R \left( \prod_{\lambda} M_\lambda \right) \subseteq \{p \in \text{Spec } R \mid p \subseteq q \in \text{ass}_R M_\lambda \text{ for some } \lambda \}.$$ 

Indeed, the inclusion on the left holds since each $M_\lambda$ is a submodule of the product module. For the one on the right: if a prime $p$ is the annihilator of an element $(m_\lambda)$, then it is contained in the annihilator of each $m_\lambda$; pick one that is non-zero.

Construction 4. Let $R$ be a commutative noetherian ring of the form $Q[x]/(x^2)$. We view $Q$ as a subring of $R$; it is also a quotient ring: $Q = R/Rx$, so it is noetherian. We assume $\dim R \geq 1$, and fix a non-minimal prime ideal $n$ in $R$.

Since each prime ideal in $R$ contains $x$ the natural map $\text{Spec } R \to \text{Spec } Q$ is bijective, with the inverse map assigning to a prime $q$ in $Q$ the prime $(q, x)$ in $R$. In particular, the prime ideal $n \cap Q$ in $Q$ is non-minimal as well.

Let $E$ be the injective hull of $Q/(n \cap Q)$ over $Q$ and set $E = \text{Hom}_Q(R, \bar{E})$, with the induced $R$-module structure. Note that $E$ is the injective hull of $R/n$ over $R$. It is convenient to view it as the $Q$-module $\bar{E} \oplus \bar{E}x^{-1}$ with the obvious action by $x$ and then by $R$. It is then evident that the complex of $R$-modules

$$J = 0 \longrightarrow E \xrightarrow{x} E \xrightarrow{x} E \xrightarrow{x} \cdots,$$

situates in degree 0 and higher, is exact with $H^0(J) = \bar{E}$, where $\bar{E}$ is viewed as an $R$-module via the surjection $R \to Q$. The natural inclusion $\iota: \bar{E} \to J$ is thus an injective resolution of $\bar{E}$ over $R$.

In what follows, the $i$th suspension of a complex $M$ is denoted $\Sigma^i M$.

Proposition 5. Let $X$ denote the complex of $R$-modules $\prod_{i \in \mathbb{Z}} \Sigma^i \bar{E}$, and let $I$ be the complex of $R$-modules $\prod_{i \in \mathbb{Z}} \Sigma^i J$. The following statements hold.

1. The complex $I$ is semi-injective and minimal.
2. The natural map $\prod_{i \in \mathbb{Z}} \Sigma^i \iota: X \to I$ is a quasi-isomorphism.
3. $\text{supp } X = \{n\} \subseteq \text{ass}_R I^n$, for each integer $n$.
4. For any prime $p$ in $\text{ass}_R I^n$ with $p \neq n$, the complex of injective $R_p$-modules $I_p$ is acyclic but not contractible, and hence not homotopically injective.

Proof. Recall that $\iota: \bar{E} \to J$ is a quasi-isomorphism.

1. The complex $\Sigma^i J$ consists of injective $R$-modules and $(\Sigma^i J)^n = 0$ for $n < -i$, hence $\Sigma^i J$ is semi-injective. Therefore the same holds for $I$, since a product of semi-injective complexes is semi-injective.

As to the minimality, note that the differential $\partial^n : I^n \to I^{n+1}$ is

$$I^n = \prod_{i \geq n} E \xrightarrow{x} \left( \prod_{i \geq n} E \right) \oplus E = \prod_{i \geq n-1} E = I^{n+1}.$$ 

Evidently $\ker(\partial^n)$ is the submodule $\prod_{i \geq n} \bar{E}$ of $I^n$. It is now easy to verify that the extension $\ker(\partial^n) \subset I^n$ is essential. Therefore $I$ is a minimal complex.

2. holds because a product of quasi-isomorphisms is a quasi-isomorphism.
(3) It is easy to see that $\text{supp}_R \overline{E} = \{n\}$; for example, $J$ is the minimal injective resolution of $E$ over $R$, so $\text{supp}_R \overline{E} = \text{ass}_R E = \{n\}$. Observe that there is an isomorphism of complexes $X \cong \bigoplus_{i \in \mathbb{Z}} \Sigma^i \overline{E}$, so $\text{supp}_R X = \{n\}$.

Since the $R$-module $I^n$ is isomorphic to $\prod_{i \geq n} E$, Remark 3 yields $\{n\} = \text{ass}_R E \subseteq \text{ass}_R I^n$.

The claim is that this inclusion is strict; equivalently, that there exist elements in $I^n = \prod_{i \geq n} E$ that are not $n$-torsion.

Indeed, $E$ is the injective hull of $R/n$, so it is a module over the local ring $R_n$; see [9, 18.4]. Since $n$ is not a minimal prime ideal in $R$, by hypothesis, $R_n$ does not have finite length, and hence neither does the $R_n$-module $E$; see [9, 18.6]. However, $E$ is Artinian, again by [9, 18.6], so for each integer $i \geq 0$ there must be an element $e_i$ in $E$ such that $n^i \cdot e_i \neq 0$ (cf. the proof of [9, 18.6(iv)]). Evidently, the element $(e_i - n)_{i \geq n}$ in $I^n$ is not $n$-torsion.

(4) Fix a prime $p$ as in the hypothesis. By Remark 3, one has $p \subset n$ so $\overline{E}_p = 0$, since $\overline{E}$ is $n$-torsion, and hence $X_p = 0$. As $I$ is quasi-isomorphic to $X$, the complex $I_p$ is quasi-isomorphic to $X_p$, and hence an acyclic complex of injective $R_p$-modules. It is also minimal since localization preserves minimality. Since the complex $I_p$ is non-zero, by the choice of $p$, it follows form the minimality that it is not contractible. □

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**References**