SUPPORT AND INJECTIVE RESOLUTIONS OF COMPLEXES OVER COMMUTATIVE RINGS

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Abstract. Examples are given to show that the support of a complex of modules over a commutative noetherian ring may not be read off the minimal semi-injective resolution of the complex. These also give examples of semi-injective complexes whose localization need not be homotopically injective.

Let R be a commutative noetherian ring. Recall that the support of a finitely generated R-module M is the set of prime ideals $\mathfrak{p}$ in R such that $M_{\mathfrak{p}} \neq 0$. For arbitrary modules and, more generally, for complexes of modules, there are various possible notions of support. Among them it is by now clear that the right definition, from a homological perspective, is the one introduced by Foxby in [3], and recalled below. With this notion, Foxby [3, 2.8,2.9] proved that when X is a complex with $\mathrm{H}^n(X) = 0$ for $n \ll 0$, a prime $\mathfrak{p}$ is in the support of X if and only if the injective hull of $R/\mathfrak{p}$ appears in the minimal semi-injective resolution of X.

The purpose of this note is to describe examples that show that such a result does not extend to arbitrary complexes, contrary to expectation; see Remark 2.

Support. We write $\operatorname{Spec} R$ for the set of all the prime ideals in R. For each $\mathfrak{p}$ in $\operatorname{Spec} R$, the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ of the local ring $R_{\mathfrak{p}}$ is denoted by $k(\mathfrak{p})$. The *support* of a complex X of R-modules is the subset

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(X \otimes^{\mathbf{L}}_R k(\mathfrak{p})) \neq 0\}\,.$$

This notion was introduced by Foxby [3, p.157], under the name 'small support', to distinguish it from the 'big support', namely, the set $\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(X)_{\mathfrak{p}} \neq 0\}$. They coincide when the R-module $\mathrm{H}(X)$ is finitely generated—see [3, 2.1]—but not in general. Also, $\operatorname{supp}_R X$ and $\operatorname{supp}_R \mathrm{H}(X)$ need not coincide; see [2, 9.4].

For each R-module M we write $\operatorname{ass}_R M$ for the set of its associated primes and $E_R(M)$ for its injective hull; see Matsumura's book [9, §§6,18].

Injective modules. Using [9, 18.4], it is easy to verify that the support of $E_R(R/\mathfrak{p})$ equals $\{\mathfrak{p}\}$, which also equals $\operatorname{ass}_R E_R(R/\mathfrak{p})$. By the structure theorem for injective R-modules [9, 18.5] any injective R-module is of the form $\bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} E(R/\mathfrak{p})^{\mu(\mathfrak{p})}$, where each $\mu(\mathfrak{p})$ is a non-negative integer (possibly ∞) which depends only on E. It then follows that there are equalities:

$$\operatorname{supp}_R E = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mu(\mathfrak{p}) \neq 0\} = \operatorname{ass}_R E\,.$$

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It is this observation that suggests the possibility of reading the support of a complex from its injective resolutions.

Injective resolutions. We say that a complex I of R-modules is *homotopically injective* if $\mathrm{Hom}_R(-, I)$ preserves quasi-isomorphisms; it is *semi-injective* if in addition each R-module I^n is injective. For example, a complex I of injective R-modules with $I^n = 0$ for $n \ll 0$ is semi-injective. Each complex X of R-modules admits a *semi-injective resolution*: a quasi-isomorphism $X \to I$, where I is semi-injective. Moreover, one can choose an I such that for each integer n the extension $\mathrm{Ker}\,(\partial^n) \subseteq I^n$ is essential; here ∂ is the differential on I. Such a *minimal* semi-injective resolution of X is unique, up to isomorphism of complexes. For details see [1] and [6, Appendix B].

In the result below the additional hypotheses on I hold if R is regular, for then any complex of injectives is semi-injective, by [5, 2.4,2.8]. They hold also when I is a minimal and $\mathrm{H}^n(X) = 0$ for $n \ll 0$, for then $I^i = 0$ for $i \ll 0$ so I and its localizations are semi-injective. Thus, it extends Foxby's result mentioned earlier.

Proposition 1. *Let R be a commutative noetherian ring and X a complex of R-modules. If a complex I of injective modules is quasi-isomorphic to X, then*

$$\mathrm{supp}_R\, X \subseteq \bigcup_{n\in\mathbb{Z}} \mathrm{ass}_R\, I^n\,.$$

Equality holds if $I_{\mathfrak{p}}$ is minimal and homotopically injective for each $\mathfrak{p} \in \mathrm{Spec}\, R$.

Remark 2. It is claimed in [7, 5.1] that the inclusion above is an equality whenever I is a minimal semi-injective resolution of X. This is however not the case; see Proposition 5 for counter-examples. The proof of [7, 5.1] breaks down in the penultimate line, where it is asserted that a certain complex is homotopically injective; what can be salvaged from it is Proposition 1. The latter result is also implicit in [4], so we provide only a sketch. In the same vein, the last line of [2, 9.2] is incorrect: only conditions (2)–(4) in op. cit. are equivalent, and are implied by condition (1).

Given an ideal $\mathfrak{a}$ in R, we write $\Gamma_{\mathfrak{a}}(-)$ for the $\mathfrak{a}$-torsion functor on the category of R-modules, and $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$ for its right derived functor; see [3] or [8].

Sketch of proof. By localization, it suffices to prove the following statement: Suppose that the ring R is local, with maximal ideal $\mathfrak{m}$ and residue field k. If $\mathfrak{m}$ is in $\mathrm{supp}_R\, X$, then the complex $\Gamma_{\mathfrak{m}}(I)$ is non-zero; the converse holds if I is minimal semi-injective. It follows from [4, 2.1,4.1] that the conditions below are equivalent:

(i) $\mathrm{H}(X \otimes^{\mathbf{L}}_R k) \neq 0$;
(ii) $\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k, X)) \neq 0$;
(iii) $\mathrm{H}(\mathbf{R}\Gamma_{\mathfrak{m}}(X)) \neq 0$.

Since the complex I consists of injective modules and is quasi-isomorphic to X, the complexes $\mathbf{R}\Gamma_{\mathfrak{m}}(X)$ and $\Gamma_{\mathfrak{m}}(I)$ are quasi-isomorphic; see [8, 3.5.1]. Therefore, if $\mathfrak{m}$ is in $\mathrm{supp}_R\, X$, the complex $\Gamma_{\mathfrak{m}}(I)$ must be non-zero.

Suppose $\mathfrak{m} \notin \mathrm{supp}_R\, X$ holds, so that $\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k, X)) = 0$. When I is semi-injective $\mathbf{R}\mathrm{Hom}_R(k, X)$ is quasi-isomorphic to $\mathrm{Hom}_R(k, I)$, which is isomorphic to $\mathrm{Hom}_R(k, \Gamma_{\mathfrak{m}}(I))$; when I is also minimal the differential on $\mathrm{Hom}_R(k, I)$ is zero, and hence $\mathrm{H}(\mathrm{Hom}_R(k, I)) = 0$ implies $\Gamma_{\mathfrak{m}}(I) = 0$. □

Next we focus on our main task; namely, giving examples that show that Proposition 1 is probably the best possible. Their construction is motivated by an observation of Neeman [10, 6.5]; see also Iacob and Iyengar [5]. First, we record an elementary remark about associated primes of products.

Remark 3. Let R be a commutative noetherian ring and let $\{M_\lambda\}$ be a family of R-modules. There are inclusions

$$\bigcup_\lambda \operatorname{ass}_R M_\lambda \subseteq \operatorname{ass}_R \big(\prod_\lambda M_\lambda\big) \subseteq \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \subseteq \mathfrak{q} \in \operatorname{ass}_R M_\lambda \text{ for some } \lambda\}.$$

Indeed, the inclusion on the left holds since each M_λ is a submodule of the product module. For the one on the right: if a prime $\mathfrak{p}$ is the annihilator of an element (m_λ), then it is contained in the annihilator of each m_λ; pick one that is non-zero.

Construction 4. Let R be a commutative noetherian ring of the form $Q[x]/(x^2)$. We view Q as a subring of R; it is also a quotient ring: $Q = R/Rx$, so it is noetherian. We assume $\dim R \geq 1$, and fix a non-minimal prime ideal $\mathfrak{n}$ in R.

Since each prime ideal in R contains x the natural map $\operatorname{Spec} R \to \operatorname{Spec} Q$ is bijective, with the inverse map assigning to a prime $\mathfrak{q}$ in Q the prime $(\mathfrak{q}, x)$ in R. In particular, the prime ideal $\mathfrak{n} \cap Q$ in Q is non-minimal as well.

Let $\bar{E}$ be the injective hull of $Q/(\mathfrak{n} \cap Q)$ over Q and set $E = \operatorname{Hom}_Q(R, \bar{E})$, with the induced R-module structure. Note that E is the injective hull of $R/\mathfrak{n}$ over R. It is convenient to view it as the Q-module $\bar{E} \oplus \bar{E}x^{-1}$ with the obvious action by x and then by R. It is then evident that the complex of R-modules

$$J = 0 \longrightarrow E \xrightarrow{x} E \xrightarrow{x} E \xrightarrow{x} \cdots,$$

situated in degree 0 and higher, is exact with $\mathrm{H}^0(J) = \bar{E}$, where $\bar{E}$ is viewed as an R-module via the surjection $R \to Q$. The natural inclusion $\iota \colon \bar{E} \to J$ is thus an injective resolution of $\bar{E}$ over R.

In what follows, the ith suspension of a complex M is denoted $\Sigma^i M$.

Proposition 5. *Let X denote the complex of R-modules $\prod_{i\in\mathbb{Z}} \Sigma^i \bar{E}$, and let I be the complex of R-modules $\prod_{i\in\mathbb{Z}} \Sigma^i J$. The following statements hold.*

(1) *The complex I is semi-injective and minimal.*
(2) *The natural map $\prod_{i\in\mathbb{Z}} \Sigma^i \iota \colon X \to I$ is a quasi-isomorphism.*
(3) $\operatorname{supp}_R X = \{\mathfrak{n}\} \subsetneq \operatorname{ass}_R I^n$*, for each integer n.*
(4) *For any prime $\mathfrak{p}$ in $\operatorname{ass}_R I^n$ with $\mathfrak{p} \neq \mathfrak{n}$, the complex of injective $R_\mathfrak{p}$-modules $I_\mathfrak{p}$ is acyclic but not contractible, and hence not homotopically injective.*

Proof. Recall that $\iota \colon \bar{E} \to J$ is a quasi-isomorphism.

(1) The complex $\Sigma^i J$ consists of injective R-modules and $(\Sigma^i J)^n = 0$ for $n < -i$, hence $\Sigma^i J$ is semi-injective. Therefore the same holds for I, since a product of semi-injective complexes is semi-injective.

As to the minimality, note that the differential $\partial^n \colon I^n \to I^{n+1}$ is

$$I^n = \prod_{i\geq n} E \xrightarrow{\begin{bmatrix} x \\ 0 \end{bmatrix}} \big(\prod_{i\geq n} E\big) \oplus E = \prod_{i \geq n-1} E = I^{n+1}.$$

Evidently $\operatorname{Ker}(\partial^n)$ is the submodule $\prod_{i\geq n} \bar{E}$ of I^n. It is now easy to verify that the extension $\operatorname{Ker}(\partial^n) \subset I^n$ is essential. Therefore I is a minimal complex.

(2) holds because a product of quasi-isomorphisms is a quasi-isomorphism.

(3) It is easy to see that $\operatorname{supp}_R \bar{E} = \{\mathfrak{n}\}$; for example, J is the minimal injective resolution of E over R, so $\operatorname{supp}_R \bar{E} = \operatorname{ass}_R E = \{\mathfrak{n}\}$. Observe that there is an isomorphism of complexes $X \cong \bigoplus_{i\in\mathbb{Z}} \Sigma^i \bar{E}$, so $\operatorname{supp}_R X = \{\mathfrak{n}\}$.

Since the R-module I^n is isomorphic to $\prod_{i\geqslant n} E$, Remark 3 yields

$$\{\mathfrak{n}\} = \operatorname{ass}_R E \subseteq \operatorname{ass}_R I^n .$$

The claim is that this inclusion is strict; equivalently, that there exist elements in $I^n = \prod_{i\geqslant n} E$ that are not $\mathfrak{n}$-torsion.

Indeed, E is the injective hull of $R/\mathfrak{n}$, so it is a module over the local ring $R_\mathfrak{n}$; see [9, 18.4]. Since $\mathfrak{n}$ is not a minimal prime ideal in R, by hypothesis, $R_\mathfrak{n}$ does not have finite length, and hence neither does the $R_\mathfrak{n}$-module E; see [9, 18.6]. However, E is Artinian, again by [9, 18.6], so for each integer $i \geq 0$ there must be an element e_i in E such that $\mathfrak{n}^i \cdot e_i \neq 0$ (cf. the proof of [9, 18.6(iv)]). Evidently, the element $(e_{i-n})_{i\geqslant n}$ in I^n is not $\mathfrak{n}$-torsion.

(4) Fix a prime $\mathfrak{p}$ as in the hypothesis. By Remark 3, one has $\mathfrak{p} \subset \mathfrak{n}$ so $\bar{E}_\mathfrak{p} = 0$, since $\bar{E}$ is $\mathfrak{n}$-torsion, and hence $X_\mathfrak{p} = 0$. As I is quasi-isomorphic to X, the complex $I_\mathfrak{p}$ is quasi-isomorphic to $X_\mathfrak{p}$, and hence an acyclic complex of injective $R_\mathfrak{p}$-modules. It is also minimal since localization preserves minimality. Since the complex $I_\mathfrak{p}$ is non-zero, by the choice of $\mathfrak{p}$, it follows form the minimality that it is not contractible. □

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