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Caveat

This is a (very) rough translation of the article.

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Contents

1. Introduction

The results.

The original method introduced in this work consists of calculating the local cohomology groups $\mathrm{H}^i_I()$, for an ideal I in a noetherian local ring A, such that $\mathrm{V}(I)$ is the support of an A-module of finite type and finite projective dimension. To attain this goal we combined two points of view:

a) the point of view of M. Auslander that consists of carefully studying modules of finite projective dimension;

b) the point of view of local cohomology introduced by A. Grothedieck.

To see the close link between these two notions, it suffices to know the following equality, due to M. Auslander:

Let A be a notherian local ring, M a A-module of finite type and finite projective dimension; then

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} A \ .$$

Indeed, the depth is entirely characterized in terms of local cohomolgy.

This method allows the approach of two kinds of problems:

- In an ambient scheme with singularities we study the properties of closed defined as supports of modules of finite projective dimension. The general idea of M. Auslander is that these closed behave as closed in a non-singular scheme. In this direction we will prove in Chapter II the following theorem, called the theorem of intersection:

Let A be a local ring essentially of finite type over a field, M an non zero A-module of finite projective dimension and of finite type, N an A-module of finite type such that $\dim(M \otimes_A N) = 0$*; then:* $\dim N \leq \operatorname{pd} M$*. (here* $\dim(.)$ *is the Krull dimension)*

This theorem of intersection permits us to prove the following two theorems:

(A) (conjectured by M. Auslander). Let A be a local ring essentially of finite type over a field, M an A-module of finite projective dimension and of finite type; then every M-regular sequence is A-regular [**2**].

(B) (conjectured by H. Bass). Let A be a local ring essentially of finite type over a field; then, for this A to be a Cohen-Macualay ring, it is necessary and sufficient that there exists an non zero A-module of finite type, and finite injective dimension [**6**].

When the ambient scheme is regular, we study the finiteness conditions for the cohomology of coherent sheaves on an open. The complementary closed is indeed defined by an ideal of finite projective dimension (theorem of syzygies). In this direction, we prove, in Chapter III, the following theorem, called finiteness theorem:

Let X be a closed sub-scheme of the projective space $\mathbb{P}^n_k$, where k is a field of characteristic $p > 0$. Let d be the least of the dimensions of the irreducible components of X. Let us assume that $\mathcal{O}_X$ satisfies the Serre conditions S_i, with $i \leq d$. Then, for all integers $s \geq n - i$ and all coherents sheaves $\mathcal{F}$ on $P - X$:

(i) $\mathrm{H}^s(\mathbb{P} - X, \mathcal{F})$ is a k-vector space of finite dimension;
(ii) $\mathrm{H}^s(\mathbb{P} - X, \mathcal{F}(n)) = 0$ for sufficiently larger integers n.

This statement had been conjectured by A. Grothendieck when X is locally complete intersection [**10**] (cf. also [**15**], chap. III, §6).

The method.

One may be surprised that in the statements of theorems (A) and (B) as well as in that of the intersection theorem there is a finiteness hypotheses over a field. In fact, we prove the above theorems for all rings of characteristic $p > 0$ and deduce the announced theorems from it.

As we will see, to calculate the local cohomology groups, the knowledge of a single module M of finite projective dimension, with annihilator an ideal I of a ring A is not enough. It is necessary to consider a sequence M_n of A-modules such that $\operatorname{pd} M_n = \operatorname{pd} M$ and such that $\operatorname{ann}(M_n)$ define the I-adic topology. This is what we do for an *ring A containing* $\mathbf{F_p}$, where p is a prime number, thanks to the following lemma:

Let $0 \longrightarrow L_s \xrightarrow{\phi_s} L_{s-1} \longrightarrow \ldots \longrightarrow L_1 \xrightarrow{\phi_1} L_0$ be an exact sequence of free A-modules of finite rank; then the sequence obtained by raising the coefficients of the matrices ϕ_i (in the bases given by L_i) to a p-th power, is also exact.

This lemma allows us to prove the theorem of interseciton in characeteristic $p > 0$. In characeteristic zero, we reduce it to characeteristic $p > 0$ thanks to a theorem of comparision between the projective dimension over a general fibre and over a special fibre, of a module of finite type over a ring that is essentially of finite type over a discrete valuation ring V, whose uniformiser is not a zero divisor in a certain , finite, of modules of finite type. (For more details see chap. II, §2).

The lemma that we have stated interprets in the following manner: the Frobenius homomorphism is flat in the category of modules of finite projective dimension. When the ring is regular we know that the Frobenius homomorphism is flat. Besides, this property characterizes regular rings [**18**]; this way of viewing permits the proof of the theorem of finiteness. (chap. III, §4).

Off course, one is justified in thinking that the theorem of intersection and its consequnces (A) and (B) are true for any local noetherian ring, and that another method would allow the proof. This is not doubt true, but the first method that seems natural is "to lift", in the sense of chapter I,§2, the modules of finite projective dimension, over regular rings. Unfortunately, as it is shown in chapter I, §2, this lifting is impossible. Anyway, we hope that the present work will convince that the utilization of local cohomology is a very useful tool for studying the support of modules of finite projective dimension, and, reciprocally, the finiteness of the projective dimension of an ideal permits one to obtain precise information on the local cohomology groups with support the closed that it defines.

The study of modules of finite projective dimension, as a algebraic geometric tool, arose in the theory of of syzygies of Hilbert. We cite the recent uses that are made by J.-P.Serre [**23**] to analyse the multiplicities of intersection, and M. Auslander for proving the theorem of purity [**4**]. In his theorem of finitiness for the finitness for an immersion morphism [**10**], A. Grothendieck utilises as essential techiniques the theorem of local duality and the spectral sequence associated to a module of finite projective dimension. We employ here, systematically and simulateanously, these two approaches.

We indicate briefly the general organization of this work. For more details the reader is referred to paragraph 0 of the three chapters.

-In chapter I, we develop the stability properties of modules of finite projective dimension which are essential to prove the results of the following chapters.

-In chapter II we state the fundamental questions related to modules of finite type and finite projective dimension. In this context, we prove the theorem of intersection and the theorem of M. Auslander and H. Bass which are deduced from it.

-In chapter III, we study the cohomolgy of opens $\mathbb{P} - X$ of a projective space $\mathbb{P}$ over a field, defined as complementary of closed X having good properties of regularity (non singular, locally complete intersection, Cohen-Macaulay, Serre conditions S_i). To do this, we use the apex of the cone of the projective space $\mathbb{P}$ as above. In fact, we prove local results more general than the stated globals that correspond to them. Finally, we prove in the connected local case, that the annihilation of the penultimate local cohomology groups is equivalent to its finiteness.

CHAPTER 1

Construction of modules of finite projective dimension

0. Introduction

The present chapter is entirely algebraic, and develops certain properties of modules of finite projective dimension. The goal of all this work is to calculate the local cohomology groups with support equal to that of a module of finite projective dimension. A certain behaviour of the local cohomology groups shows that the knowledge of one such module is not sufficient. We can convince ourselves by regarding the expression

$$\mathrm{H}_I^i(M) = \varinjlim \mathrm{Ext}_A^i\,(A/I^n, M)$$

where I is an ideal of the ring A. The Frobenius functor (§7) allows the construction of enough of such modules in characteristic $p > 0$.

§2 and 3 are variations on the theme: do the modules of finite type and finite projective dimension behave as modules of finite type over a regular ring? In §2 we see that the study of the former is not the same as the one of the latter (counter-example to lifiting). Nevertheless, we will see in Chapter II that the formula of dimension of intersections has an analogue.

§4 and 5 will show that the study of modules of finite type and finite injective dimension is practically the same as the one of modules of finite type and finite projective dimension.

Finally, in §6, the approximation theorem of Artin [**1**], allows the approximation of a module of finite projective dimension over $\widehat{A}$, by a module of finite projective dimension over the henselization $\widetilde{A}$ of a ring A satisfying the hypotheses of the theorem of Artin. The most remarkable fact is that the projective dimension of the approximation is equal to the projective dimension of the approximated, and thus that their depths are equal.

1. The functor of Frobenius

1.1. *Definition.We will say that a ring A is of characteristic p, where p is a prime, if there exists an injection $\mathbb{Z}/p\mathbb{Z} \longrightarrow A$ which is a homomorphism of rings.*

Let A be a ring of characteristic $p > 0$. The endomorphism $f\colon A \longrightarrow A$ defined by $f(x) = x^p$ for all x is a homomorphism of rings, called the homomorphism of Frobenius. We denote by ${}^fA$ the bi-A-algebra A with the structure of the A-algebra on the left defined by f, and the structure of the A-algebra on the right defined by the identity. That is to say that for $\alpha \in A$ and $x \in {}^fA$, we have $\alpha \cdot x = \alpha^p x$ and $x \cdot \alpha = x\alpha$.

Consider the category $\mathfrak{C}$ of A-modules.

1.2. *Definition.We call the functor of Frobenius, the functor* $\mathbf{F}$ *from* $\mathfrak{C}$ *to* $\mathfrak{C}$ *defined by* $\mathbf{F}(\cdot) = \cdot \otimes_{\mathbf{A}} {}^{\mathbf{f}}\mathbf{A}$ *with structure from the right. More precisely, for every* A*-module* M*, we have* $\mathbf{F}(\mathbf{M}) = \mathbf{M} \otimes_{\mathbf{A}} {}^{\mathbf{f}}\mathbf{A}$*, with with* A*-module structure given by the structure of the right* A*-algebra* ${}^fA$*.*

Example 1.3. a) $\mathbf{F}(\mathbf{A}) = \mathbf{A}$.

b) If α_x is multiplication by x in A, then $\mathbf{F}(\alpha_{\mathbf{x}}) = \alpha_{\mathbf{x}^{\mathbf{p}}}$.

c) Let n and m be positive integers, and let $\phi\colon A^n \longrightarrow A^m$ be a homomorphism. If (ϕ_{ij}) is a matrix representing ϕ, then $\mathbf{F}(\phi)$ is represented by (ϕ_{ij}^p).

d) Let I be an ideal of A and $(x_1, \ldots, x_s)$ a system of generators of I, then $\mathbf{F}(\mathbf{A}/\mathbf{I}) = \mathbf{A}/\mathbf{I}_{\mathbf{p}}$, where I_p is the ideal generated by $x_1^p, \ldots, x_s^p$.

Proposition 1.4. *The functor* $\mathbf{F}$ *commutes with the localization at a prime ideal. Said otherwise, for every prime ideal* $\mathfrak{p}$ *of* A*, we have an isomorphism of functors:*

$$\mathbf{F}(\cdot) \otimes_{\mathbf{A}} \mathbf{A}_{\mathfrak{p}} \simeq \mathbf{F}(\cdot \otimes_{\mathbf{A}} \mathbf{A}_{\mathfrak{p}}) \ .$$

We remark that the homomorphism of Frobenius $f\colon A \longrightarrow A$ corresponds to an entire morphism of affine schemes $\overline{f}\colon \operatorname{Spec} A \longrightarrow \operatorname{Spec} A$ which induces the identity on the underlying set. We deduce from it $A_{\mathfrak{p}} \otimes_A {}^fA = {}^fA \otimes_A A_{\mathfrak{p}}$, thus the announced isomorphism from the definition of $\mathbf{F}$.

Proposition 1.5. *For every* A*-module* M*, we have* $\operatorname{Supp} \mathbf{F}(\mathbf{M}) \subset \operatorname{Supp} \mathbf{M}$*. Moreover, if* M *is of finite type, we have* $\operatorname{Supp} \mathbf{F}(\mathbf{M}) = \operatorname{Supp} \mathbf{M}$*.*

The first part of the proposition is an immediate consequence of 1.4. Moreover, still from 1.4, to prove the second part, it suffices to show that when A is local and M is an non zero A module of finite type, then $\mathbf{F}(\mathbf{M}) \neq \mathbf{0}$. But then, if $\mathfrak{m}$ is the maximal ideal of A, we know that there exists a surjection $M \longrightarrow A/\mathfrak{m} \longrightarrow 0$. We deduce from it a surjection $\mathbf{F}(\mathbf{M}) \longrightarrow \mathbf{F}(\mathbf{A}/\mathfrak{m}) \longrightarrow \mathbf{0}$. But according to example 1.3.d, $\mathbf{F}(\mathbf{A}/\mathfrak{m}) \neq \mathbf{0}$, thus $\mathbf{F}(\mathbf{M}) \neq \mathbf{0}$.

1.6. In this section our goal is to prove that the functor of Frobenius is well behaved, more precisely, is exact, on the category of modules of finite type and finite projective dimension.

As an aside, we remark that when A is regular, it is elementary to prove that the morphism of Frobenius is flat. In fact, we have the following general theorem of Kunz.

Theorem.Let A *be a noetherian ring of characteristic* $p > 0$*. Then for* A *to be regular, it is necessary and sufficient that the morphism of Frobenius is flat, i.e. that the Frobenius functor is exact.*

Theorem 1.7. *Let* A *be a noetherian ring of characteristic* $p > 0$*. Let* M *be an* A *module of finite type and finite projective dimension. Then,* $\operatorname{Tor}_i^A\left(M, {}^fA\right) = 0$ *for* $i \geqslant 1$*, and* $\mathbf{F}(\mathbf{M})$ *is an* A*-module of finite type and finite projective dimension, such that for every prime ideal* $\mathfrak{p}$ *of* A *one has*

$$\operatorname{pd}_{A_{\mathfrak{p}}} \mathbf{F}(\mathbf{M})_{\mathfrak{p}} = \operatorname{pd}_{\mathbf{A}_{\mathfrak{p}}} \mathbf{M}_{\mathfrak{p}} \ .$$

We will utilize the following result:

1.8. **Lemma d'acyclicitè.** *Let A be a noetherian local ring. Consider a finite complex of A-modules of finite type $0 \to L_s \to L_{s-1} \to \cdots \to L_1 \to L_0 \to 0$. Suppose that for each integer $i > 0$, we have:*

1) $\operatorname{depth} L_i \geqslant i$;
2) $\operatorname{depth} \mathrm{H}_i(L) = 0$ *or* $\mathrm{H}_i(L) = 0$.

Then, $\mathrm{H}_i(L) = 0$ *for* $i \geqslant 1$.

Proof. Evidently, we can assume that $s \geqslant 1$, if not there is nothing to prove.

For $i \leqslant s$, we set $S_i = \operatorname{Coker}(L_{i+1} \to L_i)$. By a descending induction, we prove that for $1 \leqslant i \leqslant s$, we have $\operatorname{depth} S_i \geqslant i$ and $\mathrm{H}_i(L) = 0$.

We remark that we have $S_s = L_s$, thus $\operatorname{depth} S_s \geqslant s$. Moreover, as $\mathrm{H}_s(L) \hookrightarrow L_s$, one cannot have $\operatorname{depth} \mathrm{H}_s(L) = 0$. Thus, $\mathrm{H}_s(L) = 0$.

Now we suppose that $\operatorname{depth} S_i \geqslant i$ and $\mathrm{H}_i(L) = 0$ for $i > r$, with $1 \leqslant r < s$. We have then an exact sequence:

$$0 \to S_{i+1} \to L_r \to S_r \to 0. \tag{*}$$

Let $\mathfrak{m}$ be the maximal ideal of A; the exact sequence given by the derived functors of $\mathrm{H}^0_{\mathfrak{m}}(\cdot)$, applied to (*), proves that $\mathrm{H}^e_{\mathfrak{m}}(S_r) = 0$ for $e < r$, thus that $\operatorname{depth} S_r \geqslant r$. Let K_r be the kernel of the map $L_r \to L_{r-1}$. We have an exact sequence $0 \to S_{r+1} \to K_r \to \mathrm{H}_r(L) \to 0$. As $K_r \hookrightarrow L_r$, we have $\operatorname{depth} K_r \geqslant 1$. But then, again by the exact sequence of local cohomology, $\operatorname{depth} \mathrm{H}_r(L) = 0$ implies that $\operatorname{depth} S_{r+1} = 1$. But as $r + 1 \geqslant 2$, we know that $\operatorname{depth} S_{r+1} \geqslant 2$, we deduce $\mathrm{H}_r(L) = 0$ from it, and the lemma is proved. □

Corollary 1.9. *Let A be a noetherian local ring of depth r.*

Let $0 \to L_s \to \cdots \to L_0 \to 0$ be a complex of free A-modules of finite type, with $s \leqslant r$. The following conditions are equivalent:

(i) *For $i \geqslant 1$, the A-modules $\mathrm{H}_i(L)$ are of finite length.*
(ii) *For $i \geqslant 1$, one has $\mathrm{H}_i(L) = 0$.*

The following corollary was communicated to us, with a different proof, by M. Auslander. In the context of this section, it is interesting as it illuminates the proof of Theorem 1.7.

Corollary 1.10. *Let $\phi\colon A \to B$ be a homomorphism of noetherian rings. Let M be a finite A-module and of finite projective dimension. Let us suppose that for every prime ideal $\mathfrak{p}$ of B, appearing in the support of $M \otimes_A B$, one had $\operatorname{depth} B_{\mathfrak{p}} \geqslant \operatorname{depth} A_{\phi^{-1}(\mathfrak{p})}$. Then, $\operatorname{Tor}^A_i(M, B) = 0$, for every integer $i \geqslant 1$, and $M \otimes_A B$ is a B-module of finite projective dimension.*

Conversely, if A is local with maximal ideal $\mathfrak{m}$, and if for every A-module M finite type and finite projective dimension, $\operatorname{Tor}^A_i(M, B) = 0$ for $i \geqslant 1$, then, for every prime ideal $\mathfrak{p}$ lying over $\mathfrak{m}$, one has $\operatorname{depth} B_{\mathfrak{p}} \geqslant \operatorname{depth} A$.

Proof. The converse does not present any difficulty. Let $f_1, \dots, f_r$ by a A-regular sequence. The Koszul complex over A, associated to the sequence $f_1, \dots, f_r$ is exact. This complex being functorial, $\operatorname{Tor}^A_i(A/(f_1, \dots, f_r), B_{\mathfrak{p}}) = 0$, implies that the Koszul complex, over $B_{\mathfrak{p}}$, associated to the sequence $f_1, \dots, f_r$ is exact, thus that this sequence is $B_{\mathfrak{p}}$-regular.

We prove now the direct proposition. We suppose that the $\operatorname{Tor}^A_i(M, B)$ are not all zero. Then let $\mathfrak{p}$ be a prime ideal of B which is minimal in the union of the supports of the B-modules $\operatorname{Tor}^A_i(M, B)$ for $i \geqslant 1$. Let $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$. Consider a minimal free resolution (L_i) of the $A_{\mathfrak{q}}$-module $M_{\mathfrak{q}}$, of finite projective dimension. We apply the functor $\cdot \otimes_{A_{\mathfrak{q}}} B_{\mathfrak{p}}$ to the complex (L_i). We obtain a complex of free $B_{\mathfrak{p}}$-modules having $\operatorname{Tor}_i^{A_{\mathfrak{q}}}(M_{\mathfrak{q}}, B_{\mathfrak{p}})$ for homology. According to the choice of $\mathfrak{p}$, this homology has finite length for degrees $\geqslant 1$. But (L_i), being a minimal free resolution of $M_{\mathfrak{q}}$ over $A_{\mathfrak{q}}$, the length of the complex (L_i) is equal to $\operatorname{pd}_{A_{\mathfrak{q}}} M_{\mathfrak{q}}$, which is less than or equal to the depth of $A_{\mathfrak{q}}$. As $\operatorname{depth} A_{\mathfrak{q}} \leqslant \operatorname{depth} B_{\mathfrak{p}}$, we deduce from it that the length of the complex $(L_i \otimes_{A_{\mathfrak{q}}} B_{\mathfrak{p}})$ is less than or equal to depth of $B_{\mathfrak{p}}$. By 1.9, this implies $\mathrm{H}_i(L \otimes_{A_{\mathfrak{q}}} B_{\mathfrak{p}}) = 0$ for $i \geqslant 1$, that is that $\operatorname{Tor}_i^{A_{\mathfrak{q}}}(M_{\mathfrak{q}}, B_{\mathfrak{p}}) = 0$ for $i \geqslant 1$, and we have the contradiction that we are looking for. □

Proof of theorem 1.7. We know that the Frobenius morphism f induces the identity on the spectrum, that is to say that for every prime ideal $\mathfrak{p}$, we have $f^{-1}(\mathfrak{p}) = \mathfrak{p}$. Thus, according to 1.10, we have $\operatorname{Tor}_i^A\left(M, {}^f A\right) = 0$ for all $i \geqslant 1$ and $M \otimes_A {}^f A$ is an ${}^f A$-module of finite type and finite projective dimension. Said otherwise, $\mathbf{F}(\mathbf{M})$ is an A module of finite type and finite projective dimension.

Moreover, we remark that if A is local of maximal ideal $\mathfrak{m}$, and L is a free resolution of M, then for that L to be a minimal resolution, it is necessary and sufficient that $\mathbf{F}(\mathbf{L})$ is a minimal resolution of $\mathbf{F}(\mathbf{M})$. Indeed, suppose $L_i = A^{r_i}$, and consider the exact complex:

$$0 \longrightarrow A^{r_s} \xrightarrow{\phi_s} A^{r_{s-1}} \longrightarrow \cdots \longrightarrow A^{r_1} \xrightarrow{\phi_1} A^{r_0} \longrightarrow M \longrightarrow 0 \qquad (*)$$

Represent the mapping of ϕ_i, $i = 1, \ldots, s$, by the matrices $(\phi_i^{n,m})$. To say that (*) is a minimal resolution, is to say that for i, n, m, the element $\phi_i^{n,m}$ appears in the maximal ideal $\mathfrak{m}$ of A. But then $\mathbf{F}(*)$ can be written as:

$$0 \longrightarrow A^{r_s} \xrightarrow{(\phi_s)_p} A^{r_{s-1}} \longrightarrow \cdots \longrightarrow A^{r_1} \xrightarrow{(\phi_1)_p} A^{r_0} \longrightarrow \mathbf{F}(\mathbf{M}) \longrightarrow \mathbf{0} \ ,$$

where, according to 1.3.c, for every $i = 1, \cdots, s$, the mapping $(\phi_i)_p$ is represented by the matrix $(\phi_i^{n,m})^p$. Evidently, we have $\phi_i^{n,m} \in \mathfrak{m} \leftrightarrow (\phi_i^{n,m})^p \in \mathfrak{m}$, thus

L minimal is a resolution of $M \leftrightarrow \mathbf{F}(\mathbf{L})$ is a minimal resolution of $\mathbf{F}(\mathbf{M})$.

We know that the Frobenius functor commutes with localization, we deduce from it the end of the theorem, that is to say that for every prime ideal $\mathfrak{p}$ of A, we have

$$\operatorname{pd}_{A_{\mathfrak{p}}} \mathbf{F}(\mathbf{M})_{\mathfrak{p}} = \operatorname{pd}_{\mathbf{A}_{\mathfrak{p}}} \mathbf{M}_{\mathfrak{p}} \, .$$

□

Corollary 1.11. *Let A be a noetherian local ring of characteristic $p > 0$, and let $\mathfrak{m}$ be its maximal ideal. Let L be a free A-module of finite type, and let K be a sub-module of L. Suppose that K has finite projective dimension. Then, there exists a sequence of sub-modules K_i of L such that :*

- $K_0 = K$,
- *for every prime ideal $\mathfrak{p}$ of A we have* $\operatorname{pd}_{A_{\mathfrak{p}}}(L/K_i)_{\mathfrak{p}} = \operatorname{pd}_{A_{\mathfrak{p}}}(L/K)_{\mathfrak{p}}$,
- *if $K \subset \mathfrak{m}L$, then $K_i \subset \mathfrak{m}^{p^i} L$ for all $i \geqslant 0$.*

We set $M = L/K$ and we take $K_i = \operatorname{Ker}(\mathbf{F^i}(\mathbf{L}) \to \mathbf{F^i}(\mathbf{M}))$. As $\mathbf{F^i}(\mathbf{L}) = \mathbf{L}$, we note that K_i is a sub-module of L as well. Condition (1) is obvious. Condition (2) only translates theorem 1.7. Condition (3) is proved by considering a finite presentation $L_1 \to L \to M$ of M. To say that $K \hookrightarrow \mathfrak{m}L$ is to say that a matrix representing ϕ has its coefficients in $\mathfrak{m}$. But then, for every i, by 1.3.c, a matrix representing $\mathbf{F^i}(\phi)$ will have its coefficients in $\mathfrak{m}^{p^i}$. In the particular case when $L = A$, and where K is an ideal of finite projective dimension, we have a stronger result which will be particularly interesting to calculate, more precisely than by means of preceding result, the local cohomology with support in the closed set defined by an ideal of finite projective dimension.

Corollary 1.12. *Let A be a noetherian local ring of characteristic $p > 0$. Let I be an ideal of A, of finite projective dimension. Then, there exists a decreasing sequence of ideals (I_n) of A, defining on A the same topology as the I-adic topology, and such that for every prime ideal $\mathfrak{p}$ of A, we have $\operatorname{pd}_{A_\mathfrak{p}}(A/I_n)_\mathfrak{p} = \operatorname{pd}_{A_\mathfrak{p}}(A/I)_\mathfrak{p}$ for every $n \geqslant 0$.*

Indeed, we simply take as before $I_n = \operatorname{Ker}(\mathbf{F^n}(\mathbf{A}) \to \mathbf{F^n}(\mathbf{A}/\mathbf{I}))$. As $\mathbf{F^n}(\mathbf{A}) = \mathbf{A}$, we apply theorem 1.7, and it only remains to check that the ideals I_n define the same topology as the I-adic, in A. But we have seen 1.3.d, that if $x_1, \ldots, x_s$ is a system of generators for I, then $I_n = (x_1^{p^n}, \ldots, x_s^{p^n})$. Evidently we have $I_n \subset I^n$, and $I_n \subset I^{p^n}$, and the corollary is proved.

We end this section by giving a more visual form of theorem 1.7.

Theorem 1.13. *Let A be a noetherian local ring of characteristic $p > 0$. Consider a exact complex of free A-modules of finite type*

$$0 \longrightarrow L_s \xrightarrow{\phi_s} L_{s-1} \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{\phi_1} L_0.$$

Denote by $\phi_{i,n,m}$ the coefficients of the matrices representing the ϕ_i for a bases of L_i.

Let $\phi_i^{(p)}$ be the matrix of which the coefficients are $\phi_{i,n,m}^p$; then the complex

$$0 \longrightarrow L_s \xrightarrow{\phi_s^{(p)}} L_{s-1} \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{\phi_1^{(p)}} L_0$$

is exact.

2. Counter-example to lifting

2.1. *Definition. Let R be a regular local ring, and A a quotient of R. Let M be an A-module of finite type and finite projective dimension. We say that M lifts itself to R, if there exists an R-module N of finite type such that $M \simeq A \otimes_R N$, and $\operatorname{Tor}_i^R(A, N) = 0$ for $i \geqslant 1$.*

Remark 1: This amounts to saying that one will obtain a projective resolution of M over A, by tensoring a projective resolution of N over R by A.

Remark 2: Modules of projective dimension $\leqslant 1$ lift themselves.

Recall that we say that an A-module M of finite type and finite projective dimension is *rigid*, if it verifies the "conjecture of Tors", said otherwise, if for an A-module Q and integer r, we have $\operatorname{Tor}_r^A(M, Q) = 0$, then $\operatorname{Tor}_i^A(M, Q) = 0$ for $i \geqslant r$. We know that over a regular ring every module of finite type is rigid [**2**], [**19**]. We deduce from it the following proposition which says very well why the problem of lifting places it in the class of questions studied here.

Proposition 2.2. *Let A be a local ring which is a quotient of a regular ring R. If an A-module M, of finite type and finite projective dimension, lifts itself to R, then M is rigid.*

In chapter II, we will return to the question of rigidity and its consequences.

Our intention is to show that one can construct A-modules of finite type and finite projective dimension which cannot lift themselves to a regular ring.

PROPOSITION 2.3. *Let A be a local ring which is a quotient of a regular ring R. Let r be an integer $\geqslant 2$, and let M be an A-module of finite type and projective dimension r. We suppose that M is r-spherical, that is to say that $\operatorname{Ext}_A^i(M,A)=0$ for $i\neq 0,r$. Then, if M lifts itself to R, the A-module $\operatorname{Ext}_A^r(M,A)$ is of finite projective dimension and lifts itself to R.*

Indeed, let N be an R-module of finite type lifting M. Consider the spectral sequence:

$$\operatorname{Ext}_A^p\left(\operatorname{Tor}_q^R(N,A),A\right)={}^2\mathrm{E}_{p,q},$$

whose abutment is $\operatorname{Ext}_R^n(N,A)$. As $\operatorname{Tor}_i^R(N,A)=0$ for $i\neq 0$, it degenerates, and we obtain from it the isomorphisms:

$$\operatorname{Ext}_A^n(N\otimes_R A,A)\simeq \operatorname{Ext}_R^n(N,A)\ .$$

But, as $N\otimes_R A\simeq M$, and as M is r-spherical, we deduce from it $\operatorname{Ext}_R^n(N,A)=0$ for $p\neq 0,r$.

This implies, in the first place, that N is of projective dimension r, thus that:

$$\operatorname{Ext}_A^r(M,A)\simeq \operatorname{Ext}_R^r(N,A)\simeq \operatorname{Ext}_R^r(N,R)\otimes_R A\ .$$

Thus, to prove that $\operatorname{Ext}_A^r(M,A)$ is an A-module of finite projective dimension which lifts itself to R, it will suffice to show that $\operatorname{Tor}_i^R(\operatorname{Ext}_R^r(N,R),A)=0$ for $i\geqslant 1$. But we know that over a regular ring every module of finite type is rigid, thus $\operatorname{Ext}_R^r(N,R)$ is a rigid R-module, and it suffices to prove that $\operatorname{Tor}_1^R(\operatorname{Ext}_R^r(N,R),A)=0$. Let I be an ideal of R such that $A=R/I$. Consider the following commutative diagram where the rows are exact:

$$\begin{array}{ccccccc}
\operatorname{Ext}_R^{r-1}(N,A) & \longrightarrow & \operatorname{Ext}_R^r(N,I) & \longrightarrow & \operatorname{Ext}_R^r(N,R) & \longrightarrow & \operatorname{Ext}_R^r(N,A)\\
 & & \simeq\big\downarrow & & \simeq\big\downarrow & & \simeq\big\downarrow\\
 & & \operatorname{Ext}_R^r(N,R)\otimes_R I & \longrightarrow & \operatorname{Ext}_R^r(N,R)\otimes_R R & \longrightarrow & \operatorname{Ext}_R^r(N,R)\otimes_R A
\end{array}$$

As we have seen that $\operatorname{Ext}_R^{r-1}(N,A)=0$, we have an exact sequence:

$$0\to \operatorname{Ext}_R^r(N,R)\otimes_R I\to \operatorname{Ext}_R^r(N,R)\otimes_R R\to \operatorname{Ext}_R^r(N,R)\otimes_R A\to 0$$

which proves that $\operatorname{Tor}_1^R(\operatorname{Ext}_R^r(N,R),A)=0$.

Thus, to construct a counter-example to lifting, it will suffice to construct an r-spherical A-module Q such that $\operatorname{Ext}_A^r(Q,A)$ is not of finite projective dimension. For this we use a technique of M. Auslander on which one can find more details in [**22**] chapter II, §3.

PROPOSITION 2.4. *Let A be a local ring, and s an integer less than or equal to $\operatorname{depth} A$. Let M be a A-module of finite type such that $\operatorname{grade} M\geqslant s$. Then there exists an A-module N of finite type and s-spherical (i.e. such that $\operatorname{pd} N=s$ and $\operatorname{Ext}_A^i(N,A)=0$ for $i\neq 0,s$) such that $\operatorname{Ext}_A^s(N,A)\simeq M$. Moreover, if N' is an A-module of finite type of projective dimension s, and if η is an homomorphism of $\operatorname{Ext}_A^s(N',A)$ into $\operatorname{Ext}_A^s(N,A)$, there exists an homomorphism $\phi\colon N\to N'$ such that $\eta=\operatorname{Ext}_A^s(\phi,A)$.*

Indeed, consider a projective resolution, of length s, of M:

$$L_s \to L_{s-1} \to \cdots \to L_1 \to L_0 \to M \to 0\ . \qquad (*)$$

To say that $\operatorname{grade} M \geqslant s$ is to say that $\operatorname{Ext}^i_A(M, A) = 0$ for $i < s$, thus to say that the sequence $0 \to \check{L_0} \to \check{L_1} \to \cdots \to \check{L_s}$ is exact (where, $\check{}$ denotes the functor $\operatorname{Hom}_A(\cdot, A)$). Let $N = \operatorname{Coker}(\check{L}_{s-1} \to \check{L_s})$. Evidently, we have $\operatorname{pd} N \leqslant s$. By applying the functor $\operatorname{Hom}_A(\cdot, A)$ to the projective resolution of N:

$$0 \to \check{L_0} \to \check{L_1} \to \cdots \to \check{L_s} \to N \to 0\ ,$$

we obtain (*), and we see that $\operatorname{Ext}^i_A(N, A) = 0$ for $i \neq 0, s$ and that $\operatorname{Ext}^s_A(N, A) = M$. Thus, the first part of the proposition is proved.

Now consider a projective resolution of N':

$$0 \to P_s \to P_{s-1} \to \cdots \to P_1 \to P_0 \to N' \to 0$$

Apply the functor $\operatorname{Hom}_A(\cdot, A)$ to the complex P. We obtain a complex $\check{P}$ such that $\operatorname{Coker}(\check{P}_{s-1} \to \check{P_s}) = \operatorname{Ext}^s_A(N', A)$. The morphism $\operatorname{Ext}^s_A(N', A) \to \operatorname{Ext}^s_A(N, A)$ naturally induces a morphism of complexes of projective modules $\check{P}$ into every projective resolution of $\operatorname{Ext}^s_A(N, A)$, in particular into L, taking $\check{P_i}$ into L_{s-i}. By applying anew the functor $\operatorname{Hom}_A(\cdot, A)$, we find a morphism of complexes $\check{L} \to P$, (taking $\check{L_i}$ into P_{s-i}), thus on passing to homology, a morphism $\phi\colon N \to N'$ which by construction has the property we want.

Corollary 2.5. *Let A be a local ring of depth $\geqslant 2$, and a quotient of a regular local ring R. If every A-module of finite type, of finite projective dimension lifts itself to R, then A is regular.*

Indeed, consider the residue field k of A, which is evidently of grade $\geqslant 2$. According to the preceding proposition, there exists an 2-spherical A-module M, such that $\operatorname{Ext}^2_A(M, A) = k$. But then by 2.3, as M lifts itself and is 2-spherical, k is of finite projective dimension over A, thus A is regular.

3. Structure of ideals of projective dimension one

Our intent is to prove[1] that, over a local ring, every ideal of projective dimension one is, up to a multiplication by a regular element, is approximately a determinantal ideal. As a corollary, we will prove that if a local ring is a quotient of a regular ring, an ideal of projective dimension one lifts itself to an ideal of the regular ring.

We will use a preliminary technical lemma.

Consider a noetherian ring A. As before, we denote by $\check{\cdot}$ the functor $\operatorname{Hom}_A(\cdot, A)$.

Lemma 3.1. *Let $\phi\colon A^n \to A^{n+1}$ be a homomorphism of A-module free of finite type. Let $\check{\phi}\colon (A^{n+1})\check{} \to (A^n)\check{}$ be its dual. Let $\wedge^n\check{\phi}\colon \wedge^n (A^{n+1})\check{} \to \wedge^n(A^n)\check{}$ be the n-th exterior power of $\check{\phi}$. A choice of bases for A^{n+1} and A^n gives an isomorphism $\wedge^n(A^{n+1})\check{} \simeq A^{n+1}$. We derive from it a complex:*

$$0 \to A^n \xrightarrow{\phi} A^{n+1} \xrightarrow{\wedge^n\check{\phi}} A\ ,$$

which is exact if and only if the image of $\wedge^n\check{\phi}$ defines a closed, of $\operatorname{Spec} A$*, of grade $\geqslant 2$ (i.e. contains no prime ideal $\mathfrak{p}$ of A such that* $\operatorname{depth} A_{\mathfrak{p}} \leqslant 1$*).*

[1]Theorem 3.3, in the case of a factorial ring, has been communicated to us by D. A. Buchsbaum (cf. [**24**])

First, we show the existence of a complex as above. For this, consider ϕ as a matrix ϕ_{ij} with $(n+1)$ rows and n columns. Then $\wedge^n \check{\phi}$ is a matrix with 1 row and $(n+1)$ columns whose coefficients α_s are the determinants of the square matrices $(\phi_{ij})_{i\neq s}$ with sign $(-1)^s$. Consider then the product of the matrices $\wedge^n \check{\phi} \circ \phi$. To multiply a column of ϕ with a row of $\wedge^n \check{\phi}$ is to develop a determinant of order $(n+1)$ having two equal columns, which proves that $\wedge^n \check{\phi} \circ \phi = 0$. To demonstrate the announced equivalence, we will need the following preliminary result:

PROPOSITION 3.2. *The cokernel of $\check{\phi}$ and $\wedge^n \check{\phi}$ have the same support.*

Indeed, to say that a prime ideal is not in the support of cokernel of $\check{\phi}$ is to say that $\check{\phi} \otimes_A A_{\mathfrak{p}} : (A_{\mathfrak{p}}^{n+1})\check{} \to (A_{\mathfrak{p}}^n)\check{}$ is surjective. Thus, this is to say that there exists an minor of order n of the matrix $\check{\phi}$ which is not contained in $\mathfrak{p}$. But as the image of $\wedge^n \check{\phi}$ in A is an ideal generated by the minors of order n of $\check{\phi}$, this is indeed to say that $\wedge^n \check{\phi} \otimes_A A_{\mathfrak{p}}$ is surjective, thus that $\mathfrak{p}$ is not in the support of cokernel of $\wedge^n \check{\phi}$.

Going back to the lemma, and supposing that the complex $0 \to A^n \xrightarrow{\phi} A^{n+1} \xrightarrow{\wedge^n \check{\phi}} A$ is exact. Let A/I be the cokernel of $\wedge^n \check{\phi}$. Then the cokernel of $\check{\phi}$ is $\mathrm{Ext}^2_A(A/I, A)$. As, by the hypothesis, A/I is of finite projective dimension, for every prime ideal $\mathfrak{p}$ in A such that $\operatorname{depth} A_{\mathfrak{p}} \leqslant 1$, $A_{\mathfrak{p}}/IA_{\mathfrak{p}}$ is either zero, or of projective dimension $\leqslant 1$. Thus $\mathrm{Ext}^2_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/IA_{\mathfrak{p}}, A_{\mathfrak{p}}) = 0$. Said otherwise, $\mathrm{Ext}^2_A(A/I, A)$ is of grade $\geqslant 2$., and according to proposition 3.2, A/I also.

Reciprocally, still let A/I be the cokernel of $\wedge^n \check{\phi}$. Suppose $\operatorname{grade} A/I \geqslant 2$. Then, the local cohomology groups $\mathrm{H}^0_I(A)$ and $\mathrm{H}^1_I(A)$ are zero. Let $X = \operatorname{Spec} A$ and $U = X - \mathrm{V}(I)$. According to Proposition 3.2, $\check{\phi}_{|U}$ is surjective. Let $\mathcal{L}$ be its kernel. Thus we have an exact sequence of sheaves on U:

$$0 \to \mathcal{L} \to (\mathcal{O}_U^{n+1})\check{} \xrightarrow{\check{\phi}|U} (\mathcal{O}_U^n)\check{} \to 0. \tag{*}$$

This exact sequence shows that $\mathcal{L}$ is an invertible sheaf on U, and that we have

$$\mathcal{L} \simeq (\wedge^n \mathcal{O}_U^n) \otimes (\wedge^{n+1}(\mathcal{O}_U^{n+1})\check{}\,,$$

thus $\mathcal{L} \simeq \mathcal{O}_U$. The functor $\Gamma(U, \cdot)$ applied to (*) gives an exact sequence:

$$0 \to A \to (A^{n+1})\check{} \xrightarrow{\check{\phi}} (A^n)\check{}\,. \tag{**}$$

Now returning to the complex:

$$A^n \xrightarrow{\phi} A^{n+1} \xrightarrow{\wedge^n \check{\phi}} A\ ,$$

which gives, upon dualizing, a complex:

$$A \xrightarrow{\wedge^n \phi} (A^{n+1})\check{} \xrightarrow{\check{\phi}} (A^n)\check{}\,.$$

By comparing this last complex with the exact sequence (**), we obtain a commutative diagram:

$$\begin{tikzcd}
 & A \arrow[r, "\wedge^n \phi"] \arrow[d] & (A^{n+1})\check{} \arrow[r, "\check{\phi}"] \arrow[d, equal] & (A^n)\check{} \arrow[d, equal] \\
0 \arrow[r] & A \arrow[r, "\wedge^n \phi"] & (A^{n+1})\check{} \arrow[r, "\check{\phi}"] & (A^n)\check{}
\end{tikzcd}$$

in which the second row is exact.

Dualizing anew, we find a commutative diagram:

$$\begin{tikzcd}
 & A^n \arrow[r, "\phi"] & A^{n+1} \arrow[r, "\wedge^n \tilde{\phi}"] & A \\
0 \arrow[r] & A^n \arrow[r, "\phi"] \arrow[u, equal] & A^{n+1} \arrow[r] \arrow[u, equal] & A \arrow[u]
\end{tikzcd}$$

As the cokernel $\operatorname{Ext}^2_A(A/I, A)$ of $\tilde{\phi}$ is of grade $\geqslant 2$, the second row of this last diagram is exact, which proves again that ϕ is injective. Each A-homomorphism of A being a multiplication, let f be the element of A corresponding to the last vertical arrow of the diagram. The diagram shows that we have $I \subset fA$. If f is not invertible, f is contained in a prime ideal of height 1, which is contrary to the hypothesis that $\operatorname{grade} A/I \geqslant 2$. Thus f is invertible, and the exactness of the second row of the diagram implies the exactness of the first, which is what we were aiming to prove.

Theorem 3.3. *Let A be a noetherian local ring. A necessary and sufficient condition for an ideal I of A to be of projective dimension* 1*, is that there exists an integer n, a matrix corresponding to a homomorphism $\phi\colon A^n \to A^{n+1}$, and an element f of A, a non-zero divisor of A, such that, if I' is the ideal generated by the minors of order n of the matrix, we have* $\operatorname{grade} A/I' \geqslant 2$ *and $I = fI'$.*

This theorem is a direct consequence of 3.1. First, we remark that if a homomorphism $\phi\colon A^n \to A^{n+1}$ corresponds to a matrix of which the minors of order n generate an ideal I' such that $\operatorname{grade} A/I' \geqslant 2$, then according to lemma 3.1, there is an exact sequence:

$$0 \to A^n \xrightarrow{\phi} A^{n+1} \xrightarrow{\wedge^n \tilde{\phi}} A \to A/I' \to 0\ .$$

Thus I' is an ideal of projective dimension 1 and of course so is fI', for every element f of A, not a divisor of zero.

Conversely, let:

$$0 \to A^n \xrightarrow{\phi} A^{n+1} \xrightarrow{\psi} A \to A/I \to 0\ . \qquad (*)$$

be a projective resolution of A/I. As in the proof of lemma 3.1 we can see that $\operatorname{grade}(\operatorname{Ext}^2_A(A/I, A)) \geqslant 2$. We deduce from it that $\tilde{\phi}$ has a cokernel of grade $\geqslant 2$, and by proposition 3.2 that $\wedge^n \tilde{\phi}$ has a cokernel of grade $\geqslant 2$. Lemma 3.1 says then that we have an exact sequence:

$$0 \to A^n \xrightarrow{\phi} A^{n+1} \xrightarrow{\wedge^n \tilde{\phi}} A \to A/I' \to 0\ , \qquad (**)$$

setting $A/I' = \operatorname{Coker} \wedge^n \tilde{\phi}$.

By dualizing (*) and (**), we can construct a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\tilde{\psi}"] \arrow[d] & A^{n+1} \arrow[r, "\tilde{\phi}"] \arrow[d, equal] & A^n \arrow[r] \arrow[d, equal] & \operatorname{Ext}^2_A(A/I, A) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\wedge^n \phi"] & A^{n+1} \arrow[r, "\tilde{\phi}"] & A^n \arrow[r] & \operatorname{Ext}^2_A(A/I', A) \arrow[r] & 0
\end{tikzcd}$$

where the second row is exact, as $\operatorname{grade} A/I' \geqslant 2$. Every A-homomorphism of A being a multiplication by an element, this diagram shows that the isomorphism between I' and I, that we obtain by comparing (*) and (**), gives a multiplication in A, alternatively said that we have $I = fI'$, for an element f in A. It remains to prove that f is not a divisor of 0 in A. But since the multiplication by f defines an isomorphism between I and I', the restriction to I' of the multiplication by f is injective. As $\operatorname{grade} A/I' \geqslant 2$, I' contains at least one element regular in A, thus I' contains a sub-module isomorphic to A, on which the multiplication by f is *a fortiori* injective. Finally, f is regular in A and the theorem is proved.

Remark: The fact that A is local has been utilized in this proof only to say that an ideal of projective dimension 1 admits a free resolution of length 1, which, of course, can be true in more general cases.

Corollary 3.4. *Let A be a noetherian local ring which is a quotient of a regular local ring R. Let I be an ideal of A of projective dimension 1. Then there exists an ideal $\widetilde{I}$ of R of projective dimension 1 lifting I (i.e. such that $\widetilde{I} \otimes_R A = I$ and $\operatorname{Tor}_1^R\left(\widetilde{I}, A\right) = 0$.)*

Indeed, according to theorem 3.3 it suffices to prove the case where A/I admits a free resolution over A, of the form:

$$0 \to A^n \xrightarrow{\phi} A^{n+1} \xrightarrow{\wedge^n \tilde{\phi}} A \to A/I \to 0\ , \tag{*}$$

Let Φ be a matrix with coefficient in R, lifting the matrix ϕ. According to 3.1, we have a complex of R-modules:

$$0 \to R^n \xrightarrow{\Phi} R^{n+1} \xrightarrow{\wedge^n \tilde{\Phi}} R \to R/\widetilde{I} \to 0\ , \tag{**}$$

where $R/\widetilde{I}$ is the cokernel of $\wedge^n \tilde{\Phi}$. Still by 3.1, to prove that (**) is exact, it suffices to prove that $R/\widetilde{I}$ has grade $\geqslant 2$ in R. In a regular ring, the grade is equal to the codimension. We know [**23**] that we have:

$$\operatorname{codim}_R R/\widetilde{I} + \operatorname{codim}_R A = \operatorname{codim}_R A/I\ .$$

But, $\operatorname{grade}_A(A/I) \geqslant 2$, implies:

$$\operatorname{codim}_R A/I \geqslant \operatorname{codim}_R A + 2\ .$$

Thus, we have $\operatorname{codim}_R R/\widetilde{I} \geqslant 2$, which is what we wanted to show. Finally, (**) is exact as well, and to say that it remains exact upon tensorizing by A, is to say that $R/\widetilde{I}$ lifts to A/I, thus that the ideal $\widetilde{I}$ lifts I.

4. Modules of finite type and finite injective dimension

In this section, we propose to emphasize three important properties of modules of finite *injective* dimension over a noetherian local ring.

For this, we first recall the results proved by H. Bass in [**6**]. Following this, we give some examples to clarify the problem. Finally, if T is a non zero module of finite injective dimension over a noetherian local ring A, we show that T possesses the following three properties, the first two of which will appear in a primitive form in Chapter II, to prove a conjecture of H. Bass.

a) $\operatorname{grade} T + \dim T = \operatorname{depth} A$. This situation is minimal, because for every A-module T of finite type, we have $\operatorname{depth} A \leqslant \operatorname{grade} T + \dim T \leqslant \dim A$.

b) If $\widehat{A}$ is the completion of A, there exists an $\widehat{A}$ module of finite type and finite projective dimension having the same support as that of $\widehat{T}$.

c) We have the following theorem, of the Hilbert-Serre type: for every A-module of finite type M, we have a relation:

$$\operatorname{depth} M + \sup\{i \mid \text{ such that } \operatorname{Ext}^i_A(M,T) \neq 0\} = \operatorname{depth} A \ .$$

We recall the central conjecture of H. Bass, that we prove in Chapter II, for local rings from algebraic geometry:

If there exists an non zero A-module of finite type and finite injective dimension, the A is a Cohen-Macaulay ring.

a) and b) show that the following conjecture of M. Auslander implies the conjecture of Bass:

For every A-module M of finite type and finite projective dimension, we have

$$\operatorname{grade} M + \dim M = \dim A \ .$$

Finally, we remark, regarding result c), that one is naturally lead to look for an A-module of depth as big as possible, and in particular that if one can find an A-module of finite type of depth equal to the dimension of the ring (this we can find for rings of dimension $\leqslant 2$.), then the existence of T implies that A is Cohen-Macaulay.

Proposition 4.1. (Bass). *Let A be a noetherian local ring. For every non zero A-module T of finite type and finite injective dimension, we have* $\operatorname{inj.dim} T = \operatorname{depth} A$.

This result follows directly from the following two lemma, that we will not prove.

Lemma 4.2. *If $i = \operatorname{inj.dim} T$, we have $\operatorname{Ext}^i_A(k,T) \neq 0$, where k is the residue field of A.*

Lemma 4.3. *If $r = \operatorname{depth} A$, and if M is an A-module of finite type and projective dimension r; we have $\operatorname{Ext}^r_A(M,N) \neq 0$ for every A-module N of finite type.*

Proposition 4.4. (Bass). *Let T be an A-module of finite type over a noetherian local ring A. Let E be a minimal injective resolution of T. Then, for every i, we have:*

$$E^i \simeq \prod_{\mathfrak{p}\in \operatorname{Spec} A} \mu_i(\mathfrak{p},T)E(A/\mathfrak{p}) \ ,$$

with $E(A/\mathfrak{p})$ an injective envelope of $A/\mathfrak{p}$ and $\mu_i(\mathfrak{p},T) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_{A_\mathfrak{p}}(k(\mathfrak{p}),T_\mathfrak{p})$.

Proposition 4.5. (Bass). *Let M be an A-module of finite type over a noetherian local ring A, and let $\mathfrak{p}$ and $\mathfrak{q}$ two successive prime ideals of A (i.e. such that $\mathfrak{p} \subset \mathfrak{q}$ and $\dim A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q} = 1$. Then, if $\mu_i(\mathfrak{p},M) \neq 0$, we have $\mu_{i+1}(\mathfrak{q},M) \neq 0$.*

4.6. *Examples of modules of finite type and finite injective dimension.*

A) For a local ring R to be regular, it is necessary and sufficient that every R-module of finite type has finite injective dimension. In fact, it suffices that its residue field be of finite injective dimension.

B) Let A be a Gorenstien local ring (i.e., such that if $\mathfrak{m}$ is the maximal ideal of A, one has $\mathrm{H}^i_\mathfrak{m}(A) = 0$ for $i \neq 0, \dim A$, and for which $\mathrm{H}^n_\mathfrak{m}(A)$ is injective). Every A-module of finite projective dimension is of finite injective dimension, and vice versa.

C) Let A be a Cohen-Macaulay local ring, which is a quotient of a regular local ring R. Then, if s is the codimension of A in R, the A-module $\operatorname{Ext}^s_R(A, R)$ is of finite type and finite injective dimension. Let E be a dualizing module for A (i.e. an injective envelope of the residue field k of A), and let n the dimension of A and let $\mathfrak{m}$ the maximal ideal of A; then we know that $\operatorname{Hom}_A(\mathrm{H}^n_{\mathfrak{m}}(A), E)$ is isomorphic to the completion of $\operatorname{Ext}^s_R(A, R)$, and that this A-module that we denote $\Omega^0(A)$ is independent of the regular embedding.

PROPOSITION 4.7. *Let A be a noetherian local ring, and let T be an A-module of finite type and finite injective dimension. Then, for every prime ideal $\mathfrak{p}$ in the support of T, we have:*

$$\dim A/\mathfrak{p} + \operatorname{depth} A_{\mathfrak{p}} = \operatorname{depth} A\ .$$

Let $\mathfrak{p}$ be a prime ideal of A such that $T_{\mathfrak{p}} \neq 0$. According to 4.1, we have

$$\operatorname{inj.dim}_{A_{\mathfrak{p}}} T_{\mathfrak{p}} = \operatorname{depth} A_{\mathfrak{p}}\ .$$

Let $s = \operatorname{depth} A_{\mathfrak{p}}$. Then $\operatorname{Ext}^s_{A_{\mathfrak{p}}}(k(\mathfrak{p}), T_{\mathfrak{p}}) \neq 0$, said otherwise $\mu_s(\mathfrak{p}, T) \neq 0$. Let $\mathfrak{m}$ be the maximal ideal of A, and let $d = \dim A/\mathfrak{p}$. By 4.5, we have $\mu_{s+d}(\mathfrak{m}, T) \neq 0$. From it we easily deduce $\dim A/\mathfrak{p} + \operatorname{depth} A_{\mathfrak{p}} \leqslant \operatorname{depth} A$. But taking into account the fact that

$$\operatorname{depth} A_{\mathfrak{p}} \geqslant \operatorname{grade}_A A/\mathfrak{p}\ ,$$

the proposition will be an immediate consequence of the following general lemma.

LEMMA 4.8. *For every module M of finite type over a noetherian local ring A, we have the double inequality:*

$$\operatorname{depth} A \leqslant \operatorname{grade} M + \dim M \leqslant \dim A\ .$$

Let $\mathfrak{p}$ be a prime ideal from the support of M such that $\dim A/\mathfrak{p} = \dim M$. Obviously, we have $\dim A_{\mathfrak{p}} + \dim A/\mathfrak{p} \leqslant \dim A$. But as $\operatorname{grade} M \leqslant \operatorname{depth} A_{\mathfrak{p}} \leqslant \dim A_{\mathfrak{p}}$, we immediately deduce from it the inequality on the right.

We prove the inequality on the left by induction on $\operatorname{grade} M$.

To say that $\operatorname{grade} M = 0$ is to say that $\inf_{\mathfrak{p} \in \operatorname{Supp} M} \operatorname{depth} A_{\mathfrak{p}} = 0$, thus this to say that there exists a prime ideal $\mathfrak{p}$ in the support of M, associated to 0 in A. We have then

$$\dim M \geqslant \dim A/\mathfrak{p}\ ,$$

and as for every prime ideal $\mathfrak{q}$ associated to 0 in A, we have $\dim A/\mathfrak{q} \geqslant \operatorname{depth} A$, we can conclude.

Suppose now that the inequality on the left is proved for $\operatorname{grade} M < n$, with $n > 0$. Let N be an A-module of grade n. There exists an A-regular element α in the annihilator of N. Thus, N is an $A/\alpha A$-module of grade $n-1$. We deduce from it

$$\operatorname{grade}_{A/\alpha A} N + \dim N \geqslant \operatorname{depth}(A/\alpha A)\ ,$$

and evidently $\operatorname{grade}_A N + \dim N \geq \operatorname{depth} A$.

COROLLARY 4.9. *If T is an module of finite type and finite injective dimension over a noetherian local ring A, we have:*

$$\operatorname{grade} T + \dim T = \operatorname{depth} A\ .$$

Let $\mathfrak{p}$ be a prime ideal from the support of T such that $\dim A/\mathfrak{p} = \dim T$. According to 4.7, we have $\dim T + \operatorname{depth} A_{\mathfrak{p}} = \operatorname{depth} A$. But as $\operatorname{grade} T \leqslant \operatorname{depth} A_{\mathfrak{p}}$, according to 4.8, we have $\operatorname{depth} A_{\mathfrak{p}} = \operatorname{grade} T$, and $\dim T + \operatorname{grade} T = \operatorname{depth} A$.

THEOREM 4.10. *Let T be an non-zero module of finite type and finite injective dimension over a noetherian local ring A. Let E be the injective envelope of the residue field k of A. Let $\widehat{A}$ be the completion of A, and let $r = \operatorname{depth} A = \operatorname{depth} \widehat{A}$. Then $M = \operatorname{Ext}^r_A(E, T)$ is an $\widehat{A}$-module having the following properties:*

(i) *M is of finite type and finite projective dimension over A, and its projective dimension is $r - \operatorname{depth} T$.*
(ii) *M has the same support as the completion $\widehat{T}$ of T.*
(iii) *Let $\mathfrak{m}$ be the maximal ideal of A, and let R be a Gorenstien local ring of dimension n of which $\widehat{A}$ is a quotient; then, for every i, there are isomorphisms:*

$$\operatorname{Ext}^i_{\widehat{A}}\left(M, \widehat{A}\right) \simeq \operatorname{Hom}_A\left(\mathrm{H}^{r-i}_{\mathfrak{m}}(T), E\right) \simeq \operatorname{Ext}^{n-(r-i)}_R\left(\widehat{T}, R\right) .$$

First we prove (i). For this, consider a minimal injective resolution of T:

$$0 \to T \to I^0 \to I^1 \to \cdots \to I^r \to 0. \tag{*}$$

We want to apply the functor $\operatorname{Hom}_A(E, \cdot)$ to this exact sequence. We know that for every $i \geqslant 0$, the module I^i can be written as a direct product $\prod_{\mathfrak{p} \in \operatorname{Spec} A} \mu_i(\mathfrak{p}, T) E(A/\mathfrak{p})$, where $E(A/\mathfrak{p})$ is the injective envelope of $A/\mathfrak{p}$. We prove that for $\mathfrak{p} \neq \mathfrak{m}$, we have

$$\operatorname{Hom}_A(E, E(A/\mathfrak{p})) = 0 .$$

Let $f \colon E \to E(A/\mathfrak{p})$. If $x \in E$, the module Ax is of finite length, thus $Af(x)$ is a sub-module of finite length of $E(A/\mathfrak{p})$, thus $f(x) = 0$ as $E(A/\mathfrak{p})$, being an essential extension of $A/\mathfrak{p}$, does not contain a non-trivial sub-module of finite length. Said otherwise, for every i:

$$\operatorname{Hom}_A\left(E, I^i\right) = \operatorname{Hom}_A\left(E, E^{\mu_i(\mathfrak{m}, T)}\right) = \operatorname{Hom}_A\left(E, \mathrm{H}^0_{\mathfrak{m}}(I^i)\right) .$$

Thus, as $\operatorname{Hom}_A\left(E, \widehat{E}\right) = \widehat{A}$, by applying $\operatorname{Hom}_A(E, \cdot)$ to (*), we obtain a complex:

$$0 \to \widehat{A}^{\mu_0(\mathfrak{m},T)} \to \widehat{A}^{\mu_1(\mathfrak{m},T)} \cdots \to \widehat{A}^{\mu_r(\mathfrak{m},T)} \to 0 , \tag{**}$$

of which the cohomology $\mathrm{H}^i(\widehat{A}^{\mu})$ is $\operatorname{Ext}^i_A(E, T)$. Finally, to prove that $\operatorname{Ext}^r_A(E, T)$ is an $\widehat{A}$ module of finite type and finite projective dimension, it will suffice to prove that $\operatorname{Ext}^i_A(E, T) = 0$ for $i < r$.

Recall that as E is also an injective envelope of the residue field k of $\widehat{A}$, we would also obtain the complex (**) by replacing T by $\widehat{T}$ and (*) by a minimal injective resolution of the $\widehat{A}$-module of finite type and finite injective dimension $\widehat{T}$. Thus, it will suffice to prove that $\operatorname{Ext}^i_{\widehat{A}}\left(E, \widehat{T}\right) = 0$ for $i < r$. But then, by classical duality [**9**] we know that $\operatorname{Hom}_{\widehat{A}}\left(\operatorname{Hom}_{\widehat{A}}\left(\widehat{T}, E\right), E\right) = \widehat{T}$. Consider $T' = \operatorname{Hom}_{\widehat{A}}\left(\widehat{T}, E\right)$. Then, by an isomorphism of duality [**8**]:

$$\operatorname{Ext}^i_{\widehat{A}}\left(E, \widehat{T}\right) \simeq \operatorname{Ext}^i_{\widehat{A}}(E, \operatorname{Hom}(T', E)) \simeq \operatorname{Hom}\left(\operatorname{Tor}^{\widehat{A}}_i(E, T'), E\right) .$$

By construction, we know that T' is an inductive limit of $\widehat{A}$-modules of finite length $T'_n = \operatorname{Hom}_{\widehat{A}}\left(\widehat{T}/\mathfrak{m}^n\widehat{T}, E\right)$. Utilizing the exactness of the inductive limit functor and the properties of the dualizing functor $\mathrm{Hom}_{\widehat{A}}(\cdot, E)$, we deduce from it isomorphisms:

$$\begin{aligned}&\mathrm{Hom}_{\widehat{A}}\left(\mathrm{Tor}_i^{\widehat{A}}\left(E, \varinjlim T'_n\right), E\right) \simeq \mathrm{Hom}_{\widehat{A}}\left(\varinjlim\ \mathrm{Tor}_i^{\widehat{A}}(E, T'_n), E\right)\\ &\simeq \varprojlim\ \mathrm{Hom}_{\widehat{A}}\left(\mathrm{Tor}_i^{\widehat{A}}(E, T'_n), E\right) \simeq \varprojlim\ \mathrm{Ext}^i_{\widehat{A}}(T'_n, \mathrm{Hom}(E,E)) \simeq \varprojlim\ \mathrm{Ext}^i_{\widehat{A}}\left(T'_n, \widehat{A}\right).\end{aligned}$$

But as $\operatorname{depth} \widehat{A} = r$, and as for every n, T'_n is an $\widehat{A}$ module of finite length, $\mathrm{Ext}^i_{\widehat{A}}\left(T'_n, \widehat{A}\right) = 0$ for $i < r$, thus $\mathrm{Ext}^i_{\widehat{A}}\left(E, \widehat{T}\right) = 0$ for $i < r$.

Now note that to finish to prove (i) and to prove (ii), it will suffice to prove (iii), as by (iii):

$$\mathrm{Ext}^i_{\widehat{A}}\left(M, \widehat{A}\right) = 0 \text{ for } i > r - \operatorname{depth} T \text{ and } \mathrm{Ext}^i_{\widehat{A}}\left(M, \widehat{A}\right) \neq 0 \text{ for } i = r - \operatorname{depth} T\ ,$$

thus $\mathrm{pd}_{\widehat{A}} M = r - \operatorname{depth} T$.

On the other hand, still by (iii):

$$\operatorname{Supp} M = \bigcup_{i \geqslant 0} \operatorname{Supp}(\mathrm{Ext}^i_{\widehat{A}}\left(M, \widehat{A}\right)) = \bigcup_{i \geqslant 0} \operatorname{Supp}(\mathrm{Ext}^{n-(r-i)}_R\left(\widehat{T}, R\right)) = \operatorname{Supp} \widehat{T}\ .$$

To prove (iii), we consider the functor $\mathrm{Hom}_A\left(\mathrm{Hom}_A(\cdot, E), \widehat{A}\right)$.

Lemma 4.11. *There exists a canonical morphism of functors:*

$$\mathrm{Hom}_A(E, \cdot) \longrightarrow \mathrm{Hom}_A\left(\mathrm{Hom}_A(\cdot, E), \widehat{A}\right)$$

which induces an isomorphism on these functors restricted to the category of artinian A-modules. (i.e. such that for every artinian module C, the homomorphism $\mathrm{Hom}_A(E, C) \longrightarrow \mathrm{Hom}_A\left(\mathrm{Hom}_A(C, E), \widehat{A}\right)$ is an isomorphism).

Indeed, recall that we have $\widehat{A} \simeq \mathrm{Hom}_A(E, E)$. By the classical isomorphism of duality, we deduce from it:

$$\begin{aligned}\mathrm{Hom}_A\left(\mathrm{Hom}_A(\cdot, E), \widehat{A}\right) &\simeq \mathrm{Hom}_A(\mathrm{Hom}_A(\cdot, E), \mathrm{Hom}_A(E, E))\\ &\simeq \mathrm{Hom}_A(\mathrm{Hom}_A(\cdot, E) \otimes_A E, E)\\ &\simeq \mathrm{Hom}_A(E \otimes_A \mathrm{Hom}_A(\cdot, E), E)\\ &\simeq \mathrm{Hom}_A(E, \mathrm{Hom}_A(\mathrm{Hom}_A(\cdot, E), E))\end{aligned}$$

Thus, the canonical morphism of functors $\mathrm{id} \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(\cdot, E), E)$ induces as well a morphism $\mathrm{Hom}_A(E, \cdot) \longrightarrow \mathrm{Hom}_A\left(\mathrm{Hom}_A(\cdot, E), \widehat{A}\right)$. But these two functors are covariant, left exact, and take the same value on E. Artinian modules, being modules which admit an injective resolution of the type E^{μ}, where the μ_i are integers, we deduce from it the announced isomorphism.

Return to the proof of the theorem, and recall that it is by applying the functor $\mathrm{Hom}_A\left(E, \mathrm{H}^0_{\mathfrak{m}}(\cdot)\right)$ to an injective resolution I of T, that we obtained the projective resolution:

$$0 \longrightarrow \widehat{A}^{\mu_0} \longrightarrow \widehat{A}^{\mu_1} \cdots \longrightarrow \widehat{A}^{\mu_r}\ , \tag{**}$$

of the $\widehat{A}$-module $M = \mathrm{Ext}^r_A(E, T)$. By the lemma, this resolution is also to be obtained by applying the functor $\mathrm{Hom}_A\left(\mathrm{Hom}_A\left(\mathrm{H}^0_{\mathfrak{m}}(\cdot), E\right), \widehat{A}\right)$ to the injective resolution I of T. By applying the functor $\operatorname{Hom}_{\widehat{A}}\left(\cdot,\widehat{A}\right)$, to (**), we obtain the complex of free $\widehat{A}$ modules:

$$\widehat{A}^{\mu_r} \longrightarrow \widehat{A}^{\mu_{r-1}} \cdots \longrightarrow \widehat{A}^{\mu_0} \longrightarrow 0\ . \tag{***}$$

Thus, this last complex can also be obtained by applying the functor

$$\operatorname{Hom}_{\widehat{A}}\left(\operatorname{Hom}_A\left(\operatorname{Hom}_A\left(\mathrm{H}^0_{\mathfrak{m}}(\cdot),E\right),\widehat{A}\right),\widehat{A}\right)$$

to the injective resolution I of T. But as $\operatorname{Hom}_A\left(\mathrm{H}^0_{\mathfrak{m}}(I),E\right)$ is a complex of free $\widehat{A}$-modules of finite type, and as $\operatorname{Hom}_A\left(\widehat{A},\widehat{A}\right)=\operatorname{Hom}_{\widehat{A}}\left(\widehat{A},\widehat{A}\right)$, the morphism of complexes of free A-modules of finite type:

$$\operatorname{Hom}_A\left(\mathrm{H}^0_{\mathfrak{m}}(I),E\right) \longrightarrow \operatorname{Hom}_{\widehat{A}}\left(\operatorname{Hom}_A\left(\operatorname{Hom}_A\left(\mathrm{H}^0_{\mathfrak{m}}(I),E\right),\widehat{A}\right),\widehat{A}\right)$$

is an isomorphism, that is that the complex $\operatorname{Hom}_A\left(\mathrm{H}^0_{\mathfrak{m}}(I),E\right)$ is isomorphic to the complex (***). We know that the cohomology of the complex $\mathrm{H}^0_{\mathfrak{m}}(I)$ is $\mathrm{H}^i(\mathrm{H}^0_{\mathfrak{m}}(I))=\mathrm{H}^i_{\mathfrak{m}}(T)$. The functor $\operatorname{Hom}_A\left(\cdot,E\right)$ being of course exact, we deduce from it that the homology of the complex (***) is:

$$\operatorname{Ext}^{r-i}_{\widehat{A}}\left(M,\widehat{A}\right)=\mathrm{H}_i(\widehat{A}^{\mu})\simeq\operatorname{Hom}_A\left(\mathrm{H}^i_{\mathfrak{m}}(T),E\right)$$

and the first isomorphism announced is proved.

The second, the isomorphism $\operatorname{Hom}_A\left(\mathrm{H}^i_{\mathfrak{m}}(T),E\right)\simeq\operatorname{Ext}^{n-i}_R\left(T,R\right)$ is nothing other than the theorem of local duality.

Remark 4.12. The isomorphism of free A-modules of finite type:

$$\operatorname{Hom}_A\left(E,\mathrm{H}^0_{\mathfrak{m}}(I)\right) \longrightarrow \operatorname{Hom}_A\left(\operatorname{Hom}_A\left(\mathrm{H}^0_{\mathfrak{m}}(I),E\right),\widehat{A}\right)$$

gives an isomorphism of complexes:

$$\operatorname{Hom}_A\left(\operatorname{Hom}_A\left(E,\mathrm{H}^0_{\mathfrak{m}}(I)\right),\widehat{A}\right) \longrightarrow \operatorname{Hom}_A\left(\mathrm{H}^0_{\mathfrak{m}}(I),E\right)$$

Thus in the course of the proof of the theorem, we have seen that by applying the functor $\operatorname{Hom}_A\left(\operatorname{Hom}_A\left(E,\mathrm{H}^0_{\mathfrak{m}}(\cdot)\right),\widehat{A}\right)$ to an injective resolution of T, we obtain a complex of free $\widehat{A}$-modules of finite type, having for homology the $\widehat{A}$-modules $\operatorname{Ext}^{n-i}_R\left(\widehat{T},R\right)$.

We know that if the local ring A admits a dualizing complex $C^{\cdot}$ (in particular, if A is a quotient of a regular local ring), the homology Ω of the complex $\operatorname{Hom}_A\left(T,C^{\cdot}\right)$ is such that $\widehat{\Omega}\simeq\operatorname{Ext}^{\cdot}_R\left(\widehat{T},R\right)$. It is this fact that we use in the following section to prove that in the case where A admits a dualizing complex, we can descend the $\widehat{A}$-module $M=\operatorname{Ext}^r_A\left(E,T\right)$ to an A-module (§5, th. (5.7))

Finally, we give the following corollary, which proves in a particular case the conjecture of Bass which we have stated above.

Corollary 4.13. *Let A be a noetherian local ring. Let T be a non zero A-module of finite type, of finite injective dimension, and of grade* 0 *(i.e. such that* $\operatorname{Ass}A\cap\operatorname{Supp}T\neq\emptyset$*). Then the annihilator of T is* (0)*, and A is a Cohen-Macaulay ring.*

Indeed, let $\alpha \in A$ be such that $\alpha T = 0$. Then, if $r = \operatorname{depth} A$, $\alpha \operatorname{Ext}^r_A(E,T) = 0$. But by the theorem, $M = \operatorname{Ext}^r_A(E,T) = 0$ is an $\widehat{A}$ module of finite type and of finite projective dimension. The first part of the corollary will then be a consequence of proposition 4.14. The second part is a consequence of 4.7; indeed, $\operatorname{ann} T = 0$ implies that $\operatorname{Supp} T = \operatorname{Spec} A$. But then let $\mathfrak{p} \in \operatorname{Spec} A$ be such that $\dim A/\mathfrak{p} = \dim A$. By 4.7, as $\mathfrak{p} \in \operatorname{Supp} T$, e have $\operatorname{depth} A_{\mathfrak{p}} + \dim A/\mathfrak{p} = \operatorname{depth} A$. This obviously implies $\dim A = \dim A/\mathfrak{p} = \operatorname{depth} A$, thus A is Cohen-Macaulay.

PROPOSITION 4.14. (Auslander) *Let M be a module of finite type and finite projective dimension over a noetherian local ring A. Then if M is of grade 0, the annihilator of M is (0).*

Recall that to prove this result, one takes a projective resolution of M. By reasoning on the ranks, one proves that $\operatorname{Supp} M = \operatorname{Spec} A$. If I is the annihilator of M, one deduces from it that for every prime ideal $\mathfrak{p} \in \operatorname{Ass} A$, one has $IA_{\mathfrak{p}} = 0$, thus $I = 0$.

We end this section with a theorem giving some conditions for annihilation of cohomology. We will give this theorem in the most general form possible, because it will show that if we have a local ring A, it is not only interesting to find an A-module of finite type and biggest depth possible, but that one can, in fact, allow it to exist at a certain "distance" from A.

THEOREM 4.15. *Let T be an module of finite type and finite injective dimension over a noetherian local ring A. Let B be an A-algebra such that the structure homomorphism $A \to B$ is local, and such that if $\mathfrak{m}$ is the maximal ideal of A, the ring $B/\mathfrak{m}B$ is artinian. Then for every B-module M of finite type, we have:*

$$\operatorname{depth}_B M + \sup\{i \in \mathbb{Z} \mid \text{ such that } \operatorname{Ext}^i_A(M,T) \neq 0\} = \operatorname{depth} A\ .$$

First, recall that under the hypotheses of the theorem, the depth of M as a B-module is the same as its depth as an A-module. Indeed, the only difficulty is to show that if $\operatorname{depth}_B M > 0$, there exists an M-regular element in $\mathfrak{m}$. If not, $\mathfrak{m}$ is contained in the union of the prime ideals associated to M. By the avoidance lemma, $\mathfrak{m}B$ is contained in a prime ideal associated to M. As $\mathfrak{m}B$ is an ideal of definition of B, this is contrary to the hypotheses that $\operatorname{depth}_B M > 0$.

Now we reason by induction on $\operatorname{depth} M$. If $\operatorname{depth} M = 0$, there exists an exact sequence of A-modules $0 \to k \to M$ where k is the residue field of A. If $r = \operatorname{depth} A$, we deduce from it an exact sequence $\operatorname{Ext}^r_A(M,T) \to \operatorname{Ext}^r_A(k,T) \to 0$. By 4.2, $\operatorname{Ext}^r_A(k,T) \neq 0$, thus $\operatorname{Ext}^r_A(M,T) \neq 0$. Now suppose $\operatorname{depth} M > 0$, that is that there exists an element $\alpha \in \mathfrak{m}$ which is M-regular. Consider the exact sequence:

$$0 \to M \xrightarrow{\alpha} M \to M/\alpha M \to 0\ . \tag{*}$$

Note that we can assume that A is complete, as we will change neither the given, nor the conclusion of the theorem on replacing A and B by their completions.

Then let E be an injective envelope of the residue field k of A. We know that the functor $\operatorname{Hom}_A(\operatorname{Hom}_A(.,E),E)$, is reduced, up to isomorphism, to the identity on the category of A-modules of finite type. Setting $T' = \operatorname{Hom}(T,E)$, we have then by duality:

$$\operatorname{Ext}^i_A(M,T) \simeq \operatorname{Ext}^i_A(M, \operatorname{Hom}(T',E)) \simeq \operatorname{Hom}_A\left(\operatorname{Tor}^A_i(M,T'),E\right)\ . \tag{**}$$

Suppose depth $M = s$. Thus we have depth $M/\alpha M = s-1$, and by the induction hypotheses $\operatorname{Ext}^i_A(M/\alpha M, T) = 0$ for $i > r-(s-1)$. By (**), we deduce from it:

$$\operatorname{Tor}^A_i(M/\alpha M, T') = 0 > r-(s-1) .$$

The exact sequence of Tors associated to (*) says then that we have an exact sequence:

$$0 \longrightarrow \operatorname{Tor}^A_i(M, T') \xrightarrow{\alpha} \operatorname{Tor}^A_i(M, T') \geqslant r-(s-1) .$$

But as T' is the inductive limit of countable modules of finite length, the same holds true of $\operatorname{Tor}^A_i(M, T')$ for every i, thus if $\operatorname{Tor}^A_i(M, T') \neq 0$ there is not a regular element for $\operatorname{Tor}^A_i(M, T')$.

We deduce from it that:

$$\operatorname{Tor}^A_i(M, T') = 0 \geqslant r-(s-1)$$

that is by (**)

$$\operatorname{Ext}^i_A(M, T) = 0 \geqslant r-(s-1) .$$

On the other hand, still by the induction hypotheses,

$$\operatorname{Ext}^{r-(s-1)}_A(M/\alpha M, T) \neq 0 .$$

But then the exact sequence:

$$\operatorname{Ext}^{r-s}_A(M, T) \xrightarrow{\alpha} \operatorname{Ext}^{r-s}_A(M, T) \longrightarrow \operatorname{Ext}^{r-(s-1)}_A(M/\alpha M, T) \longrightarrow 0$$

shows that $\operatorname{Ext}^{r-s}_A(M, T) \neq 0$, and the theorem is proved.

5. Theorem of descent for modules of finite projective dimension

We have stated in the preceding section that, starting with a module T of finite injective dimension, of finite type over a noetherian local ring A, of depth r, the $\widehat{A}$ module of finite type $\operatorname{Ext}^r_A(E, T)$ has finite projective dimension. Here we intend to prove that, over "good" rings, this is the completion of an A-module of finite type and finite projective dimension. For this we will need the theorem of local duality stated in the language of derived categories, that one finds in R. Hartshorne [**13**]. The reading of this section is not necessary for what follows. Nevertheless, it allows us to prove that for "good rings" the study of modules of finite injective dimension leads itself back to the study of modules of finite projective dimension. The technique employed allows also one us recover in a natural way a theorem of G. Horrocks [**16**]. For the reasons stated before, we will use without recall the terminology of derived categories for which one can refer to [**13**].

Proposition 5.1. *Let A be a noetherian local ring, $\widehat{A}$ it completion, and M an $\widehat{A}$ module of finite type and finite projective dimension. Let L be a finite projective resolution of M. The following statements are equivalent:*

(i) *there exists an A-module N such that $M = N \otimes_A \widehat{A}$;*

(ii) *there exists a complex P in $\mathrm{D}^-_c(A)$ such that $P \otimes_A \widehat{A} \simeq \operatorname{Hom}_{\widehat{A}}\left(L, \widehat{A}\right)$ in $\mathrm{D}^-_c(\widehat{A})$.*

It is clear that (i) implies (ii) because such an N will be of finite type and finite projective dimension over A. Then we take for P the complex $\operatorname{Hom}_A(Q, A)$, where Q is a finite projective resolution of N over A. There is a homotopy equivalence $Q \otimes_A \widehat{A} \longrightarrow L$, which, by virtue of [**13**], gives the result that we seek.

Reciprocally, we can always assume that P is a complex of projective modules. Setting $Q = \operatorname{Hom}_A(P, A)$ we have $Q \otimes_A \widehat{A} \simeq L$, still by virtue of of [**13**]. As tensorization with $\widehat{A}$ commutes with homology, there is only a single non zero homology module in Q: this is the required module N.

Remark 5.2. Let A be a noetherian local ring of depth r and maximal ideal $\mathfrak{m}$. Starting from a module T of finite injective dimension over A, we have constructed in the preceding section, an $\widehat{A}$ module M of finite type and finite projective dimension with the same support as that of $\widehat{T}$. We have seen that $\operatorname{Ext}^i_{\widehat{A}}\left(M, \widehat{A}\right)$ is isomorphic to $\operatorname{Hom}_{\widehat{A}}\left(\mathrm{H}^{r-i}_{\mathfrak{m}}(T), E\right)$ where E if the injective envelope of the residue field of A. In fact if I is an injective resolution of T, we obtain a projective resolution of M by applying to I the functor $\operatorname{Hom}_{\widehat{A}}\left(\operatorname{Hom}_{\widehat{A}}\left(\mathrm{H}^0_{\mathfrak{m}}(-), E\right), \widehat{A}\right)$. That is that if L is a finite projective resolution of M over $\widehat{A}$, we have:

$$\operatorname{Hom}_{\widehat{A}}\left(L, \widehat{A}\right) \simeq \operatorname{Hom}_{\widehat{A}}\left(\mathrm{H}^0_{\mathfrak{m}}(I), E\right) \quad \text{in } \mathrm{D}^-_c(\widehat{A}).$$

Now, the theorem of local duality in the derived category form is used to calculate the complex $\mathrm{H}^o_{\mathfrak{m}}(I)$. More precisely (cf. [**13**]):

5.3. *Definition. Let A be a local ring. A dualizing complex for A is an element $R \in \mathrm{D}^-_c(A)$ such that:*

(i) *R is a finite injective resolution;*
(ii) *for every $F \in \mathrm{D}^+_c(A)$ the natural homomorphism*

$$F \to \boldsymbol{R}\operatorname{Hom}_{.}\left(\boldsymbol{R}\operatorname{Hom}_{.}(F, R), R\right)$$

is an isomorphism.

Example 5.4. Every quotient A of a local Gorenstein ring R has a dualizing complex namely: if I is a finite injective resolution of R as an R-module, $\operatorname{Hom}_R(A, I)$ is a dualizing complex for A.

Proposition 5.5. *Let A be a noetherian local ring. If A has a dualizing complex R, then $R \otimes_A \widehat{A}$ is a dualizing complex over $\widehat{A}$.*

Theorem 5.6. *(***local duality***) Let A be a ring possessing a dualizing complex R, let $\mathfrak{m}$ be its maximal ideal and E the injective envelope of the residue field $A/\mathfrak{m}$ of A; then there exists an isomorphism of functors from $\mathrm{D}^+_c(A)$ into $\mathrm{D}^+_c(A)$:*

$$\mathrm{H}^0_{\mathfrak{m}}(-) \simeq \operatorname{Hom}_A\left(\boldsymbol{R}\operatorname{Hom}(-, R), E\right) .$$

([**13**], chap. V, th. (6.2))

We have now all the material necessary to prove the following theorem:

Theorem 5.7. *Let A be a ring possessing a dualizing complex. Let T be an A-module of finite type and finite injective dimension; then there exists an A-module of finite type and finite projective dimension such that* $\operatorname{Supp} M = \operatorname{Supp} T$.

Proof. By virtue of proposition (6.1) and the remark (6.2) it is suffices to find F in $\mathrm{D}^-_c(A)$ such that $\operatorname{Hom}_{\widehat{A}}\left(\mathrm{H}^0_{\mathfrak{m}}(I), E\right)$ is isomorphic to $F \otimes_A \widehat{A}$ in $\mathrm{D}^-_c(\widehat{A})$, where I is a finite injective resolution of T.

Take $E = \operatorname{Hom}_A(I, R)$. By virtue of the theorem of local duality:

$$\operatorname{Hom}_{\widehat{A}}\left(\mathrm{H}^0_{\mathfrak{m}}(I), E\right) \simeq \operatorname{Hom}_{\widehat{A}}\left(\mathrm{H}^0_{\mathfrak{m}}(I \otimes_A \widehat{A}), E\right) \simeq \operatorname{Hom}_{\widehat{A}}\left(I \otimes_A \widehat{A}, R \otimes_A \widehat{A}\right)$$

in $\mathrm{D}_c^+(\widehat{A})$ where R is a dualizing complex over A. Now;

$$\operatorname{Hom}_{\widehat{A}}\left(I \otimes_A \widehat{A}, R \otimes_A \widehat{A}\right) \simeq \operatorname{Hom}_A(I, R) \otimes_A \widehat{A}$$

by virtue of [**13**], and thus the theorem is proved. $\square$

Proposition (6.1) allows in certain cases to descend a module of finite projective dimension knowing that its $\operatorname{Ext}^i_{\widehat{A}}\left(M, \widehat{A}\right)$ descend themselves for $i > 0$. Here are some examples.

Recall that an A-module M is *r-spherical* if:

a) $\operatorname{pd} M = r$;
b) $\operatorname{Ext}^i_A(M, A) = 0$ for $i \neq 0$ and $i \neq r$.

PROPOSITION 5.8. *Let A be a noetherian local ring, M an r-spherical $\widehat{A}$ module of finite type; then, if $\operatorname{Ext}^r_{\widehat{A}}\left(M, \widehat{A}\right)$ is the completion of an A-module of finite type, there exists an r-spherical A-module N such that $N \otimes_A \widehat{A} = M$.*

PROOF. We can note that nothing is supposed on $\operatorname{Hom}_{\widehat{A}}\left(M, \widehat{A}\right)$. Let:

$$0 \to L_r \to L_{r-1} \to \cdots \to L_1 \to L_0 \to M \to 0$$

be a free resolution of M, then writing $\check{} = \operatorname{Hom}_{\widehat{A}}\left(., \widehat{A}\right)$

$$\check{L_0} \to \check{L_1} \to \cdots \to \check{L}_{r-1} \to \check{L_r} \to \operatorname{Ext}^r_{\widehat{A}}\left(M, \widehat{A}\right) \to 0$$

is a free resolution of $\operatorname{Ext}^r_{\widehat{A}}\left(M, \widehat{A}\right)$. Choose:

$$F_0 \to F_1 \to \cdots \to F_{r-1} \to F_r \to K \to 0$$

a projective resolution of the A-module K such that $K \otimes_A \widehat{A} \overset{\phi}{\simeq} \operatorname{Ext}^r_{\widehat{A}}\left(M, \widehat{A}\right)$. We obtain a morphism of complexes $\check{L} \xrightarrow{f} \widehat{F}$ such that $\mathrm{H}_r(f) = \phi$.

As $\operatorname{grade} \operatorname{Ext}^r_{\widehat{A}}\left(M, \widehat{A}\right) \geqslant r$, we have an exact sequence:

$$0 \to (\widehat{F}_r)\check{} \to \cdots \to (\widehat{F}_1)\check{} \to (\widehat{F}_0)\check{} \to \widehat{N} \to 0,$$

where $N = \operatorname{Coker}(\widehat{F}_1 \to \widehat{F}_0)$. We obtain in this way an homomorphism $g\colon \widehat{N} \to M$ such that $\operatorname{Ext}^r_{\widehat{A}}\left(g, \widehat{A}\right) = \phi$. Even if it means adjoining to N a free module over A we can suppose that g is surjective. Thus we have an exact sequence:

$$0 \to C \to \widehat{N} \to M \to 0$$

from which we can deduce that:

a) $\operatorname{pd}_{\widehat{A}} C < \infty$;
b) $\operatorname{Ext}^i_{\widehat{A}}\left(C, \widehat{A}\right) = 0$ for all $i > 0$;

thus, by (5.3) C is a free $\widehat{A}$-module.

But, as the map $\operatorname{Ext}^1_{\widehat{A}}\left(M, \widehat{A}\right) \to \operatorname{Ext}^1_{\widehat{A}}\left(N, \widehat{A}\right)$ is an isomorphism, the sequence $0 \to \check{M} \to \check{\widehat{N}} \to \check{C}$ is exact and split. Thus, $C \simeq \check{C}$ is a direct summand of $\check{\widehat{N}}$, thus of $\widehat{N}$. Thus, $M \oplus C \simeq \widehat{N}$. As we can suppose that M does not have a free direct summand, the theorem of Krull-Schmidt-Azumaya implies that M is isomorphic to the completion of the factor of N that does not have a free direct summand. cf. [**9**] □

The proposition that follows was proved in another way by Horrocks [**16**].

PROPOSITION 5.9. *(Horrocks) Let A be a noetherian local ring, $X = \operatorname{Spec} A - \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of A. Set $Y = \operatorname{Spec} \widehat{A} - \widehat{\mathfrak{m}}$. Let M be an $\widehat{A}$-module of finite type and finite projective dimension that is locally free over Y. Then there exists an A-module N of finite type and finite projective dimension such that $\widehat{N} = M$.*

PROOF. Indeed, we can suppose that M does not have a non-trivial direct summand. Let r be the projective dimension of M. Set $t = \inf\{i > 0$, such that $\operatorname{Ext}^i_{\widehat{A}}\left(M, \widehat{A}\right) \neq 0\}$. We argue by induction on $r - t$. If $r - t = 0$, then M is r-spherical and the preceding proposition allows us to conclude.

Thus we suppose $r > t$. Consider a projective resolution of M:

$$0 \to L_r \to \cdots \to L_1 \to L_0 \to M \to 0 \ .$$

Denoting as before $\check{\cdot}$ the functor $\operatorname{Hom}_{\widehat{A}}\left(\cdot, \widehat{A}\right)$, we set $T = \operatorname{Coker}(\check{L}_{t-1} \to \check{L}_t)$. Then, we have a natural injection:

$$\operatorname{Ext}^t_{\widehat{A}}\left(M, \widehat{A}\right) \hookrightarrow T \ .$$

Consider a projective resolution of $\operatorname{Ext}^t_{\widehat{A}}\left(M, \widehat{A}\right)$ which is an A-module of finite length:

$$F_0 \to F_1 \to \cdots \to F_t \to \operatorname{Ext}^t_{\widehat{A}}\left(M, \widehat{A}\right) \to 0 \ .$$

The injection extends itself to a morphism of complexes:

$$(\widehat{F}_i)_{i \geqslant 0} \to (\check{L}_i)_{0 \leqslant i \leqslant t} \ .$$

On applying the functor $\check{\cdot}$ to the morphism of complexes, we find a morphism of complexes:

$$(L_i)_{0 \leqslant i \leqslant t} \to (\check{\widehat{F}}_i)_{i \geqslant 0}$$

which induces a homomorphism $M \xrightarrow{g} \widehat{N}$, where $N = \operatorname{Coker}(\check{F}_1 \to \check{F}_0)$. Note that N is an t-spherical A-module such that $\operatorname{Ext}^t_{\widehat{A}}\left(\widehat{N}, \widehat{A}\right) = \operatorname{Ext}^t_{\widehat{A}}\left(M, \widehat{A}\right)$. The map:

$$\operatorname{Ext}^t_{\widehat{A}}\left(g, \widehat{A}\right) : \operatorname{Ext}^t_{\widehat{A}}\left(\widehat{N}, \widehat{A}\right) \to \operatorname{Ext}^t_{\widehat{A}}\left(M, \widehat{A}\right)$$

factors the injection $\operatorname{Ext}^t_{\widehat{A}}\left(\widehat{N}, \widehat{A}\right) \hookrightarrow T$, thus this is an injection, and as it concerns modules of finite length, it is an isomorphism. By adjoining, if need be, to M, a free $\widehat{A}$-module P of finite rank, we can assume that g is surjective. Let K be the kernel of g. We can see easily that K is locally free on Y, that K is of projective dimension r, and that $\operatorname{Ext}^i_{\widehat{A}}\left(K, \widehat{A}\right) = 0$ for $1 \leqslant i \leqslant t$. By the hypotheses of induction, K descends itself, that is that there exists an A-module C such that $K = \widehat{C}$. As N is locally free over X, the A-module $\operatorname{Ext}^1_A(N, C)$ is of finite length, thus

$$\operatorname{Ext}^1_A(N, C) \simeq \operatorname{Ext}^1_{\widehat{A}}\left(\widehat{N}, \widehat{C}\right) \ ,$$

that is that every extension of $\widehat{N}$ by K provides an extension of N by C. Thus $M \oplus P$ itself descends, i.e. there exists an A-module D such that $M \oplus P = \widehat{D}$, and we deduce from it, by way of the theorem of Krull-Schmidt-Azumaya, that M itself descends. $\square$

Remark 5.10. If A is regular of dimension $\geqslant 2$, this proposition allows Horrocks to show that every vector bundle $\mathcal{F}$ over Y is the reciprocal image $f^*(\mathcal{G})$ of a vector bundle $\mathcal{G}$ over X, where $f\colon Y \to X$ is the morphism of schemes induced from the canonical homomorphism $A \to \widehat{A}$. Indeed, under the hypotheses, the global sections functor gives an equivalence of categories between vector bundles over X (resp. Y) and the A-module (resp. $\widehat{A}$-modules) of finite type, reflexive, locally free over X (resp. Y).

6. Theorem of approximation for modules of finite type and finite projective dimension

Introduction. Let A be a noetherian local ring. To give an A-module of finite type and finite projective dimension, is to give a complex of free A-modules of finite type, say: $0 \to L_r \xrightarrow{\phi_r} L_{r-1} \to \cdots \to L_1 \xrightarrow{\phi_1} L_0$, such that:

(i) $\operatorname{Coker} \phi_1 = M$;
(ii) the homology modules $\mathrm{H}_i(L)$ are zero for $i > 0$.

We have given in 1.8 a lemma (called acylicity) which gives a method to verify that a finite complex of free modules of finite type have zero homology in degree > 0. We have used this lemma in 1.7 to verify the exactness of a complex obtained beginning with geometric considerations (morphism of Frobenius). In this section, we are going to repeat this operation to show the exactness of a complex obtained via the approximation theorem of Artin, which we recall here.

Theorem 6.1. *(M. Artin) Let R be a field or an excellent discrete valuation ring. Let A be a local ring essentially of finite type over R, and let A^h be its henselization. Let (f_i) be a system of polynomial equations with coefficients in A^h. Let $\mathfrak{m}$ an ideal of A^h, and let $\widehat{A}$ the $\mathfrak{m}$-adic completion of A^h. Then if $\overline{y} = (\overline{y}_1, \ldots, \overline{y}_N)$ is a solution of the equations (f_i), from elements in $\widehat{A}$, for every integer c, there exists a solution $y = (y_1, \ldots, y_N)$, from elements of A^h, such that $y_j \cong \overline{y}_j(\mathfrak{m}^c)$ for $i = 1, \ldots, N$.*

We place ourselves in the situation of the theorem of Artin. That is to say that we consider the completion $\widehat{A}$ of a local ring A essentially of finite type over R, where R is either a field, or an excellent discrete valuation ring. Let then L be a finite complex of free modules of finite rank over $\widehat{A}$. Fixing a bases for every L_i, and denoting $\mu_i = \operatorname{rank} L_i$, we can write this complex in the following form:

$$\widehat{A}^{\mu_i} \xrightarrow{\phi_i} \widehat{A}^{\mu_{i-1}} \to \cdots \xrightarrow{\phi_1} \widehat{A}^{\mu_0} \tag{*}$$

where to the maps ϕ_i correspond the matrices $((\phi_i^{p,q})_{p,q})$. For every $i = 1, \ldots, r$, consider Φ_i the matrix in μ_i columns and μ_{i-1} rows, and with coefficients indeterminates $X_i^{p,q}$. To say that $(*)$ is a complex is to say that the matrices $((\phi_i^{p,q})_{p,q})$ are solutions of the equations $\Phi_i \circ \Phi_{i-1} = 0$, for $i = 1, \ldots, r$. Let $\mathfrak{m}$ be the maximal ideal of A, and let c an integer. By the theorem of Artin, there exists matrices $((\Psi_i^{p,q})_{p,q})_i$, with coefficients in the henselization A^h of A, such that $\Psi_i \circ \Psi_{i-1} = 0$ and that $\Psi_i^{p,q} = \phi_i^{p,q}(\mathfrak{m}^c)$.

THEOREM 6.2. *Let R be a field or an excellent discrete valuation ring. Let A be a local ring essentially of finite type over R. Let $\widehat{A}$ be the completion of A for the topology defined by the maximal ideal $\mathfrak{m}$ of A. Let M be an $\widehat{A}$-module of finite type and finite projective dimension. Consider a minimal free resolution of M over $\widehat{A}$:*

$$0 \longrightarrow L_r \xrightarrow{\phi_r} L_{r-1} \longrightarrow \ldots \longrightarrow L_1 \xrightarrow{\phi_1} L_0 \longrightarrow M \longrightarrow 0 \ .$$

Then, for every integer c, there exists an exact complex:

$$0 \longrightarrow P_r \xrightarrow{\Psi_r} P_{r-1} \longrightarrow \ldots \longrightarrow P_1 \xrightarrow{\Psi_1} P_0$$

of free modules of finite type over the henselization A^h of A, such that $P \otimes_{A^h} (A^h/\mathfrak{m}^c A^h) = L \otimes_{\widehat{A}} (\widehat{A}/\mathfrak{m}^c \widehat{A})$.

COROLLARY 6.3. *Let A be a local ring essentially of finite type over a field or a excellent discrete valuation ring. Let $\mathfrak{m}$ be its maximal ideal. Let M be a module of finite type and finite projective dimension over the completion $\widehat{A}$ of A. Then, for every integer c, there exists a local ring A', localized at a closed point of a etale covering of A, and an A'-module M' of finite type and finite projective dimension, such that*

(i) *$M'/\mathfrak{m}^c M' \simeq M/\mathfrak{m}^c M$;*
(ii) $\operatorname{pd}_{A'} M' = \operatorname{pd}_{\widehat{A}} M$, *thus* $\operatorname{depth}_{A'} M' = \operatorname{depth}_{\widehat{A}} M$.

We note right away that the corollary is an immediate consequence of the theorem. Indeed, we take a minimal resolution (L, ϕ) of M by free $\widehat{A}$-modules. We construct the exact complex (P, Ψ) of free modules of finite type over the henselization A^h of A. As A^h is an inductive limit of pointed etale coverings of A, we can find a pointed etale covering A' which contains all the coefficients of the matrices $\Psi_i^{p,q}$. As A^h is faithfully flat over A', the complex (P, Ψ) then descends itself naturally to a exact complex (P', Ψ') of free A'-modules. We take $M = \operatorname{Coker}(P_1' \xrightarrow{\phi_i'} P_0')$, and, taking into account the theorem, we easily verify that M' possesses the properties requested.

We now prove the theorem. We have seen that we have constructed a complex (P, Ψ) of free A^h-modules of finite type such that $(P/\mathfrak{m}^c P, \Psi/\mathfrak{m}^c \Psi) = (L/\mathfrak{m}^c L, \phi/\mathfrak{m}^c \phi)$. We show that if c is large enough, the complex constructed is exact. For that we use the acyclicity lemma. As the length r of the complex P is equal to the $\operatorname{pd} M$, we have $r \leqslant \operatorname{depth} \widehat{A} = \operatorname{depth} A$. Thus, by the acyclicity lemma, to show that P is exact in degrees > 0, it suffices to show that its homology is of finite length in degrees > 0. This is a consequence of the next lemma. $\square$

LEMMA 6.4. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. Consider a three term exact sequence of free A-modules of finite type: $L' \xrightarrow{\phi'} L \xrightarrow{\phi} L''$. There exists an integer c such that if $L' \xrightarrow{\Psi'} L \xrightarrow{\Psi} L''$ is a complex such that*

$$\Psi' \cong \phi' \ mod \ \mathfrak{m}^c \quad \text{and } \Psi \cong \phi \ mod \ \mathfrak{m}^c \ ,$$

then $\operatorname{Ker} \Psi / \operatorname{Im} \Psi'$ *is an A-module of finite length.*

PROOF. Consider $\operatorname{Im} \phi$ (resp. $\operatorname{Im} \phi'$), sub-module of L'' (resp. L). The Artin-Rees lemma says that there exists μ (resp. ν) such that:

$$\mathfrak{m}^n L'' \cap \operatorname{Im} \phi \hookrightarrow \mathfrak{m}^{n-\mu} (\operatorname{Im} \phi)$$

(resp. $\mathfrak{m}^n L \cap \operatorname{Im}\phi' \hookrightarrow \mathfrak{m}^{n-\nu}(\operatorname{Im}\phi')$)

Let's take $c > \sup(\mu, \nu)$, and show that if $\psi' \cong \phi' \bmod \mathfrak{m}^c$, and $\psi \cong \phi \bmod \mathfrak{m}^c$, then we have:

$$\mathfrak{m}^s L \cap \operatorname{Ker}\psi \hookrightarrow \operatorname{Im}\psi' + (\mathfrak{m}^{s+1} L \cap \operatorname{Ker}\psi) > c\ , \tag{*}$$

which proves the lemma. Indeed, we deduce from it that for every integer $i > 0$

$$\mathfrak{m}^s L \cap \operatorname{Ker}\psi \hookrightarrow \operatorname{Im}\psi' + (\mathfrak{m}^{s+i} L \cap \operatorname{Ker}\psi)\ .$$

Thus $\mathfrak{m}^s \operatorname{Ker}\psi \subset \operatorname{Im}\psi'$. Let $x \in \mathfrak{m}^s L \cap \operatorname{Ker}\psi$. Then $\psi(x) = 0$, and :

$$x = \sum_i m_i x_i \qquad \text{where} \quad x_i \in L \quad \text{and} \quad m_i \in \mathfrak{m}^s\ .$$

Thus we have:

$$\phi(x) = \phi(x) - \psi(x) = \sum_i m_i(\phi - \psi)(x_i) \in \mathfrak{m}^{s+c} L'' \cap \operatorname{Im}\phi.$$

We know that we have $\mathfrak{m}^{s+c} L'' \cap \operatorname{Im}\phi \hookrightarrow \mathfrak{m}^{s+c-\mu}(\operatorname{Im}\phi)$.

Thus, there exist elements $n_i \in \mathfrak{m}^{s+c-\mu}$, and elements $y_i \in L$ such that:

$$\phi(x) = \phi(\sum_i n_i y_i)\ .$$

Said otherwise, $\phi(x - \sum_i n_i y_i) = 0$, thus by the exactness of the sequence $L' \xrightarrow{\phi'} L \xrightarrow{\phi} L''$, there exists an element y' of L' such that $\phi'(y') = x - \sum_i n_i y_i$. As $x \in \mathfrak{m}^s L$ by hypotheses, and as $n_i \in \mathfrak{m}^{s+c-\mu} \subset \mathfrak{m}^s$, we deduce from it:

$$x - \sum_i n_i y_i \in \mathfrak{m}^s L \cap \operatorname{Im}\phi'\ .$$

But as $s > c > \nu$, we have $\phi`(y`) = x - \sum_i n_i y_i \in \mathfrak{m}^{s-\nu} \operatorname{Im}\phi'$. Thus, there exists $y'' \in \mathfrak{m}^{s-\nu} L'$ such that $x - \sum_i n_i y_i = \phi'(y'')$. From where:

$$x - \phi'(y'') = \sum_i n_i y_i + (\phi' - \psi')(y'')\ ,$$

which implies $x - \phi'(y'') \in (\mathfrak{m}^{s+c-\mu} L + \mathfrak{m}^{s+c-\nu} L) \cap \operatorname{Ker}\psi$.

We easily deduce from it $x \in \operatorname{Im}\psi' + (\mathfrak{m}^{s+1} L \cap \operatorname{Ker}\psi)$, that is (*), that we want want to prove. □

Remark 6.5. Lemma (7.4) allows us to approach M more precisely if we want it. For example, if M' is a module of finite type and finite projective dimension over A^h, we can request $\operatorname{grade}_{A^h} M' \geqslant \operatorname{grade}_{\widehat{A}} M$. Indeed, to say that $\operatorname{grade} M \geqslant g$, is to say that the following complex is exact:

$$0 \longrightarrow L_0\check{} \xrightarrow{\phi_1\check{}} L_1\check{} \longrightarrow \cdots \xrightarrow{\phi_g\check{}} L_g\check{}\ ,$$

where $\check{}$ denotes the functor $\operatorname{Hom}_{\widehat{A}}\left(\cdot, \widehat{A}\right)$. But then, by the lemma, there exists c such that $(P\check{}, \psi\check{}) \cong (L\check{}, \phi\check{})$ modulo $\mathfrak{m}^c$ which implies that $0 \longrightarrow P_0\check{} \xrightarrow{\psi_1\check{}} P_1\check{} \longrightarrow \cdots \xrightarrow{\psi_g\check{}} P_g\check{}$, is exact. (of course, here $\check{}$ denoted $\operatorname{Hom}_{\widehat{A}}\left(\cdot, \widehat{A}\right)$.

CHAPTER 2

Theorem of Intersection Application to the proof of conjectures of M. Auslander and H. Bass

0. Fundamental questions on modules of finite type and finite projective dimension, over noetherian local rings.

Recall the following theorem proved by Serre in [**23**]:

THEOREM 0.1. *Let R be a regular local ring. If M and N are two R-modules of finite type such that $M \otimes N$ is of finite length, then* $\dim M + \dim N \leq \dim R$.

This theorem of course contains the formula for codimension of intersections in algebraic geometry: The codimension of intersection of two sub-varieties of a non-singular algebraic variety is less than or equal to the sum of the codimensions of the two sub-varieties.

To prove this theorem, Serre generalizes the method of the diagonal by means of completed Tor and deduces from it the case when R is equicharacteristic. To prove the case where R is unramified, then the general case, he utilizes the Euler-Poincaré characteristic associated to the Tor functors.

0.2. We see easily that the result cannot be extended if we completely ignore the hypotheses of regularity. The homological methods being determined here, one is natural constrained to conjecture that every module $M \neq 0$, of finite type and finite projective dimension over a noetherian local ring A, possesses the following property:

(a)*For every A-module of finite type N, and for every prime ideal $\mathfrak{p}$ minimal in* $\operatorname{Supp} M \cap \operatorname{Supp} N$ *we have*

$$\dim N_{\mathfrak{p}} \leq \dim A_{\mathfrak{p}} - \dim M_{\mathfrak{p}}$$

We remark that the theorem of Serre is equivalent to the following theorem:

THEOREM 0.3. *Let M be a finite module over a regular local ring R, then every system of parameters for M can extended itself to a system of parameters for R.*

We prove $0.2 \Rightarrow 0.1$

Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ be a system of parameters for M. As $M/\alpha M$ is of finite length, $\dim(R/\alpha) \leq \dim R - \dim M$. Set $d = \dim R$. As $\dim M = s$, we can find elements $\alpha_{s+1}, \ldots, \alpha_d$ of R generating an ideal of definition in R/α. Said otherwise, $R/(\alpha_1, \ldots, \alpha_d)$ is of finite length, which obviously implies that $(\alpha_1, \ldots, \alpha_d)$ is a system of parameters for R. Conversely, let M and N be two R-modules of finite type having their tensor product of finite length. Let $\mathfrak{a}$ be the annihilator of N. The module $M \otimes (R/\mathfrak{a})$ is also of finite length. We deduce from it that $\mathfrak{a}$ contains a system of parameters $\alpha = (\alpha_1, \ldots, \alpha_s)$ for M. By 0.3, if $d = \dim R$, we have

$\dim R/\alpha = d-s$. We deduce from it $\dim N = \dim R/\mathfrak{a} \leq d-s$, and the equivalence is proved.

In this latter form, the theorem of Serre of course contains the following result:

THEOREM 0.4. *If M is a module of finite type over a regular local ring R, then every M-regular sequence is R-regular.*

It is clear that, here also the hypothesis of regularity cannot be forgotten completely. Nevertheless, in relaxing the hypothesis, it is natural to conjecture that every module M of finte type and finite projective dimension over a noetherian local ring A, has the property:

(b)*Every M regular sequence is A regular.*

0.5. In [**3**], M.Auslander has proved that this is a corollary of his conjecture, called the Tor conjecture, of a more evidently homological nature, that we formulate in the following way:

If M is a module of finite type and finite projective dimension over a noetherian local ring A, then

(c)*For every A-module N of finite type, and every integer s, if* $\mathrm{Tor}_s^A(M,N) = 0$*, then*

$$\mathrm{Tor}_j^A(M,N) = 0, \geq s.$$

(In other words, M is rigid.)

0.6. It is for settling a conjecture related to (b) and a conjecture of H. Bass cited before (Chap I, §4), that we will show "almost generally" that every module M of finite type and finite projective dimension over a local ring A, possesses the property:

(d)*For every A-module N of finite type, and for every prime ideal $\mathfrak{p}$ in A such that $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$ is a non-zero $A_{\mathfrak{p}}$ module of finite length, we have* $\dim N_{\mathfrak{p}} \leqslant \mathrm{pd}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$.

We will show, in §3, that modules of finite type and finite projective dimension over a local ring which is either of characteristic $p > 0$, or essentially of finite type over a field of characteristic 0, or a henselization of a ring of this type, possesses the property (d).

0.7. It is obvious that (d) $\Rightarrow$ (a) when M is C.M. or more generally when $\mathrm{pd}_A M = \mathrm{grade}_A M$ (such a module is called perfect, because it is the most natural generalization of a quotient of a ring A by an A-regular sequence, of which the topological properties of the support are well know thanks to the Kozul complex). It is for this reason that we conjecture that a module $M \neq 0$ of finite type and finite projective dimension over a local ring A has the following property:

(e) *For every A-module N of finite type and for every prime ideal $\mathfrak{p}$ such that $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$ is a non-zero module of finite length, we have* $\dim N_{\mathfrak{p}} \leq \mathrm{grade}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$

0.8. We prove that (e) implies not only (a) but also the following fact conjectured by Auslander: Every module M of finite type and finite projective dimension over a local ring A has the property that:

(f)*For every prime ideal $\mathfrak{p}$ in* $\mathrm{Supp}\, M$*, we have*

$$\dim M_{\mathfrak{p}} + \mathrm{grade}_{A_{\mathfrak{p}}} M_{\mathfrak{p}} = \dim A_{\mathfrak{p}}$$

0.9. Before we show precisely what the relations between these six properties are, we emphasize that, in our view, these are essentially questions concerning the topology of a closed sub-schemes defined as supports of coherent sheaves of finite projective dimension, in an ambient scheme possibly having singularities.

THEOREM 0.10. *Let M be a non-zero module of finite type and finite projective dimension over a local ring A. We have the following relations between the properties defined above:*

Moreover, if for every prime ideal $\mathfrak{p}$ in $\operatorname{Supp} M$, the completion $\widehat{M_{\mathfrak{p}}}$ of $M_{\mathfrak{p}}$ possesses property (c) as an $\widehat{A}_{\mathfrak{p}}$ module, then M possesses the property (d).

PROOF. We prove first (e) $\Leftrightarrow$ (a) + (f).

We know (Chap I,§4, no. 8) that we have always $\dim M_{\mathfrak{p}} + \operatorname{grade}_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \dim A_{\mathfrak{p}}$. Thus $\dim N_{\mathfrak{p}} \leq \operatorname{grade}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$ implies $\dim M_{\mathfrak{p}} + \dim N_{\mathfrak{p}} \leq \dim A_{\mathfrak{p}}$, and so (e) $\Rightarrow$ (a). Let $\alpha = (\alpha_1, \ldots, \alpha_s)$, where $s = \dim M_{\mathfrak{p}}$, be a system of parameters of $M_{\mathfrak{p}}$. As $M_{\mathfrak{p}}/\alpha M_{\mathfrak{p}}$ is of finite length, (e) implies $\dim A_{\mathfrak{p}}/\alpha A_{\mathfrak{p}} \leq \operatorname{grade} M_{\mathfrak{p}}$. But, by (a), α can be extended to a system of parameters of $A_{\mathfrak{p}}$, thus

$$\dim A_{\mathfrak{p}}/\alpha A_{\mathfrak{p}} = \dim A_{\mathfrak{p}} - \dim M_{\mathfrak{p}}.$$

Moreover, $\dim M_{\mathfrak{p}} + \operatorname{grade}_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \geq \dim A_{\mathfrak{p}}$, thus (e) $\Rightarrow$ (f).

Conversely, assume (a) + (f). Let N be an A-module of finite type, and let $\mathfrak{p}$ a prime ideal minimal in $\operatorname{Supp} M \cap \operatorname{Supp} N$. Then (a) implies

$$\dim N_{\mathfrak{p}} \leq \dim A_{\mathfrak{p}} - \dim M_{\mathfrak{p}}.$$

But, by (f), $\dim A_{\mathfrak{p}} - \dim M_{\mathfrak{p}} = \operatorname{grade} M_{\mathfrak{p}}$. Then $\dim N_{\mathfrak{p}} \leq \operatorname{grade} M_{\mathfrak{p}}$, thus (a) + (f) $\Rightarrow$ (e).

(c) $\Rightarrow$ (b) is proved by M. Auslander in [**3**].

(e) $\Rightarrow$ (d) is clear on account of $\operatorname{grade} M_{\leq} \operatorname{pd} M_{\mathfrak{p}}$.

(d) $\Rightarrow$ (b) We remark that if α is a non-invertible element of A, regular in A and M, $A/\alpha A$ module of finite type and finite projective dimension $M/\alpha M$, possesses the property (d). Indeed, if N is an $A/\alpha A$ module of finite type and $\mathfrak{p}/\alpha A$ a prime ideal of $A/\alpha A$, minimal in $\operatorname{Supp} N \cap \operatorname{Supp} M/\alpha M$, then N has a natural structure of an A module of finite type and $\mathfrak{p}$ is a prime ideal minimal in $\operatorname{Supp} N \cap \operatorname{Supp} M$. By (d), we deduce that $\dim N_{\mathfrak{p}} \leq \operatorname{pd}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. But as $\operatorname{pd}_{A_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{pd}_{A_{\mathfrak{p}}/\alpha A_{\mathfrak{p}}}(M_{\mathfrak{p}}/\alpha M_{\mathfrak{p}})$, we have shown that $M/\alpha M$ possesses property (d).

Proceeding by induction, it thus remains to show that every M-regular element is A-regular. But this is equivalent to the following assertion, that we prove by induction on $\dim M$:

Every prime ideal associated to A is contained in a prime ideal associated to M. If $\dim M = 0$, as $\operatorname{Ass} M$ is reduced to the maximal ideal $\mathfrak{m}$ of A, this is evident.

If $\dim M > 0$, let $\mathfrak{q}$ be a prime ideal associated to A. If there exists a prime ideal $\mathfrak{p}$ in $\operatorname{Supp} M$, non-maximal containing $\mathfrak{q}$, then as $\dim M_{\mathfrak{p}} < \dim M$, and as $M_{\mathfrak{p}}$ possesses property (d) as an $A_{\mathfrak{p}}$ module, there exists a prime ideal $\mathfrak{q}' A_{\mathfrak{p}}$ associated to $M_{\mathfrak{p}}$, containing $\mathfrak{q} A_{\mathfrak{p}}$. But then $\mathfrak{q}'$ is associated to M and contains $\mathfrak{q}$.

If the only prime ideal in $\operatorname{Supp} M$ containing $\mathfrak{q}$ is $\mathfrak{m}$, then $A/\mathfrak{q} \otimes M$ is of finite length, thus $\dim A/\mathfrak{q} \leq \operatorname{pd} M$. But as $\mathfrak{q}$ is associated to A, we know that $\dim A/\mathfrak{q} \geq \operatorname{depth} A$. Then $\operatorname{depth} A \leq \operatorname{pd} M$, which implies $\operatorname{depth} M = 0$ and that $\mathfrak{m} \in \operatorname{Ass} M$, by the formula $\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} A$.

To complete the proof of the theorem, it suffices to show that if the completion $\widehat{M}$ of M is a rigid $\widehat{A}$ module (ie. possesses property (c) as an $\widehat{A}$ module) then, if N is an A-module of finite type such that $M \otimes N$ is of finite length, we have $\dim N \leq \operatorname{pd} M$.

Consider a minimal injective resolutin E of N. By chapter I 4.5, for every i, we have:

$$E^i = \prod_{\mathfrak{p} \in \operatorname{Supp} N} E(A/\mathfrak{p})^{\mu_i(\mathfrak{p},N)} \ ,$$

where, for each prime ideal $\mathfrak{p}$, the module $E(A/\mathfrak{p})$ is the injective envelope of $A/\mathfrak{p}$. As $E(A/\mathfrak{p})$ is an essential extension of $A/\mathfrak{p}$, we hvae $\operatorname{Ass} E(A/\mathfrak{p}) = \{\mathfrak{p}\}$. As $\operatorname{Supp} M \cap \operatorname{Supp} N$ is reduced to the maximal ideal $\mathfrak{m}$ of A, for every non maximal prime ideal $\mathfrak{p}$ of $\operatorname{Supp} N$, we have $\operatorname{Hom}_A(M, E(A/\mathfrak{p})) = 0$, because:

$$\operatorname{Ass} \operatorname{Hom}_A(M, E(A/\mathfrak{p})) = (\operatorname{Supp} M) \cap \{\mathfrak{p}\} = \emptyset \ .$$

From it we that deduce that for every i, we have:

$$\operatorname{Hom}_A\left(M, E^i\right) = \operatorname{Hom}_A\left(M, E(A/\mathfrak{m})^{\mu_i(\mathfrak{m},N)}\right) = \operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E^i)\right) \ .$$

That is that the complex $\operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E^i)\right)$ has for cohomology the A-modules $\mathrm{H}^i(\operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E^i)\right)) = \operatorname{Ext}^i_A(M, N)$. Consider the complex $\mathrm{H}^0_{\mathfrak{m}}(E^i)$ for which we know that it has the modules $\mathrm{H}^i_{\mathfrak{m}}(N)$ for cohomology. In every degree, we can decompose the exact sequence in the following way:

$$0 \to K^{i-1} \to \mathrm{H}^0_{\mathfrak{m}}(E^{i-1}) \to C^i \to 0 \tag{1}$$

$$0 \to K^i \to \mathrm{H}^0_{\mathfrak{m}}(E^i) \to \mathrm{H}^0_{\mathfrak{m}}(E^{i+1}) \tag{2}$$

$$0 \to C^i \to K^i \to \mathrm{H}^i_{\mathfrak{m}}(N) \ . \tag{3}$$

Set $r = \operatorname{pd} M$. For $i > r$, as $\operatorname{Ext}^i_A(M, N) = 0$, we have an exact sequence:

$$\operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E^{i-1})\right) \xrightarrow{f_{i-1}} \operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E^i)\right) \xrightarrow{f_i} \operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E^{i+1})\right) \ . \tag{*}$$

By applying functor $\operatorname{Hom}_A(M, \cdot)$ to the exact sequences (1), (2) and (3), we obtain the following exact sequences:

$$\operatorname{Hom}_A\left(M, K^{i-1}\right) \hookrightarrow \operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E^{i-1})\right) \to \operatorname{Hom}_A\left(M, C^i\right) \twoheadrightarrow \operatorname{Ext}^1_A\left(M, K^{i-1}\right) \tag{1'}$$

$$\operatorname{Hom}_A\left(M, K^i\right) \hookrightarrow \operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E^i)\right) \to \operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E^{i+1})\right) \tag{2'}$$

$$\operatorname{Hom}_A\left(M, C^i\right) \hookrightarrow \operatorname{Hom}_A\left(M, K^i\right) \to \operatorname{Hom}_A\left(M, \mathrm{H}^i_{\mathfrak{m}}(N)\right) \to \operatorname{Ext}^1_A\left(M, C^i\right) \ . \tag{3'}$$

and the isomorphisms:

$$\operatorname{Ext}^j_A\left(M, C^i\right) \simeq \operatorname{Ext}^{j+1}_A\left(M, K^{i-1}\right) \geqslant 1 \ . \tag{1''}$$

On account of (*) and (2'), we have:

$$\operatorname{Im}(f_{i-1}) = \operatorname{Hom}_A\left(M, K^i\right) \ .$$

But, by (1'), we have $\operatorname{Im}(f_{i-1}) \hookrightarrow \operatorname{Hom}_A\left(M, C^i\right)$, and by (3'):

$$\operatorname{Hom}_A\left(M, C^i\right) \hookrightarrow \operatorname{Hom}_A\left(M, K^i\right) \ .$$

Thus, $\operatorname{Im}(f_{i-1}) = \operatorname{Hom}_A\left(M, C^i\right) = \operatorname{Hom}_A\left(M, K^i\right)$.

We deduce from it $\operatorname{Ext}^1_A\left(M, K^{i-1}\right) = 0$ by (1'), and $\operatorname{Hom}_A\left(M, \mathrm{H}^i_{\mathfrak{m}}(N)\right) \hookrightarrow \operatorname{Ext}^2_A\left(M, K^{i-1}\right)$ by (3') and (1").

We will show that $\operatorname{Ext}^2_A\left(M, K^{i-1}\right)$ is zero. As $K^{i-1} \hookrightarrow \mathrm{H}^0(E^{i-1})$, the A-module K^{i-1} is artinian. Set $E = E(A/\mathfrak{m})$. Recall that covariant functor $\operatorname{Hom}_A(\cdot, E)$ establishes an anti-eqivalence between the category of artinian A-modules and the category of $\widehat{A}$-modules of finite type. Thus, for each $j \geqslant 0$, we have an isomorphism:

$$\operatorname{Hom}_{\widehat{A}}\left(\operatorname{Tor}^{\widehat{A}}_j\left(\widehat{M}, \operatorname{Hom}_{\widehat{A}}\left(K^{i-1}, E\right)\right), E\right) \simeq \operatorname{Ext}^j_{\widehat{A}}\left(\widehat{M}, K^{i-1}\right) \simeq \operatorname{Ext}^j_{\widehat{A}}\left(M, K^{i-1}\right) \ .$$

Then we have a sequence of implications:

$$\begin{aligned}\operatorname{Ext}^1_A\left(M, K^{i-1}\right) = 0 &\Rightarrow \operatorname{Tor}^{\widehat{A}}_1\left(\widehat{M}, \operatorname{Hom}_{\widehat{A}}\left(K^{i-1}, E\right)\right) = 0 \\ &\Rightarrow \operatorname{Tor}^{\widehat{A}}_2\left(\widehat{M}, \operatorname{Hom}_{\widehat{A}}\left(K^{i-1}, E\right)\right) = 0 \\ &\Rightarrow \operatorname{Ext}^2_A\left(M, K^{i-1}\right) = 0 \ .\end{aligned}$$

We deduce from it $\operatorname{Hom}_A\left(M, \mathrm{H}^i_{\mathfrak{m}}(N)\right) = 0$, which implies $\mathrm{H}^i_{\mathfrak{m}}(N) = 0$, as $\mathrm{H}^i_{\mathfrak{m}}(N) \neq 0$, entails $\operatorname{Ass}\operatorname{Hom}_A\left(M, \mathrm{H}^i_{\mathfrak{m}}(N)\right) = \{\mathfrak{m}\}$. Thus, we have proved $\mathrm{H}^i_{\mathfrak{m}}(N) = 0$ for $i > \operatorname{pd} M$. But by the theorem of local duality, $\dim N = \sup\{j \text{ such that } \mathrm{H}^i_{\mathfrak{m}}(N) \neq 0\}$. It follows that $\dim N \leqslant \operatorname{pd} M$, which is what we want to prove. □

1. The property of intersection

In this section, we intend to return to property (d) defined in 0.6, and examine some elementary cases.

1.1. Definition: Let A be a noetherian local ring. We say that a non zero A-module M of finite type and finite projective dimension possesses the property of intersection if, for each A-module N of finite type, we have $\dim N_{\mathfrak{p}} \leqslant \operatorname{pd} M_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ minimal in $\operatorname{Supp} M \cap \operatorname{Supp} N$.

1.2. *Examples.*

a) *Quotient of A by a A-regular sequence.* It is easy to see that such an A-module possesses the property (e) and *a fortiori* the property of intersection. Moreover, the Koszul complex shows that such a module also possesses property (c), and thus all the properties defined in §1.
b) *Module of finite projective dimension, of grade* 0. We know (I, (4.14)) that such an A-module has for support the spectrum of the ring. From it we deduce that it has property (e) and *a fortiori* the property of intersection.
c) *Module of projective dimension 1.* If $\operatorname{grade} M > 0$, then M admits a projective resolution of the form $0 \longrightarrow A^s \xrightarrow{f} A^s \longrightarrow M \longrightarrow 0$, from which we show that if α is the determinant of f, we have $\operatorname{Supp} M = \operatorname{Supp} A/\alpha$, thus, by a), the module M possesses all the properties in §1.

   If $\operatorname{grade} M = 0$, by b), M has the property (e); as M has evidently the property (c), in this case also, it has the properties (a), (b), (c), (d), (e) and (f).

THEOREM 1.3. *Each non zero A-module, of finite type and of projective dimension less than or equal to 2, over a notherian local ring A, possesses the property of intersection.*

PROOF. By example b), it suffices to prove the case $\operatorname{grade} M > 0$. For each prime ideal $\mathfrak{p}$ of $\operatorname{Supp} M$, the completion $\widehat{M_{\mathfrak{p}}}$ of $M_{\mathfrak{p}}$ is an $\widehat{A}_{\mathfrak{p}}$-module of projective dimension $\leqslant 2$. Thus, by 0.10, it will suffice to prove to following proposition: □

PROPOSITION 1.4. *Let A be a noetherian local ring. Each A-module M of finite type and finite projective dimension ≤ 2, and of grade > 0, is rigid (i.e. such that if N is an A-module of finite type and s an integer such that $\operatorname{Tor}_s^A(M,N) = 0$, we have $\operatorname{Tor}_j^A(M,N) = 0$ for $j \geqslant s$.)*

PROOF. Consider thus N an A-module of finite type and s an integer such that $\operatorname{Tor}_s^A(M,N) = 0$. The only uncertainity is in the case $s = 1$ and $\operatorname{pd}_A M = 2$. By induction on the dimension of M, we can suppose that $\operatorname{Tor}_2^A(M,N)$ is of finite length. The proposition will then be a consequence of the following two lemmas: □

LEMMA 1.5. *If Q is an A-module of depth > 0, and if $\operatorname{Tor}_s^A(M,Q)$ is of finite length, then $\operatorname{Tor}_s^A(M,Q) = 0$.*

PROOF. Indeed, consider a projective resolution $0 \to L_2 \to L_1 \to L_0 \to M$ of M. Then $\operatorname{Tor}_2^A(M,Q)$ is the kernel of $Q \otimes_A L_2 \to Q \otimes_A L_1$, but as Q is of depth > 0, the same holds true for $Q \otimes_A L_2$ which, thus, is not contained in a non trivial sub-module of finite length. □

LEMMA 1.6. *If Q is an A-module of finite length, we denote its length $l(Q)$. Thus for every A-module E, of finite length, we have $\sum_{i=0}^{2}(-1)^i l(\operatorname{Tor}_i^A(M,E)) = 0$.*

PROOF. Indeed, as M is of projective dimension 2, the Euler-Poincaré character function $\sum_{i=0}^{2}(-1)^i l(\operatorname{Tor}_i^A(M,\cdot))$, defined on the category of A-modules of finite length, is additive. Thus, it will suffice to show that this is zero on the residue field k. But, as $l(\operatorname{Tor}_i^A(M,k))$ is the rank of the i-th free module in a minimal free resolution of M, the alternating sum $\sum_{i=0}^{2}(-1)^i l(\operatorname{Tor}_i^A(M,k))$ is equal to the rank of M, thus to 0, as M is of grade > 0. □

NOW THE PROOF OF THE PROPOSITION. By lemma 1.5, it will suffice to prove that $\operatorname{Tor}_1^A(M,N) = 0$ implies $\operatorname{depth} N > 0$.

Suppose $N \neq 0$ and $\operatorname{depth} N = 0$. Then, there exists an exact sequence:

$$0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(N) \longrightarrow N \longrightarrow N' \longrightarrow 0 ,$$

where $\mathrm{H}^0_{\mathfrak{m}}(N)$ is the largest sub-module of finite length of N, and N' is either 0 or of depth > 0. From it we deduce an exact sequence:

$$\operatorname{Tor}_2^A\left(M, \mathrm{H}^0_{\mathfrak{m}}(N)\right) \hookrightarrow \operatorname{Tor}_2^A(M,N) \longrightarrow \operatorname{Tor}_2^A(M,N') \twoheadrightarrow \operatorname{Tor}_1^A\left(M, \mathrm{H}^0_{\mathfrak{m}}(N)\right) . \tag{*}$$

As $\operatorname{Tor}_2^A(M,N)$ and $\operatorname{Tor}_1^A\left(M, \mathrm{H}^0_{\mathfrak{m}}(N)\right)$ are both of finite length, (*) implies that $\operatorname{Tor}_2^A(M,N')$ is of finite length, thus zero according to (1.5). But this implies $\operatorname{Tor}_1^A\left(M, \mathrm{H}^0_{\mathfrak{m}}(N)\right) = 0$, by (*). But now, this last equality is not possible, as according to lemma (1.6), we have $l(\operatorname{Tor}_1^A\left(M, \mathrm{H}^0_{\mathfrak{m}}(N)\right)) - l(M \otimes_A \mathrm{H}^0_{\mathfrak{m}}(N)) \geqslant 0$. □

COROLLARY 1.7. *If M is a module of finite type and projective dimension $\leqslant 2$, over a noetherian local ring A, each M-regular sequence is A-regular.*

This is the implication (d) $\Rightarrow$ (b) in the theorem 0.10.

2. The theorem of intersection

This section is entirely devoted to the proof of the essential theorem of this chapter.

THEOREM 2.1. *Let A be a noetherian local ring. We suppose that one of the following conditions hold:*

(i) *A is of characteristic $p > 0$. (i.e. there exists an injective homomorphism of rings $\mathbb{Z}/p\mathbb{Z} \hookrightarrow A$).*
(ii) *A is essentially of finite type over a field of characteristic 0.*
(iii) *A is ind-smooth over a ring essentially of finite type over a field of characteristic 0 (i.e. there exists a local ring B essentially of finite type over a filed of characteristic 0, such that A is the inductive limit of locally-smooth B-algebras, the transition functions being local).*

Then, every non zero A-module of finite type and finite projective dimension possesses the property of intersection.

We will proceed in the following way: we will first prove the case where A verifies condition (i); from it we will deduce by successive reduction, the case where A verifies condition (ii); finally, case (iii) follows easily from (ii).

First we remark that properties (i), (ii) and (iii) are preserved by localization, that is to say that in the three cases, it will suffice to show that if M is a non zero A-module of finite type and finite projective dimension, and if N is a non-zero A-module of finite type such that $M \otimes_A N$ is of finite length, we have $\dim N \leqslant \operatorname{pd} M$.

First consider the case where A is of characteristic $p > 0$. To begin, we will proceed as in the proof of (c) $\Rightarrow$ (d) in theorem 0.10. That is that we consider a minimal injective resolution E of N. We know that for each i, we have $E^i = \prod_{\mathfrak{p} \in \operatorname{Supp} N} E(A/\mathfrak{p})^{\mu_i(\mathfrak{p},N)}$, where $E(A/\mathfrak{p})$ is the injective envelope of $A/\mathfrak{p}$. As $\operatorname{Supp} M \cap \operatorname{Supp} N$ is reduced to the maximal ideal $\mathfrak{m}$ of A, we show that for every non-maximal prime ideal $\mathfrak{p}$ of $\operatorname{Supp} N$, we have $\operatorname{Hom}_A(M, E(A/\mathfrak{p})) = 0$. We deduce from it that, for each i, one has:

$$\operatorname{Hom}_A\left(M, E^i\right) = \operatorname{Hom}_A\left(M, E(A/\mathfrak{m})^{\mu_i(\mathfrak{m},N)}\right) = \operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E^i)\right) \ .$$

Said otherwise, the complex $\operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E)\right)$ admits for cohomology the A-modules $\mathrm{H}^i(\operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E)\right)) = \operatorname{Ext}^i_A\left(M, N\right)$. This implies that for $i > \operatorname{pd}_A M$, we have exact sequences:

$$\operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E^{i-1})\right) \to \operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E^{i})\right) \to \operatorname{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E^{i+1})\right) \ . \tag{*}$$

Now we consider the complex $\mathrm{H}^0_{\mathfrak{m}}(E)$, of which the homology is

$$\mathrm{H}^i(\mathrm{H}^0_{\mathfrak{m}}(E)) = \mathrm{H}^i_{\mathfrak{m}}(N) \ .$$

Near degree i, we can decompose this complex into exact sequeces:

$$0 \longrightarrow K^{i-1} \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(E^{i-1}) \longrightarrow C^i \longrightarrow 0 \tag{4}$$

$$0 \longrightarrow K^{i} \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(E^{i}) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(E^{i+1}) \tag{5}$$

$$0 \longrightarrow C^{i} \longrightarrow K^i \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(N) \longrightarrow 0\ . \tag{6}$$

The exactness of (*) shows then that:

$$\mathrm{Coker}(\mathrm{Hom}_A\left(M, K^{i-1}\right) \longrightarrow \mathrm{Hom}_A\left(M, \mathrm{H}^0_{\mathfrak{m}}(E^{i-1})\right)) = \mathrm{Hom}_A\left(M, K^i\right)\ ,$$

and, *a fortiori*, that:

$$\mathrm{Hom}_A\left(M, C^i\right) = \mathrm{Hom}_A\left(M, K^i\right)\ . \tag{**}$$

This is the last fact that we are going to use to prove the theorem. We consider an exact sequence $0 \longrightarrow \Omega \longrightarrow L \longrightarrow M \longrightarrow 0$, where L is a free A-module of finite type, and $\Omega \hookrightarrow \mathfrak{m}L$. As A is of characteristic $p > 0$, we have seen in chapter I, that we can construct a sequence of sub-modules Ω_j of L such that:

(1) $\Omega_0 = \Omega$;
(2) for each j, the A-module L/Ω_j has the same support and same projective dimension as $L/\Omega = M$;
(3) $\Omega_j \subset \mathfrak{m}^{p^j} L$ for each j.

By property 2) of the modules Ω_j, we see, as for M, that for each j we have:

$$\mathrm{Hom}_A\left(L/\Omega_j, C^i\right) = \mathrm{Hom}_A\left(L/\Omega_j, K^i\right) \quad \text{for } i > \mathrm{pd}_A M = \mathrm{pd}_A L/\Omega_j\ . \tag{**j}$$

By property 3) of the modules Ω_j, we deduce from it, for each j, the equalities:

$$\mathrm{Hom}_A\left(L/\mathfrak{m}^{p^j} L, C^i\right) = \mathrm{Hom}_A\left(L/\mathfrak{m}^{p^j} L, K^i\right)\ . \tag{***}$$

But the modules C^i and K^i having support in $\mathrm{V}(\mathfrak{m})$, each element of C^i or K^i is annihilated by a power of $\mathfrak{m}$. Said otherwise,

$$\mathrm{Hom}_A\left(L, C^i\right) = \varinjlim \mathrm{Hom}_A\left(L/\mathfrak{m}^{p^n}, C^i\right)$$

and

$$\mathrm{Hom}_A\left(L, K^i\right) = \varinjlim \mathrm{Hom}_A\left(L/\mathfrak{m}^{p^n}, K^i\right)\ .$$

By (***), this implies $\mathrm{Hom}_A\left(L, C^i\right) = \mathrm{Hom}_A\left(L, K^i\right)$, thus $C^i = K^i$, and by the exact sequence (3), $\mathrm{H}^i_{\mathfrak{m}}(N) = 0$. Thus we have shown that $\mathrm{H}^i_{\mathfrak{m}}(N) = 0$ for $i > \mathrm{pd}_A M$. As by the theorem of local duality $\dim N = \sup\{s \text{ such that } \mathrm{H}^i_{\mathfrak{m}}(N) \neq 0\}$, it follows that $\dim N \leq \mathrm{pd}\, M$, what we were wanting to prove.

Now we consider the case where A is essentially of finite type over a field of characteristic 0. Thus let M be a non zero module of finite type and finite projective dimension, and let N be a non zero module of finite type such that $M \otimes_A N$ is of finite length. We want to construct a noetherian local ring $\overline{A}$ of characteristic $p > 0$, a non zero $\overline{A}$ module $\overline{M}$ of finite type and projective dimension equal to $\mathrm{pd}_A M$, and a non zero $\overline{A}$-module $\overline{N}$ of finite type, such that $\dim_{\overline{A}} \overline{N} \geqslant \dim_A N$, and that $\overline{M} \otimes_{\overline{A}} \overline{N}$ is of finite length. Then we will be able to conclude by utilizing the result already proved in characteristic $p > 0$.

LEMMA 2.2. *Let M be a non zero module of finite type and finite projective dimension r, over a noetherian local ring A essentially of finite type over a field of characteristic* 0. *Let N be a non-zero A-module of finite type such that* $\operatorname{Supp} M \cap \operatorname{Supp} N$ *is reduced to the maximal ideal of A. Then, there exists a k-algebra of finite type A', a prime ideal $\mathfrak{p}$ of A', and A'-modules of finite type M' and N' having the following properties:*

(i) $A'_{\mathfrak{p}} = A,\ M'_{\mathfrak{p}} = M,\ N'_{\mathfrak{p}} = N.$
(ii) *M' is an A'-module of finite projective dimension r, admitting a resolution of length r, by free A'-modules of finite type.*
(iii) *the prime ideal $\mathfrak{p}$ belongs to all the irreducible components of* $\operatorname{Supp} N'$, *and this is the unique minimal prime ideal of* $\operatorname{Supp} M' \cap \operatorname{Supp} N'$.

There exists a k-algebra of finite type B, and a prime ideal $\mathfrak{m}$ of B such that $A = B_{\mathfrak{m}}$. Let X and Y be two B-modules of finite type such that $X_{\mathfrak{m}} = M$ and $Y_{\mathfrak{m}} = N$. We consider a resolution of the B-module X, by free B-modules of finite type. Let Ω be the r-th syzygy of X given by this resolution. As $X_{\mathfrak{m}}$ is of projective dimension r over $B_{\mathfrak{m}}$, the $B_{\mathfrak{m}}$-module $\Omega_{\mathfrak{m}}$ is free, thus there an element s in $B - \mathfrak{m}$ such that Ω_s is free over B_s. On the other side, consider the minimal prime ideals $\mathfrak{q}_i$ $(i = 1, \ldots, n)$ of the support of the B-module Y. By arranging conveniently, we may can assume that $\mathfrak{q}_i \subset \mathfrak{m}$ for $i \leqslant l$, and $\mathfrak{q}_i \not\subset \mathfrak{m}$ for $i > l$. By the avoidance lemma, we can find an element $t \in \cap_{i>l}\mathfrak{q}_i$, such that $t \notin \mathfrak{m}$. Finally, let $\mathfrak{p}_i$ $(i = 1, \ldots, n')$ be the minimal prime ideals in $\operatorname{Supp} X \cap \operatorname{Supp} Y$, with $\mathfrak{m} = \mathfrak{p}_1$. Again by the avoidance lemma, there exists an element $u \in \cap_{i>1}\mathfrak{p}_i$ with $u \notin \mathfrak{m}$. We verify easily that

$$A' = B_{stu}, \quad \mathfrak{p} = \mathfrak{m}A' \quad M' = X_{stu} \quad \text{and} \quad N' = Y_{stu}$$

have the required properties, first by the construction, the second by the choice of s, and the third by the choice of t and u.

LEMMA 2.3. *Let A' be an algebra of finite type over a field k, and let $\mathfrak{p}$ be a prime ideal of A', and M' and N' finite type modules over A' having the following properties:*

M' is a A'-module of projective dimension r, admitting a length r resolution by free A'-modules of finite rank.

The prime ideal $\mathfrak{p}$ appears in all the irreducible components of $\operatorname{Supp} N'$ *and is the unique minimal prime ideal of* $\operatorname{Supp} M' \cap \operatorname{Supp} N'$.

Then there exists a subfield $\overline{k}$ of k, which is an extension of finite type of its prime field, a $\overline{k}$-algebra of finite type $\overline{A'}$ and $\overline{A'}$-modules of finite type $\overline{M}'$ and $\overline{N}'$ having the following properties:

(i) $A' = k \otimes_{\overline{k}} \overline{A'}$, $M' = \overline{M}' \otimes_{\overline{A'}} A'$, $N' = \overline{N}' \otimes_{\overline{A'}} A'$.
(ii) *$\overline{M}'$ is a $\overline{A'}$-module of projective dimension r, admitting a length r resolution by free $\overline{A'}$-modules of finite rank.*
(iii) *If $\overline{\mathfrak{p}} = \mathfrak{p} \cap \overline{A'}$, then $\overline{\mathfrak{p}}$ is the unique minimal prime ideal of* $\operatorname{Supp} \overline{M'} \cap \operatorname{Supp} \overline{N'}$, *the $\overline{A'}_{\overline{\mathfrak{p}}}$-module $\overline{M'}_{\overline{\mathfrak{p}}}$ is of projective dimension r, finally* $\dim \overline{N'}_{\overline{\mathfrak{p}}} \geqslant \dim N'_{\mathfrak{p}}$

We consider a free resolution of length r of the A'-module M', by free modules of finite type, and a finite presentation of the A'-module N', as:

$$0 \to L_r \xrightarrow{\phi_r} L_{r-1} \to \cdots \xrightarrow{\phi_1} L_0 \to M' \to 0 \quad \text{and} \quad F_1 \xrightarrow{\Psi} F_0 \to N' \to 0\ .$$

We choose bases for each L_i and each F_i and represent the homomorphisms ϕ_i and ψ by matrices $(\phi_i^{p,q})$ and $(\psi^{p,q})$. We know that A' is of the form $k[X_1,\dots,X_n]/I$. Take polynomials $\alpha_i^{p,q}(X)$ and $\beta^{p,q}(X)$ in $k[X_1,\dots,X_n]$ such that their images in A' are $\phi_i^{p,q}$ and $\psi^{p,q}$. Take polynomials $a_s(X)$, in $k[X_1,\dots,X_n]$, forming a finite system of generators for I. Finally, choose a family of polynomials $b_t(X)$ such that their images in A' generate the prime ideal $\mathfrak{p}$.

It is possible to find a field $\overline{k}$ contained in k, which is an extension of finite type of the prime field of k, and which contains the coefficients of polynomials $\alpha_i^{p,q}(X)$, $\beta^{p,q}(X)$, $a_s(X)$ and $b_t(X)$. Then we consider the ring $\overline{A'}=\overline{k}[X_1,\dots,X_n]/\overline{I}$, where $\overline{I}$ is the ideal generated by the polynomials $a_s(X)$ in $\overline{k}[X_1,\dots,X_n]$. Evidently, we have $A'=k\otimes_{\overline{k}}\overline{A'}$, thus A' is a faithfully flat $\overline{A'}$-algebra. For each i,p,q, let $\overline{\phi}_i^{\,p,q}$ be the image of $\alpha_i^{p,q}(X)$ in $\overline{A'}$. The coefficients $\overline{\phi}_i^{\,p,q}$ form matrices $\overline{\phi}_i$ such that $A'\otimes_{\overline{A'}}\overline{\phi}_i=\phi_i$. As A' is faithfully flat over $\overline{A'}$, the exact complex (L,ϕ) descends to an exact complex of free $\overline{A'}$-modules:

$$0\longrightarrow \overline{L}_r \xrightarrow{\overline{\phi}_r} \overline{L}_{r-1}\longrightarrow\cdots\xrightarrow{\overline{\phi}_1}\overline{L}_0\ .$$

Let $\overline{M'}$ be the cokernel of $\overline{\phi}_1$. We have $\overline{M'}\otimes_{\overline{A'}}A'=M'$, and $\overline{M'}$ admits a free resolution of length r by $\overline{A'}$-modules of finite type. We construct in the same way an $\overline{A'}$-module of finite type $\overline{N'}$ such that $\overline{N'}\otimes_{\overline{A'}}A'=N'$. Thus we have proved (i) and (ii).

We will show that (iii) is verified. Let $\overline{\mathfrak{q}}$ be a prime ideal of $\overline{A'}$ such that

$$\overline{\mathfrak{q}}\in\operatorname{Supp}\overline{M'}\cap\operatorname{Supp}\overline{N'}\ .$$

As A' is faithfully flat over $\overline{A'}$, there exists a prime ideal $\mathfrak{q}$ of A' lying over $\overline{A'}$. But then, $A'_{\mathfrak{q}}$ is faithfully flat over $\overline{A'}_{\overline{\mathfrak{q}}}$, thus $\mathfrak{q}\in\operatorname{Supp}M'\cap\operatorname{Supp}N'$, and by hypotheses (2), $\mathfrak{q}\supset\mathfrak{p}$, which implies $\overline{\mathfrak{q}}\supset\overline{\mathfrak{p}}$.

As $A'_{\mathfrak{q}}$ is faithfully flat over $\overline{A'}_{\overline{\mathfrak{q}}}$, we have $\operatorname{pd}_{\overline{A'}_{\overline{\mathfrak{p}}}}\overline{M'}_{\overline{\mathfrak{p}}}=\operatorname{pd}_{A'_{\mathfrak{p}}}M'_{\mathfrak{p}}=r$.

Finally, we have to prove $\dim\overline{N'}_{\overline{\mathfrak{p}}}\geqslant\dim N'_{\mathfrak{p}}$. Therefore, let d be the dimension of $\overline{N'}_{\overline{\mathfrak{p}}}$, and let $x=(x_1,\dots,x_d)$ a sequence of elements of $\overline{\mathfrak{p}}\overline{A'}_{\overline{\mathfrak{p}}}$ such that $\overline{N'}_{\overline{\mathfrak{p}}}/x\overline{N'}_{\overline{\mathfrak{p}}}$ is of finite length. To show that $\dim N'_{\mathfrak{p}}\leqslant d$ it suffices to show that $N'_{\mathfrak{p}}/xN'_{\mathfrak{p}}$ is of finite length. But, as $N'_{\mathfrak{p}}/xN'_{\mathfrak{p}}=A'\otimes_{\overline{A'}}\overline{N'}_{\overline{\mathfrak{p}}}/x\overline{N'}_{\overline{\mathfrak{p}}}$, we know [**7**, chap. IV,§2, th.2] that

$$\operatorname{Supp}N'_{\mathfrak{p}}/xN'_{\mathfrak{p}}=\bigcup_{\overline{\mathfrak{q}}\in\operatorname{Supp}\overline{N'}_{\overline{\mathfrak{p}}}/x\overline{N'}_{\overline{\mathfrak{p}}}}\operatorname{Supp}A'_{\mathfrak{p}}/\overline{\mathfrak{q}}A'_{\mathfrak{p}}=\operatorname{Supp}A'_{\mathfrak{p}}/\overline{\mathfrak{p}}A'_{\mathfrak{p}}\ .$$

But, we have constructed $\overline{A'}$ and $\overline{\mathfrak{p}}$ in a way that $\overline{\mathfrak{p}}A'=\mathfrak{p}$. *A fortiori*, we will have $\dim A'_{\mathfrak{p}}/\overline{\mathfrak{p}}A'_{\mathfrak{p}}=0$, thus the lemma is proved.

LEMMA 2.4. *Let R be a dedekind ring having infinitely many maximal ideals. Let B an R-algebra of finite type. Let V be a B-module of finite type and projective dimension r. Let T be a B-module of finite type. Suppose that there exists a prime ideal $\mathfrak{p}$ of B having the following properties:*

(i) $\mathfrak{p}\cap R=0$.
(ii) $\mathfrak{p}$ *is the only minimal prime ideal in* $\operatorname{Supp}V\cap\operatorname{Supp}T$*, and* $\mathfrak{p}$ *is contained in every irreducible component of* $\operatorname{Supp}T$.
(iii) $V_{\mathfrak{p}}$ *is a* $B_{\mathfrak{p}}$*-module of projective dimension* r.

Then there exists a maximal ideal $\mathfrak{m}$ of R, and an element $\alpha\in\mathfrak{m}$, which is a uniformiser of $R_{\mathfrak{m}}$, having the following properties:

(1) *α is not invertible in $B/\mathfrak{p}$.*
(2) *α is regular in B, V and T.*
(3) *If $\mathfrak{q}$ is a prime ideal of B, minimal among those containing $\mathfrak{p}+\alpha B$, then:*
   (a) *$V_\mathfrak{q}/\alpha V_\mathfrak{q}$ is a $B_\mathfrak{q}/\alpha B_\mathfrak{q}$ module of projective dimension r;*
   (b) *$\dim T_\mathfrak{q}/\alpha T_\mathfrak{q} = \dim T_\mathfrak{p}$;*
   (c) *$\operatorname{Supp} V_\mathfrak{q}/\alpha V_\mathfrak{q} \cap \operatorname{Supp} T_\mathfrak{q}/\alpha T_\mathfrak{q}$ is reduced to the ideal $\mathfrak{q}B_\mathfrak{q}/\alpha B_\mathfrak{q}$.*

As $\mathfrak{p}\cap R = 0$, the ring $B/\mathfrak{p}$ is an R-algebra of finite type, which contains R. The normalization lemma says that there exists an element s of R, such that $(B/\mathfrak{p})_s$ is integral over an polynomial algebra over R_s. That proves that all the maximal ideals of R, other than a finite number, lift themselves in $B/\mathfrak{p}$. Otherwise said, for each maximal ideal $\mathfrak{a}$ of R, other than a finite number, we have $\mathfrak{a}(B/\mathfrak{p}) \neq B/\mathfrak{p}$. As the prime ideals of B associated to B, V or T are finite in number, we see that we can find a maximal ideal $\mathfrak{m}$ of R, such that $\mathfrak{m}(B/\mathfrak{p}) \neq B/\mathfrak{p}$, and such that $\mathfrak{m}$ is not contained in the union of the prime ideals appearing in $\operatorname{Ass} B \cup \operatorname{Ass} V \cup \operatorname{Ass} T$. Thus, if $\alpha \in R$ is a uniformiser of $B_\mathfrak{m}$, we can choose $\mathfrak{m}$ of the kind that α is regular in B, V and T, and that α is not invertible in $B/\mathfrak{p}$. Let $\mathfrak{q}$ be a prime ideal of B, minimal among those containing $\mathfrak{p} + \alpha B$. As $\mathfrak{p} \subset \mathfrak{q}$, the $B_\mathfrak{q}$-module $V_\mathfrak{q}$ is of projective dimension r. As α is $B_\mathfrak{q}$ regular and $V_\mathfrak{q}$ regular, that implies that $V_\mathfrak{q}/\alpha V_\mathfrak{q}$ has projective dimension r as a $B_\mathfrak{q}/\alpha B_\mathfrak{q}$-module. As minimal prime ideal of $\operatorname{Supp} T$ is also contained in $\mathfrak{p}$, and as the ring B is catenary, we have $\dim T_\mathfrak{q} = \dim T_\mathfrak{p} + \dim B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q} = \dim T_\mathfrak{p} + 1$, thus $\dim T_\mathfrak{q}/\alpha T_\mathfrak{q} = \dim T_\mathfrak{p}$. Finally, let $\mathfrak{q}'B_\mathfrak{q}$ be a prime ideal in $B_\mathfrak{q}$ appearing in

$$\operatorname{Supp}\left(V_\mathfrak{q}/\alpha V_\mathfrak{q}\right) \cap \operatorname{Supp}\left(T_\mathfrak{q}/\alpha T_\mathfrak{q}\right) .$$

Then $\mathfrak{q}'B_\mathfrak{q} \in \operatorname{Supp} V_\mathfrak{q} \cap \operatorname{Supp} T_\mathfrak{q}$, thus $\mathfrak{q}'B_\mathfrak{q} \supset \mathfrak{p}B_\mathfrak{q}$. But $\alpha \in \mathfrak{q}'B_\mathfrak{q}$, thus $\alpha B_\mathfrak{q} + \mathfrak{p}B_\mathfrak{q} \subset \mathfrak{q}'B_\mathfrak{q}$, thus, as $\mathfrak{q}$ has been chosen minimal among the prime ideals containing $\mathfrak{p}+\alpha B$, we can only have $\mathfrak{q}'B_\mathfrak{q} = \mathfrak{q}B_\mathfrak{q}$, which well proves:

$$\operatorname{Supp} V_\mathfrak{q}/\alpha V_\mathfrak{q} \cap \operatorname{Supp} T_\mathfrak{q}/\alpha T_\mathfrak{q} = \{\mathfrak{q}B_\mathfrak{q}/\alpha B_\mathfrak{q}\} ,$$

and the lemma is proved.

Utilizing first lemma 2.2, then lemma 2.3, and localizing the situation at the prime ideal $\overline{\mathfrak{p}}$, finally delocalizing it carefully and applying anew 2.2, we find that we are back to the following problem:

Let k be a field which is an extension of finite type of $\mathbb{Q}$. Let B' be an algebra of finite type over k. Let V' and T' two B'-modules of finite type, and $\mathfrak{p}'$ a prime ideal of B', such that:

(1) V' is of projective dimension r over B'.
(2) $V'_{\mathfrak{p}'}$ is a $B'_{\mathfrak{p}'}$-module of projective dimension r.
(3) $\mathfrak{p}'$ is the only minimal prime ideal in $\operatorname{Supp} V' \cap \operatorname{Supp} T'$.
(4) $\mathfrak{p}'$ is contained in every irreducible component of $\operatorname{Supp} T'$.

To show that there exists a noetherian local ring $\overline{A}$, of characterstic $p > 0$, a $\overline{A}$-module $\overline{M}$ of finite type and projective dimension r, and a $\overline{A}$-module $\overline{M}$ of finite type $\overline{N}$ such that $\dim_{\overline{A}} \overline{N} = \dim_{B'_{\mathfrak{p}'}} T'_{\mathfrak{p}'}$, and that $\overline{M} \otimes_{\overline{A}} \overline{N}$ be of finite length.

Let n be the transendence degree of k over $\mathbb{Q}$. There exist elements $X_1, \ldots, X_n$ of k forming a transendence bases of k over $\mathbb{Q}$, that is such that k is finite over $\mathbb{Q}(X_1, \ldots, X_n)$.

Consider the integral closure D in k of:

a) $\mathbb{Z}$ if $n = 0$;

b) $\mathbb{Q}(X_1, \ldots, X_{n-1})[X_n]$ if $n > 0$.

Then D is a Dedekind domain having infintely many maximal ideals. Proceeding directly, as in the proof of 2.2, we see that we can find an element s of D, a D-algebra of finite type B, and B-modules of finite type V and T such that, if $S = D_s - \{0\}$, we will have:

1) $S^{-1}B = B'$, $S^{-1}V = V'$, $S^{-1}T = T'$.

2) V is a B-module of projective dimension r.

3) If $\mathfrak{p} = \mathfrak{p}' \cap B$, then $\mathfrak{p}$ is the only minimal prime ideal in $\operatorname{Supp} V \cap \operatorname{Supp} T$, and $\mathfrak{p}$ is contained in every irreducible component of $\operatorname{Supp} T$.

Set $R = D_s$. Then R is a Dedekind domain having infintely many maximal ideals. As $\mathfrak{p} \cap R = 0$, all the conditions required for lemma 2.4 exist. Thus, $B_{\mathfrak{q}}/\alpha B_{\mathfrak{q}}$ is an algebra essentially of finite type over the field $R_{\mathfrak{m}}/\alpha R_{\mathfrak{m}}$. Then, there are two cases:

a) If $n = 0$, then $R_{\mathfrak{m}}/\alpha R_{\mathfrak{m}}$ is a field of characteristic $p > 0$, and we are at the end of our difficulties.

b) $n > 0$, then $R_{\mathfrak{m}}/\alpha R_{\mathfrak{m}}$is an extension of finite type over $\mathbb{Q}$, of transcendence degree $n-1$ over $\mathbb{Q}$. We then delocalize carefully, thanks to lemma 2.2, and we repeat the operation $n-1$ times, in the same way to return to the preceeding case, which we have resolved.

Now we consider the case where A verifies condition (iii). That is that there is a filtered inductive system of local rings B_α such that:

1) $A = \varinjlim B_\alpha$.

2) For each α, the local ring B_α is essentially of finite type over a field.

3) Each transition map $B_\alpha \longrightarrow B_{\alpha'}$ is local, in fact $B_{\alpha'}$ is a locally smooth B_α-algebra. (ie. a B_α-algebra obtained by localization of an smooth extension of B_α)

Thus, let M be an A-module of finite type and finite projective dimension, an let N an A-module of finite ype, and such that $M \otimes_A N$ is of finite length. We want to show that $\dim_A N \leqslant \operatorname{pd} M$. We change nothing in the problem by replacing N by A/I, where I is the annihilator of N. We consider a minimal free resolution of M, with:

$$0 \longrightarrow A^{\eta_r} \xrightarrow{\phi_r} A^{\eta_{r-1}} \longrightarrow \cdots \xrightarrow{\phi_1} A^{\eta_0} \longrightarrow M \longrightarrow 0 \ .$$

For $i = 1, \ldots, r$ we represent the homomorphisms ϕ_i by matrices $(\phi_i^{p,q})$, and choose a system of generators a_j of the ideal I.

We can find α such that the image of B_α in A contains the coefficients of the matrices ϕ_i $(i = 1, \ldots, r)$, and the elements a_j. As the coefficients $\phi_i^{p,q}$ of the matrices ϕ_i are in B_α, the maps $B_\alpha^{\eta_i} \xrightarrow{\phi_i} B_\alpha^{\eta_{i-1}}$ are defined, and as A is faithfully flat over B_α, the existence and exactness of the complex (A^{η_i}, ϕ) implies that:

$$0 \longrightarrow B_\alpha^{\eta_r} \xrightarrow{\phi_r} B_\alpha^{\eta_{r-1}} \longrightarrow \cdots \xrightarrow{\phi_1} B_\alpha^{\eta_0} \ ,$$

is an exact complex of free B_α-modules.

Let $M_\alpha = \operatorname{Coker}(B_\alpha^{\eta_1} \xrightarrow{\phi_1} B_\alpha^{\eta_0})$. Then M_α is a B_α-module of finite type and projective dimension equal to $\operatorname{pd} M$ such that $M_\alpha \otimes_{B_\alpha} A = M$. Let I_α be the ideal generated by the a_j in B_α. We consider $M_\alpha \otimes_{B_\alpha} (B_\alpha/I_\alpha)$. Then, as

$$M_\alpha \otimes_{B_\alpha} (B_\alpha/I_\alpha) \otimes_{B_\alpha} A = M \otimes_A (A/I) \ ,$$

and as A is faithfully flat over B_α, we conclude that $M_\alpha \otimes_{B_\alpha} (B_\alpha/I_\alpha)$ is an B_α-module of finite length. As B_α is essentially of finite type over a field, we deduce from it $\dim B_\alpha/I_\alpha \leqslant \mathrm{pd}_{B_\alpha} M_\alpha = \mathrm{pd}\, M$. To prove the theorem, it suffices to show $\dim B_\alpha/I_\alpha = \dim A/I$. But, as A/I is faithfully flat over B_α/I_α, we obviously have $\dim B_\alpha/I_\alpha \leqslant \dim A/I$. On the other hand, as A/I is a localization of an ring integral over B_α/I_α, we have $\dim B_\alpha/I_\alpha \geqslant \dim A/I$, by the theorem of Cohen-Seidenberg.

Corollary 2.5. *Let X be a scheme locally of finite type over a field. Let $\mathcal{F}$ an $\mathcal{O}_X$-module, locally of finite projective dimension. Let Y be a irreducible closed sub-scheme of X. We assume that* $\mathrm{Supp}\,\mathcal{F} \cap Y$ *is not empty. Then, if C is a irreducible component of* $\mathrm{Supp}\,\mathcal{F} \cap Y$*, we have* $\dim Y - \dim C \leqslant \inf_{x\in C} \mathrm{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x$.

This corollary is the geometric form of theorem (3.1).

3. Proof of a conjecture of M. Auslander

This section essentially boils down to the following statement:

Theorem 3.1. *Let A be a noetherian local ring. Supposing one of the following conditions holds:*

(i) A is of characteristic $p > 0$.

(ii) A is ind-smooth over a ring essentially of finite type over a field.

Then, if $M \neq 0$ is an A-module of finite type of finite projective dimension, each M-regular sequence is an A-regular sequence.

On account of the theorem of intersection, this is a consequence of the implication (d) $\Rightarrow$ (b) of the theorem 0.10.

Remark. Let A be a ring, which is the completion of a local ring, essentially of finite type over a field of characteristic 0. Then A is also the completion of a excellent henselian ring B, for which the theorem of intersection is true. The canonical morphism $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ defines a bijection between $\mathrm{Ass}\, A$ and $\mathrm{Ass}\, B$. Therefore, let $\mathfrak{p}$ be a prime ideal associated to A, and M an A-module of finite type and finite projective dimension such that $M/\mathfrak{p}M$ is of finite length. We set $\mathfrak{q} = \mathfrak{p} \cap B$. By utilizing the theorem of approximation for modules of finite projective dimension, we can construct a B-module of finite type N, of projective dimension equal to $\mathrm{pd}_A M$, such that $N/\mathfrak{q}N$ is of finite length. By the theorem of intersection for B, we deduce from it that N is of depth 0, and thus that M is of depth 0.

By utilizing this remark and the technique of lemma 5.2, as below, we prove the following result, by induction on $\dim M$:

Proposition 3.2. *Let B be a local ring essentially of finite type over a field, and A its completion. Then, if M is an A-module of finite type and finite projective dimension, every M-regular element is A-regular.*

We remark that in the more general case the difficulty in generalizing to regular sequences arises from that fact that if x is an A-regular element, then A/xA is no longer the completion of a ring essentially of finite type over a field. Nevertheless, (4.2) is sufficient to prove the following result due to M. Auslander.

Corollary 3.3. *Let A be a noetherian local ring. Supposing one of the following conditions is verified:*

(i) *A is of characteristic $p > 0$.*

(ii) *The completion of A is isomorphic to the completion of a ring essentially of finite type over a field.*

Then, in order that A to be a domain, it is necessary and sufficient that there exists a prime ideal $\mathfrak{p}$ of A which has finite projective dimension over A.

The condition is obviously necessary, as (0) is prime ideal, if A is a domain.

Reciprocally, let $\mathfrak{p}$ be a prime ideal such that $A/\mathfrak{p}$ has finite projective dimension. Then, by (4.1) or (4.2), every element of $A-\mathfrak{p}$ is A-regular. That implies that the canonical map $A \longrightarrow A_{\mathfrak{p}}$ is an injection. But, as $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is of finite projective dimension, $A_{\mathfrak{p}}$ is regular, thus a domain. Then the injection $A \hookrightarrow A_{\mathfrak{p}}$ shows that A is a domain.

4. Perfect modules

We recall the following terminology which generalizes the notion of perfect ideals due to Macaulay.

4.1. Let A be a noetherian local ring. We will say that an A-module M of finite type is perfect if it is of finite projection dimension, and if its projective dimension is equal to its grade. (i.e. $\operatorname{Ext}^i_A(M, A) = 0$ for $i \neq \operatorname{pd} M$).

Examples.

1) If $\alpha = (\alpha_1, \dots, \alpha_s)$ is an A-regular sequence, $A/\alpha A$ is perfect A-module.
2) If M is an A-module of finite length and finite projective dimension, it is a perfect A-module.
3) If M is a Cohen-Macaulay A-module of finite projective dimension, it is a perfect A-module, because if α is M-regular and A-regular, M is perfect if and only if $M/\alpha M$ is perfect.
4) If M is a perfect A-module and $\mathfrak{p}$ is a prime ideal in the support of M, then $M_{\mathfrak{p}}$ is a perfect $A_{\mathfrak{p}}$-module.

From the very definition of perfect modules we derive the following theorem which shows that over a local ring for which the theorem of intersection is true, a perfect module verifies the conjectures (a), (b), (d), (e) and (f).

THEOREM 4.2. *Let A be a local ring for which the theorem of intersection is true. Let M be a perfect A-module. Then,*

1) *Each system of parameters for M extends to a system of parameters for A.*
2) *Each M-regular sequence is a A-regular sequence.*
3) $\operatorname{grade} M + \dim M = \dim A$.
4) *If N is an A-module of finite type, and $\mathfrak{p}$ is a prime ideal minimal in* $\operatorname{Supp} N \cap \operatorname{Supp} M$*, then* $\dim N_{\mathfrak{p}} + \dim M_{\mathfrak{p}} \leqslant \dim A_{\mathfrak{p}}$.

The theorem is an immediate consequence of 0.10, taking into account the fact that the theorem of intersection is true for A, and the fact that for each prime ideal $\mathfrak{p}$ in $\operatorname{Supp} M$, we have $\operatorname{grade}_{A_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{pd}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$.

Finally, we note the following fact which links the regularity of a module of finite projective dimension to that of the ring.

PROPOSITION 4.3. *Let A be a noetherian local ring for which the theorem of intersection is true. Let M be a finite A-module of finite projective dimension. For each prime ideal $\mathfrak{p}$ of* $\operatorname{Supp} M$*, if $M_{\mathfrak{p}}$ is C.M. then the same holds for $A_{\mathfrak{p}}$.*

Indeed, if $\alpha = (\alpha_1, \ldots, \alpha_s)$ is an M-regular sequence of maximal length, $M_\mathfrak{p}/\alpha M_\mathfrak{p}$ is a finite length $A_\mathfrak{p}$ module of finite projective dimension. The theorem of intersection says that $\dim A_\mathfrak{p} \leqslant \operatorname{pd}_{A_\mathfrak{p}} M_\mathfrak{p}/\alpha M_\mathfrak{p}$. But, as each module of finite projective dimension, has projective dimension less than or equal to the depth of the ring, we have $\dim A_\mathfrak{p} \leqslant \operatorname{depth} A_\mathfrak{p}$, which implies that $\dim A_\mathfrak{p} = \operatorname{depth} A_\mathfrak{p}$.

Finally, we give a homological characterization of system of parameters, not necessarily maximal, which are regular sequence:

PROPOSITION 4.4. *Let A be a noetherian local ring, $(x_1, \ldots, x_s)$ a system of parameters for A, i.e.* $\dim A/(x_1, \ldots, x_s) = \dim A - s$*; then a necessary and sufficient condition that $(x_1, \ldots, x_s)$ is a A-regular sequence is that* $\operatorname{pd}_A A/(x_1, \ldots, x_s) < \infty$.

The necessity is well known. To prove the sufficiency, we are goint to show that the Koszul complex $K.(x_1, \ldots, x_s; A)$ is exact. This complex is equal to:

$$0 \to \wedge^s A^s \to \wedge^{s-1} A^s \to \cdots ` \to \wedge^2 A^s \to A^s \to A. \tag{*}$$

(i) We are going to prove that $(*)_\mathfrak{p}$ is exact if $\operatorname{depth} A_\mathfrak{p} < s$, where $\mathfrak{p}$ is a prime ideal of A.

Indeed, if $\mathfrak{p} \not\supset (x_1, \ldots, x_s)$ we know that $(*)_\mathfrak{p}$ splits. On the other hand, we are going to show that $\operatorname{pd}_{A_\mathfrak{p}} (A/x_1, \ldots, x_s)_\mathfrak{p} \geqslant s$ for each $\mathfrak{p} \supset (x_1, \ldots, x_s)$. It will suffice to prove for $\mathfrak{p}$ minimal containing $(x_1, \ldots, x_s)$. Now, for such a $\mathfrak{p}$, $(A/(x_1, \ldots, x_s))_\mathfrak{p}$ is a $A_\mathfrak{p}$-module of finite length, thus, by (5.3), $A_\mathfrak{p}$ is a C.M. ring, thus the system of parameters $x_1, \ldots, x_s$ of $A_\mathfrak{p}$ is an $A_\mathfrak{p}$-regular sequence, thus $(*)_\mathfrak{p}$ is a minimal resolution of $(A/x_1, \ldots, x_s)_\mathfrak{p}$.

(ii) Now we will show that $(*)$ is exact. Let $\mathfrak{p}$ be a prime ideal minimal in $\cup_{i=1}^s \operatorname{Supp} \mathrm{H}_i(x_1, \ldots, x_s; A)$. Then $\mathfrak{p} \supset (x_1, \ldots, x_s)$, thus $\operatorname{depth} A_\mathfrak{p} \geqslant s$, thus $\operatorname{pd}_{A_\mathfrak{p}} (A/(x_1, \ldots, x_s))_\mathfrak{p} \geqslant s$.

Applying the acylicity lemma 1.8 to the complex $(*)_\mathfrak{p}$ of which the homology in positive degree is of finite length, we find that $\mathrm{H}_i(x_1, \ldots, x_s; A)_\mathfrak{p} = 0$ for $i = 1, \ldots, x$, which is a contradiction. Thus $\operatorname{Supp} \mathrm{H}_i(x_1, \ldots, x_s; A) = \emptyset$

5. Proof of a conjecture of H. Bass

The theorem which follows proves in the "almost" general case a result conjectured by H. Bass in [**6**].

THEOREM 5.1. *Let A be a noetherian local ring. Supposing one of the following conditions is verified:*

(i) *A is of characteristic $p > 0$.*
(ii) *A is essentially of finite type over a field of characteristic 0.*
(iii) *A is ind-smooth over a ring essentially of finite type over a field of characteristic 0.*
(iv) *The completion of A is isomorphic to the completion of a ring essentially of finite type over a field of characteristic 0.*

Then, if there exists a non zero A-module T, of finite type and finite injective dimension, A is a C.M. ring.

Remark. It is clear that for each C.M. local ring A, there is an A-module of finite type and finite injective dimension. Indeed, let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a system of parameters of A. This is an A-regular sequence, thus A/α is an A-module of finite length and finite projective dimension. Then, if E is the injective envelope of the residue field of A (i.e. a dualizing module for A) $\mathrm{Hom}_A(A/\alpha, E)$ is an A-module of finite length (thus of finite type) and finite injective dimension.

First, we recall some very simple facts that we will use in the course of the proof.

(*) In order that a local ring is C.M., it is necessary and sufficient that its completion is C.M.

(**) In order that a module T (resp. M) of finte type is of finite injective dimension (resp. of finite projective dimension) over a a noetherian local ring A, it is necessary and sufficient that its completion $\widehat{T}$ (resp. $\widehat{M}$) be of finite injective dimension (resp. of finite projective dimension) over the completion $\widehat{A}$ of A.

On account of (*) and (**), we remark that it is sufficient to prove the theorem when A verifies conditions (i) and (iv), and even more generally when the completion of A verifies conditions (i) and (iv).

Our intention is to prove the theorem by induction on $\dim T$. For this, we will the need the following lemma:

LEMMA 5.2. *Let A be a noetherian local ring. We assume that the completion $\widehat{A}$ of A verifies one of conditions (i) or (iv) of (5.1). Then for every prime ideal $\overline{\mathfrak{p}}$ of $\widehat{A}$, the completion $\widehat{\widehat{A}_{\mathfrak{p}}}$ of $\widehat{A}_{\mathfrak{p}}$ verifies one of conditions (i) or (iv).*

The property of being of characteristic $p > 0$ obviously preserves itself upon localization and completion, thus the condition (i) does not pose any problem.

Thus, we suppose that A is essentially of finite type over k, a field of characteristic 0. Let $\overline{\mathfrak{p}}$ be a prime ideal of the completion $\widehat{A}$ of A, and $\mathfrak{p}$ its image over A. As A is an excellent ring, its formal fibres are regular. That is $\widehat{A}_{\overline{p}}/\overline{\mathfrak{p}}\widehat{A}_{\overline{p}}$ is a regular local ring. If C is the completion of the local ring $\widehat{A}_{\overline{p}}$, then $C/\mathfrak{p}C$ is the completion of $\widehat{A}_{\overline{p}}/\overline{\mathfrak{p}}\widehat{A}_{\overline{p}}$, and as $C/\mathfrak{p}C$ is equicharacteristic, $C/\mathfrak{p}C$ is isomorphic to a ring of formal power series $K[[X_1, \dots, X_s]]$, where K is the residue field of C. As $\widehat{A}_{\overline{p}}$ is flat over $A_{\mathfrak{p}}$ and the local criterion of formal freeness implies that C is formally free over $A_{\mathfrak{p}}$ ([**12**], chapt. 0_{IV}). But then, C is also formally free over the completion $\widehat{A_{\mathfrak{p}}}$ of $A_{\mathfrak{p}}$. Let k' be the residue field of $A_{\mathfrak{p}}$. Then k' is a finite extension of k. Thus k' is an algebraic extension (seperable, of course) of a pure transcendtal extension of k. That implies the k' is contained in the henselization of $A_{\mathfrak{p}}$, and more precisely there exists a local ring A', localization of a smooth covering of $A_{\mathfrak{p}}$, such that A' is essentially of finite type over its residue field k'. As the homomorphism $A_{\mathfrak{p}} \to C$ factors through $\widehat{A_{\mathfrak{p}}}$, it factors through A'. We consider the commutative diagram:

As (1) is formally free, and (2) formally unramified, we know ([**12**], chap. IV, (17.1.4)) that (3) is formally free. Let $\mathfrak{p}'$ be the maximal ideal of A'. Then $C/\mathfrak{p}'C$ is isomorphic to a ring of formal power series $K[[X_1, \dots, X_s]]$. We know that there exists a ring B of finite type over k', and a maximal ideal $\mathfrak{m}$ of B such that $A' = B_{\mathfrak{m}}$ and such that $B/\mathfrak{m} = k'$. Consider the ring $B' = B \otimes_{k'} K[[X_1, \dots, X_s]]$. Let $\mathfrak{m}'$ be the maximal ideal $(\mathfrak{m}, X_1, \dots, X_s)$ of B'. As $K[[X_1, \dots, X_s]]$ is formally free over k', therefore B' is formally free over B, and $B'_{\mathfrak{m}'}$ is formally free over $B_{\mathfrak{m}} = A'$. But then, the isomorphism $\widehat{B'_{\mathfrak{m}'}}/\mathfrak{p}'\widehat{B'_{\mathfrak{m}'}} \simeq C/\mathfrak{p}C'$ lifts itself ([**12**], chapt. 0_{IV} (19.7.1.5)) to an isomorphism $\widehat{B'_{\mathfrak{m}'}} \simeq C$, which shows that C is isomorphic to the completion of a local algebra, essentially of finite type over a field of characteristic 0.

Now we prove the theorem by induction on $\dim T$. We have seen (chap. I, (4.10)) that we can construct an $\widehat{A}$-module M, of finite type and finite projective dimesion, having the same support as that of $\widehat{T}$.

If $\dim T = 0$, then $\dim M = 0$. By hypotheses, $\widehat{A}$ is the completion of a local ring B for which the theorem of intersection holds. But then, M is also a B-module of finite length and finite projective dimension. As $B \otimes_B M$ is of finite length, the theorem of intersection says that $\dim B \leqslant \operatorname{pd} M$. But as $\operatorname{pd} M \leqslant \operatorname{depth} B$, it follows that $\dim B \leqslant \operatorname{depth} B$, which says that B is C.M., thus that the completion $\widehat{A}$ of B is C.M.

We suppose that $\dim T > 0$. By the induction hypotheses, taking into account lemma (5.2), we can suppose that for each prime ideal $\overline{\mathfrak{p}}$, non-maximal in the support of the $\widehat{A}$-module $\widehat{T}$, the ring $\widehat{A}_{\overline{\mathfrak{p}}}$ is C.M. Thus, let M be an $\widehat{A}$ module of finite type and finite projective dimension having the same support as that of $\widehat{T}$. Let $\overline{\mathfrak{q}}$ be a minimal prime ideal of $\widehat{A}$ such that $\dim \widehat{A}/\overline{\mathfrak{q}} = \dim \widehat{A}$. There are two possible cases:

1) $M/\overline{\mathfrak{q}}M$ is not of finite length. In this case, there is a prime ideal $\overline{\mathfrak{p}}$ from the support of $\overline{T}$, containing $\overline{\mathfrak{q}}$. We can take $\overline{\mathfrak{p}}$ maximal among those prime ideals which are different from the maximal ideal $\mathfrak{m}$ of $\widehat{A}$. As $\widehat{A}$ is catenary,

$$\dim \widehat{A}/\overline{\mathfrak{p}} + \dim \widehat{A}_{\overline{\mathfrak{p}}}/\overline{\mathfrak{q}}\widehat{A}_{\overline{\mathfrak{p}}} = \dim \widehat{A}/\mathfrak{q} \ .$$

Thus $\dim \widehat{A}_{\overline{\mathfrak{p}}} = \dim \widehat{A} - 1$. As $\overline{\mathfrak{p}} \in \operatorname{Supp} \widehat{T}$, we know that we have $\operatorname{depth} \widehat{A}_{\overline{\mathfrak{p}}} + \dim \widehat{A}/\mathfrak{p} = \operatorname{depth} \widehat{A}$, thus $\operatorname{depth} \widehat{A} = \dim \widehat{A}_{\overline{\mathfrak{p}}} + 1$. But by the induction hypotheses, $\widehat{A}_{\overline{\mathfrak{p}}}$ is C.M., thus $\operatorname{depth} \widehat{A}_{\overline{\mathfrak{p}}} = \dim \widehat{A} - 1$. It follows that $\operatorname{depth} \widehat{A} = \dim \widehat{A}$, which says that $\widehat{A}$ is C.M.

2) $M/\overline{\mathfrak{q}}M$ is of finite length. By the hypotheses there is a ring B essentially of finite type over a field such that $\widehat{A}$ is the completion of B. Let B^h be the henselization of B. Then $\widehat{A}$ is also the completion of B^h. Consider the prime ideal of B^h, $\mathfrak{q} = \overline{\mathfrak{q}} \cap B^h$. As B^h is a excellent henselian local ring, its formal fibres are regular. Taking into account $\dim \widehat{A}/\overline{\mathfrak{q}}$, that clearly implies $\mathfrak{q}\widehat{A} = \overline{\mathfrak{q}}$. Thus, $M/(\mathfrak{q}\widehat{A})M$ is a finite length module. Otherwise said, there is an integer c, such that if $\mathfrak{m}$ is the maximal ideal of $\widehat{A}$, the canonical homomorphism:

$$(\widehat{A}/\mathfrak{m}^{c+1}) \otimes_{\widehat{A}} (M/(\mathfrak{q}\widehat{A})M) \to (\widehat{A}/\mathfrak{m}^{c}) \otimes_{\widehat{A}} (M/(\mathfrak{q}\widehat{A})M)$$

is an isomorphism. We then utilize the theorem of approximation for modules of finite projective dimension (chap. I, §6). We know that there is a B^h-module M' of finite type and finite projective dimension, equal to $\operatorname{pd} M$, such that:

$$M' \otimes_{B^h} (\widehat{A}/\mathfrak{m}^{c+1}) \simeq M/\mathfrak{m}^{c+1}M \ .$$

We deduce from it that the canonical homomorphism:

$$M' \otimes_{B^h} (\widehat{A}/\mathfrak{m}^{c+1}) \otimes_{\widehat{A}} (\widehat{A}/\mathfrak{q}\widehat{A}) \to M' \otimes_{B^h} (\widehat{A}/\mathfrak{m}^{c}) \otimes_{\widehat{A}} (\widehat{A}/\mathfrak{q}\widehat{A})$$

is an isomorphism. It follows from it that the homomorphism:

$$M'/\mathfrak{q}M' \otimes_{B^h} (\widehat{A}/\mathfrak{m}^{c+1}) \to (M'/\mathfrak{q}M') \otimes_{B^h} (\widehat{A}/\mathfrak{m}^{c})$$

is an isomorphism, thus that $M'/\mathfrak{q}M'$ is of finite length. But as the theorem of intersection is true for B^h, we deduce from it $\dim B^h/\mathfrak{q} \leqslant \operatorname{pd}_{B^h} M'$. But as

$$\dim B^h/\mathfrak{q} = \dim B^h \quad \text{and} \quad \operatorname{pd}_{B^h} M' \leqslant \operatorname{depth} B^h$$

that implies that B^h is C.M. and thus that the completion $\widehat{A}$ of B is C.M., what we want to prove.

COROLLARY 5.3. *If in the hypotheses of the theorem, we assume more over that T is cyclic, (i.e. a quotient of A of a ideal), then A is a Gorenstein ring.*

By to the theorem, already we know that A is C.M. Let n be the dimension of A, and let k its residue field. By Bass [**6**], it will suffice to show that $\operatorname{Ext}^n_A(k, A) \simeq k$. This is a consequence of the following lemma:

LEMMA 5.4. *Let T be a module of finite type and finite injective dimension over a noetherian local ring A. Let r be the depth of A, and let $x = (x_1, \dots, x_r)$ an A-regular sequence of maximal length. Then, there is an isomorphism of functors defined in the category of A-modules of finite type:*

$$\operatorname{Ext}^r_A(\operatorname{Hom}_A(\cdot, A/x), T) \simeq \cdot \otimes_A T/xT \ .$$

We can suppose that $T \neq 0$. As the injective dimension of T is r, the functor is contravariant and right exact, thus $\operatorname{Ext}^r_A(\operatorname{Hom}_A(\cdot, A/x), T)$ is covariant and right exact. It is well known ([**22**], §2.3, lemme 3) that, for such a functor, the restriction to the category of modules of finite type is isomporphic to the tensor product by its value on the base ring. As $\operatorname{Ext}^r_A(A/x, T) \simeq T/xT$, the lemma is proved.

Then, (5.3) follows simply from the lemma, because as A is C.M., $r = \operatorname{depth} A = \dim A = n$. Thus, $\operatorname{Ext}^n_A(k, A) = \operatorname{Hom}_A(k, A/x)$, where $x = (x_1, \dots, x_r)$ is a A-regular sequence of maximal length. But to say that T is cyclic, is to say that $k \otimes_A T \simeq k$, thus this implies $\operatorname{Hom}_A(k, A/x) \simeq k$, and the corollary (5.3) is proved.

To end this section, we are going to see that in the particular case where the module of finite injective dimension is cyclic (corrollary (5.3)), one can give a general proof, without restriction on the ring, of the conjecture of Bass. Once again, local cohomology will be a useful tool.

THEOREM 5.5. *Let A be a noetherian local ring. For A to be a Gorenstein ring, it is necessary and sufficient that there exists an ideal I of the ring such that A/I is of finite injective dimension.*

The condition is clearly necessary as a Gorenstein ring is of finite injective dimension over itself. To prove the reciprocal, we suppose that A is complete, which poses no problem as $\operatorname{inj.dim}_A A/I < \infty \Rightarrow \operatorname{inj.dim}_{\widehat{A}} \widehat{A}/I\widehat{A} < \infty$, and

$$\widehat{A} \text{ is Gorenstein} \qquad \Rightarrow A \text{ is Gorenstein.}$$

Consider r the depth of A, thus also the injective dimension of A/I. We are going to show that if k is the residue field of A, we have $\operatorname{Ext}^i_A(k, A) = 0$ for $i > r$, which by virtue of (1.4.4) and (1.4.5) will show that A is of finite injective dimension over itself, thus that it is Gorenstein.

For each A-regular sequence of length r, $\alpha = (\alpha_1, \dots, \alpha_r)$, lemma (5.4) gives an isomorphism of functors defined in the category of A-modules of finite type:

$$\operatorname{Ext}^r_A(\operatorname{Hom}_A(\cdot, A/\alpha), A/I) \simeq \cdot \otimes_A A/(\alpha + I) \ .$$

We easily deduce from it:

$$\operatorname{Ext}^r_A(k, A) = \operatorname{Hom}_A(k, A/\alpha) \simeq k \quad \text{and} \quad \operatorname{Ext}^r_A(k, A/I) \simeq k \ .$$

As the functor $\operatorname{Ext}^r_A(\cdot, A)$ (resp. $\operatorname{Ext}^r_A(\cdot, A/I)$) is exact on the left (resp. on the right) in the category of A-modules of finite length, we prove by induction on $l(N) =$ length of N that for each A-module N of finite length, we have:

$$l(\operatorname{Ext}^r_A(N, A)) \leqslant l(N) \quad (\text{ resp. } l(\operatorname{Ext}^r_A(N, A/I)) \leqslant l(N)) \ .$$

If N is an A/I-module of finite length, let $\alpha = (\alpha_1, \dots, \alpha_r)$ be an A-regular sequence such that $\alpha N = 0$. Then, we have:

$$\operatorname{Ext}^r_A(N, A) \simeq \operatorname{Hom}_A(N, A/\alpha) \ ,$$

and $\operatorname{Ext}^r_A(\operatorname{Hom}_A(N, A/\alpha), A/I) \simeq N \otimes_A A/(\alpha + I) = N$ therefore implies:

$$l(\operatorname{Ext}^r_A(N, A)) = l(N) \quad \text{ and } \quad l(\operatorname{Ext}^r_A(N, A/\alpha)) = l(N) \ .$$

In particular, this implies that the functor $\operatorname{Ext}^r_A(\cdot, A)$ is exact in the category of A/I-modules of finite length. Then, the exact sequence associated to the functor $\operatorname{Ext}^{\cdot}_A(\cdot, A)$ shows that the functor $\operatorname{Ext}^{r+1}_A(\cdot, A)$ is exact to the left in the category of A/I-modules of finite length. We will then use the following result:

Lemma 5.6. *Let E be a injective envelope of the residue field of A, then $\operatorname{Ext}^r_A(E, A/I)$ is a cyclic A-module of finite projective dimension.*

The fact that $\operatorname{Ext}^r_A(E, A/I)$ is an A-module of finite projective dimension is nothing other than (1.4.10). Consider a minimal injective resolution I of A/I. Then I is indeed of length r, and we know ((1.4.4) and (1.4.5)) that we have an isomorphism $I^r \simeq E^{\mu_r}$, where $\mu_r = \dim_k \operatorname{Ext}^r_A(k, A/I)$. Thus $I^r \simeq E$, and $\operatorname{Ext}^r_A(E, A/I)$ is a quotient of $\operatorname{Hom}_A(E, E) = A$, which indeed proves that it is cyclic.

Remark. In fact $\operatorname{Ext}^r_A(E, A/I) \simeq A/I$. This is a consequence of the theorem that we will prove and of ((1.4.10)(iii)).

Thus, let $A/\mathfrak{b} = \operatorname{Ext}^r_A(E, A/I)$. As it is clearly a quotient of A/I, we know that $\operatorname{Ext}^{r+1}_A(\cdot, A)$ is left exact on the category of $A/\mathfrak{b}$-modules of finite length. To prove the theorem, we are going to show that if $i > r$, and if $\operatorname{Ext}^i_A(\cdot, A)$ is left exact in the category of $A/\mathfrak{b}$-modules of finite length, then $\operatorname{Ext}^i_A(\cdot, A)$ is zero in the category of $A/\mathfrak{b}$-modules of finite length. This will indeed prove the theorem, because if $\operatorname{Ext}^i_A(\cdot, A)$ is zero in the category of $A/\mathfrak{b}$-modules of finite length, then $\operatorname{Ext}^{i+1}_A(\cdot, A/I)$ is exact to the left, thus zero in this category, which will well prove that $\operatorname{Ext}^i_A(k, A) = 0$ for $i > r$.

We recall ([**10**], chap. IV, prop. 2) that if a contravariant functor F defined in the category of modules of finte length over a noetherian local ring R with maximal ideal $\mathfrak{n}$, is left exact, there is a canonical isomorphism of functors:

$$F(\cdot) \simeq \operatorname{Hom}_R\left(\cdot, \varinjlim F(R/\mathfrak{n}^s)\right) \ .$$

In the case which we are interested, we will thus have an isomorphism of functors defined in the category of $A/\mathfrak{b}$-modules of finite length:

$$\operatorname{Ext}^i_A(\cdot, A) \simeq \operatorname{Hom}_A\left(\cdot, \varinjlim \operatorname{Ext}^i_A(A/(\mathfrak{b} + \mathfrak{m}^s), A)\right) \ .$$

Thus to prove the theorem, it will suffice to show that

$$\varinjlim \operatorname{Ext}^i_A(A/(\mathfrak{b} + \mathfrak{m}^s), A) = 0 > r \ .$$

For that, we will use the spectral sequence associated to a composed functor. We consider the functor $\mathrm{Hom}_A(A/\mathfrak{b},\cdot)$ defined in the category of A-modules, and the functor:

$$\mathrm{H}^0_{\mathfrak{m}}(\cdot)=\varinjlim \mathrm{Hom}_{A/\mathfrak{b}}(A/\mathfrak{b}+\mathfrak{m}^s,\cdot)$$

defined in the category of $A/\mathfrak{b}$-modules. Indeed, we have an isomorphism of functors:

$$\mathrm{H}^0_{\mathfrak{m}}(\cdot)\circ \mathrm{Hom}_A(A/\mathfrak{b},\cdot)=\varinjlim \mathrm{Hom}_{A/\mathfrak{b}}(A/\mathfrak{b}+\mathfrak{m}^s,\cdot)\ .$$

As $\mathrm{Hom}_A(A/\mathfrak{b},\cdot)$ transforms a injective A-module into an $A/\mathfrak{b}$-module acyclic for the functor $\mathrm{H}^0_{\mathfrak{m}}(\cdot)$, the associated spectral sequence of the composed functor converges. We verify easily that the right deriveds of the left exact functor $\varinjlim \mathrm{Hom}_{A/\mathfrak{b}}(A/\mathfrak{b}+\mathfrak{m}^s,\cdot)$ are the functors $\varinjlim \mathrm{Ext}^i_{A/\mathfrak{b}}(A/\mathfrak{b}+\mathfrak{m}^s,\cdot)$. Thus, we have an convergent spectral sequence

$$\mathrm{H}^p_{\mathfrak{m}}(\mathrm{Ext}^q_A(A/\mathfrak{b},A))\Rightarrow H^n=\varinjlim \mathrm{Ext}^n_{A/\mathfrak{b}}(A/\mathfrak{b}+\mathfrak{m}^s,A)\ .$$

Thus, to prove that $H^n=0$ for $n>r$, it will suffice to show that $\mathrm{H}^p_{\mathfrak{m}}(\mathrm{Ext}^q_A(A/\mathfrak{b},A))=0$ for $p+q>r$.

Let us recall that as A/I is of finite injective dimension, therefore, by (1.4.7), for each prime ideal $\mathfrak{p}\in \mathrm{Supp}\,A/I$ we have $\dim A/\mathfrak{p}+\mathrm{depth}\,A_{\mathfrak{p}}=\mathrm{depth}\,A=r$.

Therefore, let M be an A-module of finite type such that $\mathrm{Supp}\,M\hookrightarrow \mathrm{V}(I)$.

Let $\mathfrak{p}\in\mathrm{Supp}\,M$ be such that $\dim M=\dim A/\mathfrak{p}$. As $\mathrm{grade}\,M\leqslant \mathrm{depth}\,A_{\mathfrak{p}}$, we deduce from it $\dim M+\mathrm{grade}\,M\leqslant r$ (thus there is a equality by virtue of (1.4.8)). Take $M=A/\mathfrak{b}$. As $A/\mathfrak{b}$ is of finite projective dimension, for each integer q, we have:

$$\mathrm{grade}(\mathrm{Ext}^q_A(A/\mathfrak{b},A))\geqslant q$$

since, if $\mathrm{depth}\,A_{\mathfrak{p}}<q$ we have $\mathrm{pd}_A\,A/\mathfrak{b}<q$, thus $\mathrm{Ext}^q_A(A/\mathfrak{b},A)_{\mathfrak{p}}=0$. Thus we deduce from it $\dim \mathrm{Ext}^q_A(A/\mathfrak{b},A)\leqslant r-q$. And, the cohomological dimension being less than the Krull dimension, $\mathrm{H}^p_{\mathfrak{m}}(\mathrm{Ext}^q_A(A/\mathfrak{b},A))=0$, for $p>r-q$, which is what we want to prove.

Corollary 5.7. *Let A be a noetherian local ring. A necessary and sufficient condition for A to be Gorenstein is that there exists an ideal of definition $\mathfrak{q}$, irreducible and of finite projective dimension. If moreover it has dimension $\leqslant 2$, then q is necessarily generated by a regular sequence.*

Let E be the injective envelope of the residue field of A. As $A/\mathfrak{q}$ is a Gorenstein of dimension 0, we have $\mathrm{Hom}_A(A/\mathfrak{q},E)\simeq A/\mathfrak{q}$. Thus $A/\mathfrak{q}$ is also of finite injective dimension, and A is a Gorenstein by (5.5). If moreover, A is of dimension 2, consider a minimal projective resolution of $A/\mathfrak{q}$:

$$0\to A^{\mu_2}\to A^{\mu_1}\to A\to A/\mathfrak{q}\to 0\ .$$

As $A/\mathfrak{q}$ is Gorenstein of codimension 2 in A, we have:

$$\mathrm{Ext}^2_A(A/\mathfrak{q},A)\simeq \mathrm{Hom}_A(A/\mathfrak{q},E)\simeq A/\mathfrak{q}\ .$$

This implies $\mu_2=1$ because:

$$0\to A^*\to (A^{\mu_1})\check{}\to (A^{\mu_2})\check{}\to \mathrm{Ext}^2_A(A/\mathfrak{q},A)\to 0\ ,$$

is a minimal free resolution of $\mathrm{Ext}^2_A(A/\mathfrak{q},A)$. As $\mu_1-\mu_2-1=\mathrm{rank}(A/\mathfrak{q})=0$, we deduce from it that $\mu_1=2$, which is what we desired.

CHAPTER 3

Finiteness and vanishing theorems for cohomology of schemes

0. Introduction

In [**10**] A. Grothendieck poses the following question:

Let k be an algebraic closed field, and X a closed sub-scheme of the projective space $P = \mathbb{P}^n(k)$ on k. Suppose that X is equidimensional of dimension d and that X is locally complete intersection in P. Show that for each coherent sheaf $\mathcal{F}$ on $P - X$, the cohomology groups $\mathrm{H}^i(P - X, \mathcal{F})$ are finite over k for $i \geqslant n - d$.

In [**14**], R. Hartshorne proves this cojecture when k is of characteristic 0 and when the normal sheaf of X in P is ample (in particular, when X is non-singular of charactersitic 0). Moreover, Hartshorne proves the following result:

Theorem . *Let X be a connected closed sub-scheme of dimension $\geqslant 1$ of $P = \mathbb{P}^n(k)$. Then for each quasi-coherent sheaf $\mathcal{F}$ on $P - X$, we have:*

$$\mathrm{H}^{n-1}(P - X, \mathcal{F}) = 0 \ .$$

These global statements correspond to, more general, local statements concerning the finiteness of local cohomology groups defined by an ideal in a regular local ring. Our intention is to prove here two local theorems which will have as corollaries the conjecture of Grothendieck and the theorem of Hartshorne in characteristic $p > 0$, and which, more generally, show that the vanishing of the penultimate cohomology group is equivalent, in the connected case, to its finiteness. We will use here, as in chapter II, the iteration of the morphism of Frobenius to calculate certain local cohomology groups of formal schemes.

1. Relations between local cohomology and cohomology of projective varieties

The goal of this section is to prove that local theorems are more general then projective theorems. For this, we show first that the hypothesis on the depth, or the regularity, or the dimension, of local rings of a projective variety X embedded in a projective space P, carries over to local rings of the complement of the apex of the cone, of the projective variety as above. After that, we show that the conclusions on the finiteness or the nullity of certain local cohomology groups, in the local ring of the apex of the cone, gives conditions on the vanishing for the cohomology of coherent sheaves on the complement of X in P.

We will fix once for all the notations for this section:

Let X be a projective variety embedded in a projective space $P = \mathbb{P}^n(k)$ over a field k.

Let $A = k[T_0, \dots, T_n]$ the ring of the cone of P as above, and I an graded ideal of A defining X. We have, setting $B = A/I$, $X = \operatorname{Proj}(B)$ and $P = \operatorname{Proj}(A)$. We will denote t_i the image of T_i in B, i.e. $B = k[t_0, \dots, t_n]$. Let $\mathfrak{m} = (T_0, \dots, T_n)$ the ideal of A defining the apex of the cone of P as above. We set $R = A_{\mathfrak{m}}$; this is a regular local ring. We set $\mathfrak{a} = IR$ and $C = R/\mathfrak{a}$; we see that C is the local ring of the apex of cone of X as above.

We set $U = \operatorname{Spec} R - \{\mathfrak{m}R\}$.

Consider $\mathcal{F}$ a coherent sheaf on P and M the graded module of finite type which defines $\mathcal{F}$.

To compare $\operatorname{Spec}(B) - \{\mathfrak{m}\}$, and thus $U \cap \operatorname{Spec} B$, with X, we will use the following fundamental lemma. The three propositions which follow it are immediate corollaries.

1.1. *Preliminary lemma* ([**12**], chap. II, (8.3.6)). *For $j = 0, 1, \dots, n$, we have* $\operatorname{Spec} B_{t_j} \simeq X_{t_j}[T, T^{-1}]$*, where T is a variable.*

Proposition 1.2. *The following conditions are equivalent:*

a) *The irreducible components of X are of dimension greater than or equal to d.*
b) *The irreducible components of C are of dimension greater than or equal to $d+1$.*

Proposition 1.3. *The following conditions are equivalent:*

a) *X is non-singular (resp. locally complete intersection).*
b) *For each prime ideal $\mathfrak{p}$ of A, different from $\mathfrak{m}$, the ring $B_{\mathfrak{p}}$ is regular (resp. complete intersection in $A_{\mathfrak{p}}$).*

Proposition 1.4. *Let i be a integer. The following conditions are equivalent:*

a) *The local rings of X verify the Serre condition S_i.*
b) *For each prime ideal $\mathfrak{p}$ of A, different from $\mathfrak{m}$, the ring $B_{\mathfrak{p}}$ verifies the condition S_i, and moreover if $\mathfrak{p}$ is maximal* $\operatorname{depth} B_{\mathfrak{p}} \geqslant i+1$.

Corollary 1.5. *The following conditions are equivalent:*

a) *The local rings of X are C.M.*
b) *The local rings of* $\operatorname{Spec} B - \{\mathfrak{m}\}$ *are C.M.*

To compare the local cohomology groups with support in $\mathrm{V}(I)$ and the cohomology groups of coherent sheaves on $P - X$, the essential tool is the following result:

Lemma 1.6. ([**12**], chap. III) *With the notation introduced in the beginning of the section, we have an exact sequence:*

$$0 \to \mathrm{H}^0_I(M) \to M \to \sum_{v \in \mathbb{Z}} \mathrm{H}^0(P - X, \mathcal{F}(v)) \to \mathrm{H}^1_I(M) \to 0 \ ,$$

and isomorphisms:

$$\sum_{v \in \mathbb{Z}} \mathrm{H}^i(P - X, \mathcal{F}(v)) \simeq \mathrm{H}^{i+1}_I(M) \geqslant 1 \ .$$

COROLLARY 1.7. *Let i be an integer. Suppose that $\mathrm{H}_I^{i+1}(M)$ is an artinian A-module. Then:*

(i) *The vector spaces $\mathrm{H}^i(P-X,\mathcal{F}(v))$ are of finite dimension over k for each $v\in\mathbb{Z}$.*
(ii) *For v sufficiently big, we have $\mathrm{H}^i(P-X,\mathcal{F}(v))=0$ if $i\geqslant 1$.*

The corollary is easily deduced from the lemma. Indeed, $\mathrm{H}_I^{i+1}(M)$ has a structure of a graded A-module. A strictly decreasing sequence of k sub-modules of $\mathrm{H}^i(P-X,\mathcal{F}(v))$ generates a strictly decreasing sequence of A-sub-modules of $\mathrm{H}_I^{i+1}(M)$. This is not possible, and $\mathrm{H}^i(P-X,\mathcal{F}(v))$ is an artinian k-module, that is, of finite type. If $i\geqslant 1$, the A-module $\sum_{v\in\mathbb{Z}}\mathrm{H}^i(P-X,\mathcal{F}(v))$ is artinian. The sequence of A-submodules

$$K(l)=\sum_{v\geqslant l}\mathrm{H}^i(P-X,\mathcal{F}(v))$$

is thus stabilized for l very big, which indeed implies $\mathrm{H}^i(P-X,\mathcal{F}(v))=0$ for v very big.

As the calculation of local cohomology commutes with localization, we have $(\mathrm{H}_I^i(M))_{\mathfrak{m}}=\mathrm{H}_{\mathfrak{a}}^i(M_{\mathfrak{m}})$, thus if $\mathrm{H}_I^i(M)$ has support $\mathrm{V}(\mathfrak{m})$, we have $\mathrm{H}_{\mathfrak{a}}^i(M_{\mathfrak{m}})=\mathrm{H}_I^i(M)$. It is this remark which we will use to see that in "good cases", the finiteness of local cohomolgy groups over a regular local ring implies the finiteness of the cohomology groups of a coherent sheaf over a quasi-projective variety. To be precise, each of the following are the "good cases" which will interest us further.

PROPOSITION 1.8. *Let X be a closed sub-scheme of the projective space $P=\mathbb{P}^n(k)=\operatorname{Proj}A$, where $A=k[T_1,\dots,T_n]$, defined by the graded ideal I. Let X_j $(j=1,\dots,l)$ the irreducible components of X, and let $d=\inf_j\dim X_j$. Let $\mathfrak{m}=(T_0,\dots,T_n)$ be the maximal ideal of the origin in A. Then:*

(1) *$\mathrm{H}_I^{n+1}(A)$ has support in $\mathrm{V}(\mathfrak{m})$, and thus we have $\mathrm{H}_I^{n+1}(A)=\mathrm{H}_{IA_{\mathfrak{m}}}^{n+1}(M_{\mathfrak{m}})$ for each A-module M.*
(2) *If X is locally complete intersection, $\mathrm{H}_I^s(A)$ has support in $\mathrm{V}(\mathfrak{m})$ for each $s\geqslant n+1-d$, and thus we have $\mathrm{H}_I^{s+1}(M)=\mathrm{H}_{IA_{\mathfrak{m}}}^{s+1}(M_{\mathfrak{m}})$ for $s\geqslant n+1-d$, and for each A-module M.*
(3) *If k is of characteristic $p>0$, and if X verifies the condition S_i, with $i\leqslant d$, the $\mathrm{H}_I^s(A)$ has support in $\mathrm{V}(\mathfrak{m})$ for $s\geqslant n+1-i$, and thus we have $\mathrm{H}_I^{s+1}(M)=\mathrm{H}_{IA_{\mathfrak{m}}}^{s+1}(M_{\mathfrak{m}})$ for $s\geqslant n+1-i$, and for each A-module M.*

In the three cases, we will show that for the integers s considered, the modules $\mathrm{H}_I^s(A)$ are the inductive limits of modules with support in $\mathrm{V}(\mathfrak{m})$.

We note straight away that once we know that $\mathrm{H}_I^s(A)$ has support in $\mathrm{V}(\mathfrak{m})$ for $s\geqslant s_0$, we easily deduce from it that, for each A-module M, $\mathrm{H}_I^s(M)$ has support in $\mathrm{V}(\mathfrak{m})$ for $s\geqslant s_0$. For modules of finite type, this is seen by descending induction, and using a finite presentation, taking into account the fact that the cohomology is zero in degrees $>n+1$. As each A-module is the inductive limit of finite A-modules, the general case follows from it immediately.

Thus, it will suffice to show the related results for local cohomology groups of A.

CASE 1. We have $\mathrm{H}_I^{n+1}(A)=\varinjlim\operatorname{Ext}_A^{n+1}\left(A/I^l,A\right)$. But by (1.4), for each prime ideal $\mathfrak{p}$ of height $n+1$ (thus maximal), if $\mathfrak{p}\neq\mathfrak{m}$, we have $\operatorname{depth}(A/I^l)_{\mathfrak{p}}\geqslant 1$.

Thus it follows from it that for each prime ideal $\mathfrak{p} \neq \mathfrak{m}$, we have $\mathrm{pd}_{A_\mathfrak{p}}(A/I^l)_\mathfrak{p} < n+1$, and $\mathrm{Ext}^{n+1}_A\left(A/I^l, A\right)$ has its support reduced to $\mathfrak{m}$, for each l.

Case 2. $\mathrm{H}^s_I(A) = \varinjlim \mathrm{Ext}^s_A\left(A/I^l, A\right)$. Thus it will suffice to show that for $s \geqslant n+1-d$, the module $\mathrm{Ext}^s_A\left(A/I^l, A\right)$ has support in $\mathrm{V}(\mathfrak{m})$. For that, we will show that for $\mathfrak{p} \neq \mathfrak{m}$, we have $\mathrm{pd}_{A_\mathfrak{p}}(A/I^l)_\mathfrak{p} < n+1-d$. But by (1.3), $(A/I)_\mathfrak{p}$ is a complete intersection in $A_\mathfrak{p}$, thus $(A/I^l)_\mathfrak{p}$ is C.M. and its projective dimension over $A_\mathfrak{p}$, is equal to its codimension in $A_\mathfrak{p}$. But the integer d has been chosen in such a way that for each $\mathfrak{p}$, we have:

$$\mathrm{codim}_{A_\mathfrak{p}}(A/I)_\mathfrak{p} \leqslant n-d < n+1-d \ ,$$

which proves Case 2.

Case 3. For each integer l, we denote I_l, the ideal of A generated by the p-th powers of the elements of the ideal I. We know, by (1.1.7), that as Frobenius is flat on A, we have for each integer l, and each prime ideal $\mathfrak{p}$:

$$\mathrm{pd}_{A_\mathfrak{p}}(A/I)_\mathfrak{p} = \mathrm{pd}_{A_\mathfrak{p}}(A/I_l)_\mathfrak{p} \ . \qquad (*)$$

We see easily that the system of ideals (I_l) and (I^l) define the same topology in A, thus for each $s \in \mathbb{Z}$, $\mathrm{H}^s_I(A) = \varinjlim \mathrm{Ext}^s_A(A/I_l, A)$. Finally, to show that $\mathrm{H}^s_I(A)$ has its support reduced to $\mathrm{V}(\mathfrak{m})$ for $s \geqslant n+1-i$, it suffices to show that for $\mathfrak{p} \neq \mathfrak{m}$, we have $\mathrm{pd}_{A_\mathfrak{p}}(A/I_l)_\mathfrak{p} \leqslant n-i$ for each l. By (*), it will suffice to show:

$$\mathrm{pd}_{A_\mathfrak{p}}(A/I)_\mathfrak{p} \leqslant n-i \mathfrak{p} \neq \mathfrak{m} \ .$$

If $\mathfrak{p}$ is maximal, we know (1.4) that $\mathrm{depth}(A/I)_\mathfrak{p} \geqslant i+1$, thus:

$$\mathrm{pd}_{A_\mathfrak{p}}(A/I)_\mathfrak{p} \leqslant \dim(A/I)_\mathfrak{p} - (i+1) = n+1-(i+1) = n-i \ .$$

If $\mathfrak{p}$ is not maximal, we know that:

$$\mathrm{depth}(A/I)_\mathfrak{p} \geqslant \inf(i, \dim(A/I)_\mathfrak{p}) \ ,$$

thus,

$$\mathrm{pd}_{A_\mathfrak{p}}(A/I)_\mathfrak{p} \leqslant \sup(\dim A_\mathfrak{p} - i, \dim A_\mathfrak{p} - \dim(A/I)_\mathfrak{p}) \ .$$

But as $\dim A_\mathfrak{p} \leqslant n$, we have $\dim A_\mathfrak{p} - i \leqslant n-i$.

Also, $\dim A_\mathfrak{p} - \dim(A/I)_\mathfrak{p} = \mathrm{codim}_{A_\mathfrak{p}}(A/I)_\mathfrak{p}$; but $\mathrm{codim}_{A_\mathfrak{p}}(A/I)_\mathfrak{p}$ is equal to the codimension in A of a irreducible component of A/I, thus, by the choice of d, is less than or equal to $n-d$ and *a fortiori* to $n-i$.

We note that we have again used the iteration of the Frobenius homomorphism to study certain local cohomology groups, and more precisely to calculate their support.

We will see in §4 of this chapter, that by utilizing it more carefully we can prove the finiteness of the cohomology modules studied in Case 3 of the last proposition.

2. Relations between local cohomology and cohomology of formal schemes

Like everything in the preceeding section, we can interpret the local cohomology groups in more geometric terms, thanks to formal schemes. The ingredient is the theorem of local duality. To use this, we restrict ourselves to the consideration of Gorenstein rings. In fact, in practice we only work with regular local rings, so this restriction is not very important.

Let R be a Gorenstein ring, and I an ideal in R. We set:

$$\operatorname{fdepth}_I R = \textit{ formal depth of } \operatorname{Spec} R \textit{ along } \quad \mathrm{V}(I) = \inf_i \{ i \mid \mathrm{H}^i_{\mathfrak{m}}(\widehat{X}, \mathcal{O}_{\widehat{X}}) \neq 0 \} ,$$

where $\widehat{X}$ is the formal completion of $X = \operatorname{Spec} R$ along $\mathrm{V}(I)$. proposition (2.2) below implies that

$$\operatorname{fdepth}_I R \geqslant s \Longleftrightarrow \mathrm{H}^i_I(R) = 0 \quad \forall i > \dim R - s .$$

In proposition (2.3). we will give obvious conditions for it, which we will use in sections 4 and 5.

First we recall the following result which is a corollary of ([**12**], chap. 0_{III}, prop. (13.3.1)).

Proposition 2.1. *Let X be a noetherian scheme and let Y be a closed set in X defined by a sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_X$. Let $\mathcal{F}$ be a coherent sheaf on X and let $\mathcal{F}_n = \mathcal{F}/\mathcal{I}^{n+1}\mathcal{F}$. Let U be an open in X, and let $\widehat{U}$ its formal completion along $Y \cap U$. Then the canonical homomorphims:*

$$h_i \colon \ \mathrm{H}^i(\widehat{U}, \varprojlim \mathcal{F}_n) \to \varprojlim \mathrm{H}^i(U, \mathcal{F}_n)$$

are surjective for $i > 0$.

Moreover, if the projective system $\mathrm{H}^i(U, \mathcal{F}_n)$ verifies the Mittag-Leffler condition, then h_i is an isomorphism.

Proposition 2.2. *Let R be a complete Gorenstein ring of dimension d. Let $\mathfrak{m}$ be a maximal ideal of R and let E the injective envelope of its residue field k (thus, a dualizing module for R). Let I be an ideal of R.*

Set $U = \operatorname{Spec} R - \mathfrak{m}$ and $Y = \mathrm{V}(I) \cap U$. Then, if $\widehat{U}$ is the formal completion of U along Y, there is an exact sequence:

$$0 \to \operatorname{Hom}_R\left(\mathrm{H}^d_I(R), E\right) \to R \to \Gamma(\widehat{U}, \mathcal{O}_{\widehat{U}}) \to \operatorname{Hom}_R\left(\mathrm{H}^{d-1}_I(R), E\right) \to 0 ,$$

and isomorphims:

$$\mathrm{H}^i(\widehat{U}, \mathcal{O}_{\widehat{U}}) \simeq \operatorname{Hom}_R\left(\mathrm{H}^{d-i-1}_I(R), E\right) .$$

Indeed, we consider the projective system of exact sequences:

$$0 \to \mathrm{H}^0_{\mathfrak{m}}(R/I^n) \to R/I^n \to \Gamma(U, R/I^n) \to \mathrm{H}^1_{\mathfrak{m}}(R/I^n) \to 0. \qquad (*)$$

As the projective system (R/I^n) satisfies the M.L., we obtain, on passing to the limit, an exact system:

$$0 \to \varprojlim \mathrm{H}^0_{\mathfrak{m}}(R/I^n) \to R \to \Gamma(\widehat{U}, \mathcal{O}_{\widehat{U}}) \to \varprojlim \mathrm{H}^1_{\mathfrak{m}}(R/I^n) \to 0 . \qquad (1)$$

Also, the isomorphisms:

$$\mathrm{H}^i(U, R/I^n) \xrightarrow{\sim} \mathrm{H}^{i+1}_{\mathfrak{m}}(R/I^n) \geqslant 1 , \qquad (**)$$

give, on passing to the limit, isomorphims:

$$\varprojlim \mathrm{H}^i(U, R/I^n) \simeq \varprojlim \mathrm{H}^{i+1}_{\mathfrak{m}}(R/I^n) \geqslant 1 . \qquad (1')$$

As $\mathrm{H}^s_{\mathfrak{m}}(R/I)$ is artinian for each ideal I of R and for each integer $s \geqslant 0$, (*) and (**) show that each of the projective systems $(\mathrm{H}^i(U, R/I^n))_n$ verfies M.L.

Thus, by (2.1), we have:

$$\mathrm{H}^i(\widehat{U}, \mathcal{O}_{\widehat{U}}) = \varprojlim \mathrm{H}^i(U, R/I^n) \geqslant 0 . \qquad (2)$$

The theorem of local duality gives functorial isomorphisms:

$$\mathrm{H}^s_{\mathfrak{m}}(\cdot) = \mathrm{Hom}_R\left(\mathrm{Ext}^{d-s}_R(\cdot, R), E\right) \ .$$

Thus

$$\begin{aligned}\varprojlim \mathrm{H}^s_{\mathfrak{m}}(R/I^n) &\simeq \varprojlim \mathrm{Hom}_R\left(\mathrm{Ext}^{d-s}_R(R/I^n, R), E\right)\\ &\simeq \mathrm{Hom}_R\left(\varinjlim \mathrm{Ext}^{d-s}_R(R/I^n, R), E\right)\\ &\simeq \mathrm{Hom}_R\left(\mathrm{H}^{d-s}_I(R), E\right)\end{aligned}$$

By combining (1),(2) and (3) we find the exact sequence, and by combining (1'),(2) and (3) we find the isomorphisms.

Corollary 2.3. *Let R be a complete Gorenstein ring of dimension d. Let U be the complementary open set of the closed point in* $\operatorname{Spec} R$. *Let I be an ideal of R. Let R be an integer such that $0 \leqslant r \leqslant d$. The following conditions are equivalent:*

1) $\mathrm{H}^s_I(M)$ is an artinian R-module, for each R-module of finite type M, and each integer $s \geqslant d-r$.

2) $\mathrm{H}^s_I(R)$ is an artinian R-module for each integer $s \geqslant d-r$.

3) If $\widehat{U}$ is the formal completion of U along $\mathrm{V}(I)\cap U$, then $\mathrm{H}^i(\widehat{U}, \mathcal{O}_{\widehat{U}})$ is an R-module of finite type for $i \leqslant r-1$.

We have already proved 1) $\Leftrightarrow$ 2) by descending induction. The equivalence 2) $\Leftrightarrow$ 3) is an immediate consequence of the proposition.

3. The local theorem of Lichtenbaum-Hartshorne

It concernes the following theorem on the cohomology of local rings, proved by R. Hartshorne in [**14**].

Theorem 3.1. *Let A be a noetherian local ring of dimension n. Let I be an ideal of A having the following property:*

For each prime ideal $\mathfrak{p}$ in the completion $\widehat{A}$ of A, such that $\dim \widehat{A}/\mathfrak{p} = \dim A$, *we have*

$$\dim \widehat{A}/(I\widehat{A}+\mathfrak{p}) \geqslant 1 \ .$$

Then the local cohomology functor $\mathrm{H}^n_I(\cdot)$ is zero.

We will give a direct proof of this theorem, inspired by the one of R. Hartshorne, but shorter.

Clearly, we can suppose that A is complete, thus a quotient of a complete regular local ring $\overline{R}$. We know that $\mathrm{grade}_{\overline{R}} A + \dim A = \dim \overline{R}$, that is that there exists a $\overline{R}$-regular sequence α of length $\dim \overline{R} - n$ in the annihilator of A. Alternatively, A is the quotient of a complete intersection $R = \overline{R}/\alpha$, which has the same dimension as that of A. Finally, there exists a complete Gorenstein local ring R, and an ideal I of R such that $A = R/I$ and $\dim R = \dim A = n$.

Lemma 3.2. *Let $\mathfrak{p}_i$ $(i = 1, \ldots, s)$ be the minimal prime ideals of R. Let U be the complement of the closed point in* $\operatorname{Spec} R$. *Then, there exists prime ideals $\mathfrak{q}_i$ $(i = 1, \ldots, s)$ such that:*

1) *For each i, $\{\mathfrak{q}_i\}$ is a closed point of U.*
2) *For each i, we have $\mathfrak{p}_i \subset \mathfrak{q}_i$, and $\mathrm{V}(\mathfrak{q}_i A) \hookrightarrow \mathrm{V}(I)$.*

Indeed, as $\dim R/I = \dim R$, we can suppoe that there exists an integer i, with $1 \leqslant t \leqslant s$, such that $I \subset \mathfrak{p}_i$ for $i \leqslant t$ and that $I \not\subset \mathfrak{p}_i$ for $i > t$. Let then I' in R be the preimage of I.

If $i \leqslant t$, we consider the locally closed $U \cap \mathrm{V}(\mathfrak{p}_i + I')$ which is non empty by the hypotheses, and we can take for $\{\mathfrak{q}_i\}$ a closed point of this locally closed.

If $i > t$, we can take for $\{\mathfrak{q}_i\}$ a closed point of $U \cap \mathrm{V}(\mathfrak{p}_i)$ which is not contained in $\mathrm{V}(I)$. Such a point exists because $U \cap \mathrm{V}(\mathfrak{p}_i)$ is a Jacobson scheme.

We assert that for $i \leqslant t$, we have $I' \subset \mathfrak{q}_i$, thus $\mathrm{V}(\mathfrak{q}_i A) \hookrightarrow \mathrm{V}(I'A) = \mathrm{V}(IA)$, and for $i > t$, by construction $\mathrm{V}(\mathfrak{q}_i A)$ is reduced to the closed point of $\operatorname{Spec} A$.

The lemma being proved, let $\mathfrak{b} = \cap_{i=1}^{s} \mathfrak{q}_i$; our intention is to prove that the functor $\mathrm{H}^n_{\mathfrak{b}}(\cdot)$, defined in the category of A-modules, is zero, and deduce from it that the fuctor $\mathrm{H}^n_I(\cdot)$, defined in the category of A-modules is also zero.

For each integer l, we set $\mathfrak{b}_l = \cap_{i=1}^{s} \mathfrak{q}_i^{(l)}$, and will show that the system of ideals $(\mathfrak{b}_l)$ define the same topology as that of powers of $\mathfrak{b}$, in R. Clearly, $\mathfrak{b}^l \subset \mathfrak{b}_l$ for each l. On the other hand, a primary decomposition of $\mathfrak{b}^l$ allows us to ascertain that $\mathfrak{b}^l = \mathfrak{b}_l \cap \mathfrak{m}_l$, where $\mathfrak{m}_l$ is, for each l, an ideal primary to the maximal ideal of R. Note that if $S = \cap_{i=1}^{s}(R - \mathfrak{q}_i)$, by the choice of the $\mathfrak{q}_i$, we know that $S \cap \mathfrak{p}_i = \emptyset$ for $i = 1, \dots, s$, thus that S does not contain a divisor of 0. Said otherwise, R is contained in $S^{-1}R$. As $\cap_{l \geqslant 0} \mathfrak{b}_l S^{-1}R = 0$, we deduce from it $\cap_{l \geqslant 0} \mathfrak{b}_l = 0$. As the ring R is complete, the theorem of Chevalley says that the topology defined by the maximal ideal of R is minimal among the separated topologies. That implies that for each ideal $\mathfrak{m}$, primary to the maximal ideal of R, there exists an integer t_0 such that $\mathfrak{b}_t \hookrightarrow \mathfrak{m}$ for $t \geqslant t_0$. Thus, there exists an integer $t_0(l)$ such that $\mathfrak{b}_l \hookrightarrow \mathfrak{m}_l$ for $t \geqslant t_0(l)$. We deduce $\mathfrak{b}_l \cap \mathfrak{b}_t \hookrightarrow \mathfrak{b}_l \cap \mathfrak{m}_l = \mathfrak{b}^l$ for $t \geqslant t_0(l)$, that is $\mathfrak{b}^l \supset \mathfrak{b}_l$ for $t \geqslant \sup(l, t_0(l))$, which shows indeed that the system of ideals $(\mathfrak{b}_l)$ and $(\mathfrak{b}^l)$ define the same topology in R. That shows that for each integer j, we have:

$$\mathrm{H}^j_{\mathfrak{b}}(R) = \varinjlim \operatorname{Ext}^j_R(R/\mathfrak{b}_l, R) \ .$$

We recall the following result which is proved in [**22**], and which is moreover a corollary of (1.4.15).

Proposition 3.3. *Let R be a local Gorenstein ring. For each R-module M of finite type, we have a equality:*

$$\operatorname{depth} M + \sup\{i \in \mathbb{Z}, \textit{suchthat} \operatorname{Ext}^i_R(M, R) \neq 0\} = \dim R \ .$$

In the case we are interested in, we recall that the ideals $\mathfrak{b}_l$ have been chosen in a way that $\operatorname{depth} R/\mathfrak{b}_l = 1$, for each l. By the proposition, this implies

$$\operatorname{Ext}^n_R\left(R/\mathfrak{b}^l, R\right) = 0, \quad \text{for each} \quad l, \text{ thus } \mathrm{H}^0_{\mathfrak{b}}(R) = 0$$

and as the functor $\mathrm{H}^0_{\mathfrak{b}}(\cdot)$ is right exact, $\mathrm{H}^0_{\mathfrak{b}}(\cdot) = 0$. From it we deduce that the functor $\mathrm{H}^0_{\mathfrak{b}A}(\cdot)$ defined in the category of A-modules is zero. But as $\mathrm{V}(\mathfrak{b}A) \hookrightarrow \mathrm{V}(I)$, we have an exact sequence of functors:

$$\mathrm{H}^n_{\mathfrak{b}A}(\cdot) \to \mathrm{H}^n_I(\cdot) \to \mathrm{H}^n_{\mathrm{V}(I)-\mathrm{V}(\mathfrak{b}A)}(\operatorname{Spec} R - \mathrm{V}(\mathfrak{b}A), \widetilde{\cdot}) \ .$$

As $\dim(\operatorname{Spec} R - \mathrm{V}(\mathfrak{b}A)) < n$, the last cohomology functor of this three term exact sequence is zero, thus $\mathrm{H}^n_{\mathfrak{b}A}(\cdot) = 0$, implies $\mathrm{H}^n_I(\cdot) = 0$.

Corollary 3.4. *Let X be an algebraic quasi-projective scheme over a field k. If $\dim X = n$, the following conditions are equivalent:*

(a) $\mathrm{H}^n(X, \mathcal{F}) = 0$ *for each quasi-coherent sheaf $\mathcal{F}$ on X.*

(b) *No irreducible component of dimension n of X is projective.*

This is an immediate consequence of the preceeding theorem, applied to all closed points of the graded ring of the projective closure of X.

Remark. This corollary, the theorem of Lichtenbaum, which is a little stronger than the local theorem of Hartshorne, is true more generally. S. Kleiman has given a proof for which the hypotheses X quasi-projective is useless.

4. Theorem of finiteness in characteristic $p > 0$

In this section, we give the principal theorem of this chapter, which resolves a conjecture of Grothendieck cited in the introduction, for projective schemes over a field of non zero characteristic. The method consists in using the iteration of the Frobenius morphism. We will begin this section with a vanishing theorem for local cohomology with support in a closed defined by a C.M. ring. It is to be noted that this theorem does not extend to characteristic 0.

Proposition 4.1. *Let R be a regular local ring of characteristic $p > 0$. Let I be an ideal of R such that R/I is a C.M. ring; then:*

$$\mathrm{H}^i_I(M) = 0 \quad \textit{for each } i > \dim R - \dim R/I \ .$$

We have already noted in chapter I that the Frobenius morphism is flat on R. Let I_l be the ideal of R generated by the p^l-th powers of elements of I. Then, R/I_l is C.M. for each integer l; moreover the topology on R defined by the I_l is the same as the I-adic topology; thus for each integer i, we have:

$$\mathrm{H}^i_I(R) \simeq \varinjlim \mathrm{Ext}^i_R\,(R/I_l, R) \ .$$

Let $n = \dim R$ and $d = \dim R/I = \dim R/I_l$. We know that for each l

$$\mathrm{pd}(R/I_l) = n - d \ ,$$

thus $\mathrm{Ext}^i_R\,(R/I_l, R) = 0$ for $i > n-d$, and for each l. Thus $\mathrm{H}^i_I(R) = 0$ for $i > n-d$.

Then we show, by descending induction, and by passage to the inductive limit, that this implies:

$$\mathrm{H}^i_I(M) = 0 > n - d, \quad \text{and for each } R\text{-module M.}$$

Remark. We prove in the same way that if R is a regular local ring of non zero characteristic, and if I is an ideal of R, then $\operatorname{depth} R/I \geqslant s$ implies $\mathrm{H}^i_I(\cdot) = 0$ for $i > \dim R - s$, otherwise said $\operatorname{depth} R/I \geqslant s$ implies $\operatorname{fdepth}_I R \geqslant s$.

Corollary 4.2. *Let k be a field of characteristic $p > 0$. Let X be a projective variety of dimension d, embedded in the projective space $P = \mathbb{P}^n(k)$. Suppose that the local ring of the apex of the cone of X is a C.M ring. Then, setting $U = P - X$, we have:*

$$\mathrm{H}^i(U, \mathcal{F}) = 0 \geqslant n - d$$

for each quasi-coherent sheaf $\mathcal{F}$ on U.

Indeed, by virtue of (1.5), if I is a graded ideal of $k[X_0, \dots, X_n] = A$ defining X, the ring $k[X_0, \dots, X_n]/I$ is a C.M. ring. Thus, for each maximal ideal $\mathfrak{m}$ of A, we have $\operatorname{pd} A_\mathfrak{m}/IA_\mathfrak{m} = n - d$.

By (4.1), we deduce from it that for each maximal ideal $\mathfrak{m}$ of A, we have $\mathrm{H}^i_{IA_\mathfrak{m}}(\cdot) = 0$ for $i > n - d$. Thus, finally $\mathrm{H}^i_I(\cdot) = 0$ for $i > n - d$, and we conclude thanks to (1.7).

4.3. *Counter-example to proposition (4.2) in characteristic* 0 *(Hartshorne).*

We can construct a projective variety X of dimension 3, embedded in $\mathbb{P}^8_{\mathbb{C}}$, such that the local ring of the peak of the cone of X as above is C.M (and also Gorenstein), and such that there exists a coherent sheaf $\mathcal{F}$ on $P - X$, for which $\mathrm{H}^6(P - X, \mathcal{F}) \neq 0$.

For this we consider the projective variety X obtained by blowing up a point in $\mathbb{P}^3_{\mathbb{C}}$. Let E be the exceptional divisor on X, and let f the homomorphism $X \longrightarrow \mathbb{P}^3_{\mathbb{C}}$. We set

$$\mathcal{L} = f^*(\mathcal{O}_{P^3}(2)) \otimes \mathcal{O}_X(-E) \ .$$

Then:

a) $\mathcal{L}$ is very ample on X, and $\dim \mathrm{H}^0(X, \mathcal{L}) = 9$.

b) $\mathrm{H}^i(X, \mathcal{L}^{\otimes v}) = 0$ for $i = 1, 2$.

c) $\omega_X \simeq \mathcal{L}^{\otimes(-2)}$, where ω_X is the sheaf which appears in the theorem of projective duality.

d) $\mathrm{H}^i(X, \mathbb{C}) \simeq \mathbb{C}^{\nvDash}$ because the Chern classes of the sheaves $f^*(\mathcal{O}_{P^3}(1))$ and $\mathcal{O}_X(E)$ are linearly independent in $\mathrm{H}^2(X, \mathbb{Z}) \longrightarrow \mathrm{H}^{\nvDash}(\mathbb{X}, \mathbb{C})$.

Thus we see that the ring of the cone of X (embedded in P^8) is C.M. We remark that we have given a counter-example thanks to the following theorem, due to Barth, and of which we find a proof in Hartshorne [**19**].

THEOREM 4.4. (Barth) *Let X be a projective (algebraic) variety of dimension s contained in a non singular variety Y, proper over $\mathbb{C}$, of dimension n. Suppose that*

$$\mathrm{H}^i(Y - X, \mathcal{F}) = 0$$

for each coherent sheaf $\mathcal{F}$ on $Y - X$ and each integer $i \geqslant q$, q given. Then, the homomorphism $\mathrm{H}^i(Y, \mathbb{C}) \longrightarrow \mathrm{H}^{\sqsupset}(\mathbb{X}, \mathbb{C})$ is an isomorphism for $i < n - q$.

Here $Y = \mathbb{P}^8_{\mathbb{C}}$ and $\mathrm{H}^2(\mathbb{P}^8, \mathbb{C}) \longrightarrow \mathrm{H}^{\nvDash}(\mathbb{X}, \mathbb{C})$ is not an isomorphism thanks to d).

COROLLARY 4.5. *Let A be a polynomial ring over $\mathbb{C}$, and I an ideal of A. Then in general there does exist a entire morphism $A \xrightarrow{f} A$ such that the ideals $f^n(I)A$ define the I-adic topology.*

Indeed, such a homomorphism will be flat, by the acyclicity lemma. Upon setting $I_n = f^n(I)A$, we will have that for each n, $\mathrm{pd}\, A/I_n = \mathrm{pd}\, A/I$. We will deduce from it $\mathrm{H}^i_I(\cdot) = 0$ for $i > \mathrm{pd}\, A/I$, which is false by virtue of the counter-example which is what we want to see.

Now we will need some supplementary technical lemmas on the Frobenius functor.

The rings considered are noetherian and of characteristic $p > 0$. We will denote, as in chapter I, f the homomorphism of Frobenius and $\mathbf{F}$ the functor of Frobenius.

To simply the notation, we make the following definitions:

4.6. Let A be a ring of characteristic $p > 0$. Then for each A-module M and for each homomorphism ϕ of A-modules, we will set:

$$M_{(i)} = \mathbf{F^i(M)} \quad \textit{and} \quad \phi_{(\mathbf{i})} = \mathbf{F^i}(\phi) \ .$$

PROPOSITION 4.7. *Let A be a noetherian ring and M and A-module of finite type. Consider a finite presentation of M:*

$$A^{r_1} \xrightarrow{\phi} A^{r_0} \longrightarrow M \longrightarrow 0 \ . \tag{*}$$

Then,

$$A^{r_1} \xrightarrow{\phi_{(n)}} A^{r_0} \longrightarrow M \longrightarrow 0 \ . \tag{n*}$$

is a finite presentation of $M_{(n)}$*. Moreover, if* A *is local, of residue field* k*, and if* $r_0 = \operatorname{rank}_k(k \otimes_A M)$ *then* $r_0 = \operatorname{rank}_k(k \otimes_A M_{(n)})$*.*

Clearly, it suffices to prove the proposition for $n = 1$. We apply the functor $\mathbf{F}(\cdot) = \cdot \otimes {}^{\mathbf{f}}\mathbf{A}$ to (*) and we obtain from it the first part of the statement. Let α_{ij} be the coefficients of the matrix ϕ. Then $\phi_{(1)}$ is represented by the matrix α_{ij}^p. If A is local, to say (*) is a minimal presentation of M, is to say that the α_{ij}^p are in the maximal ideal $\mathfrak{m}$ of A. Then, (1*) is a minimal presentation of $M_{(1)}$ and from it we indeed obtain the required result.

Proposition 4.8. *Let* A *be a regular ring. For each* A*-module of finite type* M*, we have isomorphims:*

$$\operatorname{Ext}^i_A\left(M_{(n)}, A\right) \simeq \operatorname{Ext}^i_A\left(M, A_{(n)}\right) \quad \textit{foreachi} \ .$$

Indeed, we have already seen in chapter I §1, that ${}^fA$ is flat over A, thus

$$\operatorname{Ext}^i_A\left(M_{(1)}, A\right) \simeq \operatorname{Ext}^i_{{}^fA}\left(M \otimes_A {}^fA, {}^fA\right) \simeq \operatorname{Ext}^i_A\left(M, A\right) \otimes_A {}^fA \ .$$

Now we have all the material necessary to prove the theorem of finiteness in characteristic $p > 0$.

Theorem 4.9. *Let* R *be a regular local ring of characteristic* $p > 0$*. Let* I *be an ideal in* R *and let* i *be an integer such that:*

1) *For each irreducible component* Y *of* $\operatorname{Spec} R/I$*, we have* $i < \dim Y$*.*
2) *If* U *is the complementary open of a closed point in* $\operatorname{Spec} R$*, then* R/I *restricted to* U *verifies* S_i *(the Serre condition).*

Then, if n *is the dimension of* R*, for each* R*-module* M *of finite type, and each integer* $s \geqslant n - i$*, the local cohomology groups* $\mathrm{H}^s_I(M)$ *are artinian* R*-modules.*

Lemma 4.10. *It suffices to prove that* $\mathrm{H}^s_I(R)$ *is artinian for* $s \geqslant n - i$*.*

We reason by descending induction, which we can do because $\mathrm{H}^s_I(\cdot) = 0$ for $s > n$. Assume that $\mathrm{H}^s_I(R)$ is artinian for $s \geqslant n - i$, and that for each R module M of finite type, $\mathrm{H}^s_I(M)$ is artinian for $s > r \geqslant n - i$. Let N be a R-module of finite type. There exists an exact sequence:

$$0 \longrightarrow K \longrightarrow R^e \longrightarrow N \longrightarrow 0 \ .$$

We deduce from it an exact sequence:

$$\mathrm{H}^r_I(R^e) \longrightarrow \mathrm{H}^r_I(N) \longrightarrow \mathrm{H}^{r+1}_I(K) \ .$$

The two exterior modules of this three term exact sequence being artinian, the same is the case for the central module, and the lemma is proved.

As R is regular, for each integer l, $I_{(l)}$ is an ideal of R. The system of ideals $I_{(l)}$ defining the I-adic topology in R, we have for each integer s,

$$\mathrm{H}^s_I(R) = \varinjlim \operatorname{Ext}^s_R\left(R/I_{(l)}, R\right) \ .$$

Lemma 4.11. *For* $s \geqslant n - i$*, the modules* $\operatorname{Ext}^s_R\left(R/I_{(l)}, R\right)$ *are of finite length for each* $l \geqslant 0$*.*

As R is regular, $R/I_{(l)} = (R/I)_{(l)}$. By (4.8), there is an isomorphism:

$$\operatorname{Ext}^s_R\left((R/I)_{(l)}, R\right) = \operatorname{Ext}^s_R\left(R/I, R\right)_{(l)} .$$

Finally, as (1.4.5), $\operatorname{Ext}^s_R\left(R/I, R\right)_{(l)}$ has the same support as $\operatorname{Ext}^s_R\left(R/I, R\right)$, it suffices to show that $\operatorname{Ext}^s_R\left(R/I, R\right)$ has finite length for $s \geqslant n-i$. Let $\mathfrak{q}$ be a non maximal prime ideal of R. We want to prove that

$$\operatorname{Ext}^s_{R_\mathfrak{q}}\left(R_\mathfrak{q}/IR_\mathfrak{q}, R_\mathfrak{q}\right) = 0 \geqslant n-i .$$

Evidently, we can suppose that $I \subset \mathfrak{q}$, if not there is no problem. We know that we have $\operatorname{depth}(R_\mathfrak{q}/IR_\mathfrak{q}) \geqslant \inf(i, \dim R_\mathfrak{q}/IR_\mathfrak{q})$.

Let $c = \inf_i(\dim Y_i)$, for Y_i irreducible component of R/I. Then the codimension of $R_\mathfrak{q}/IR_\mathfrak{q}$ in $R_\mathfrak{q}$ is $\leqslant n-c$, thus $\dim R_\mathfrak{q}/IR_\mathfrak{q} \geqslant \dim R_\mathfrak{q} - (n-c)$. We deduce from it:

$$\operatorname{depth}(R_\mathfrak{q}/IR_\mathfrak{q}) \geqslant \inf(i, \dim R_\mathfrak{q} - (n-c))$$

thus

$$\operatorname{pd}_{R_\mathfrak{q}}(R_\mathfrak{q}/IR_\mathfrak{q}) \leqslant \sup(\dim R_\mathfrak{q} - i, n-c) ,$$

and as $n-i > \dim R - i$ and $n-i > n-c$, $\operatorname{Ext}^s_{R_\mathfrak{q}}\left(R_\mathfrak{q}/IR_\mathfrak{q}, R_\mathfrak{q}\right) = 0$ for $s \geqslant n-i$.

Lemma 4.12. *Let E be a dualizing module for R (i.e. an injective envelope for the residue field of R); then the following conditions are equivalent:*

1) $\mathrm{H}^s_I(R)$ *is artinian.*
2) $\operatorname{Hom}_R\left(\mathrm{H}^s_I(R), E\right)$ *is a module of finite type over the completion $\widehat{R}$ of R for the maximal ideal.*
3) $\varprojlim \operatorname{Hom}_R\left(\operatorname{Ext}^s_R\left(R/I_{(l)}, R\right), E\right)$ *is an $\widehat{R}$-module of finite type.*

1) $\Leftrightarrow$ 2) is a classical duality. The equivalence 2) $\Leftrightarrow$ 3) is an immediate consequence of the isomorphism of functors: $\varprojlim \operatorname{Hom}_R(\cdot, E) \simeq \operatorname{Hom}_R\left(\varprojlim, E\right)$.

Lemma 4.13. *In the category of R-modules of finite length, we have an isomorphism of functors:*

$$\operatorname{Hom}_R(\cdot, E) \simeq \operatorname{Ext}^n_R(\cdot, R) .$$

Indeed, these two functors are exact and coincide on the residue field k of R. Moreover, this isomorphism is canonical.

Combining the preceeding two lemmas, we assert that to prove the theorem, it suffices to prove that for $s \geqslant n-i$:

$$\varprojlim \operatorname{Ext}^n_R\left(\operatorname{Ext}^s_R\left(R/I_{(l)}, R\right), R\right)$$

is an $\widehat{R}$ module of finite type.

This is an immediate consequence of the lemma and of the proposition which follows:

Lemma 4.14. *Let k be the residue field of R; then the residual rank:*

$$\operatorname{rank}_k\left(k \otimes_R \operatorname{Ext}^n_R\left(\operatorname{Ext}^s_R\left(R/I_{(l)}, R\right), R\right)\right)$$

is independent of l.

By (4.8), there is an isomophism:

$$\mathrm{Ext}^n_R\left(\mathrm{Ext}^s_R\left(R/I_{(l)}, R\right), R\right) \simeq \mathrm{Ext}^n_R\left(\mathrm{Ext}^s_R\left(R/I, R\right), R\right)_{(l)} \ .$$

But, by (4.7), for each R module M of finite type, and for each integer l, we have $\mathrm{rank}_k(k \otimes_R M) = \mathrm{rank}_k(k \otimes_R M_{(l)})$.

Proposition 4.15. *Let A be a local ring of residue field k. Let (M_l) a projective system of R-modules of finite length, of bounded residual rank (i.e. such that $\mathrm{rank}_k(k \otimes_A M) \leqslant C$). Then $\varprojlim M_l$ is a module of finite type over the completion $\widehat{A}$ of A for the maximal ideal.*

For each l, we set $P_l = \cap_{l' \geqslant l} \mathrm{Im}(M_{l'} \to M_l)$. As the modules M_l are of finite length, the projective system verifies the Mittag-Leffler condition, that is that we have $P_l = \mathrm{Im}(M_{l'} \to M_l)$ for l' very big. We deduce from it $\mathrm{rank}_k(k \otimes_A P_l) \leqslant C$ for each l. We know that the modules P_l form a surjective projective system such that $\varprojlim P_l = \varprojlim M_l$. Thus, we can replace the system M_l by the system P_l, said otherwise, we can suppose that the projective system M_l is surjective. Therefore the sequence $\mathrm{rank}_k(k \otimes_A M_l)$ is increasing and bounded. Thus after "forgetting" a finite number of M_l, we can suppose that $\mathrm{rank}_k(k \otimes_A M_l)$ is constant equal to η. That is so say that we can suppose that the surjective homomorphisms:

$$M_{l'} \to M_l \qquad (l' \geqslant l) \tag{*}$$

defines isomorphisms

$$k \otimes_A M_{l'} \simeq k \otimes_A M_l \ .$$

We consider a surjection

$$A^\eta \to M_l \to 0 \ .$$

We can then lift the map (*) to a commutative diagram:

$$\begin{tikzcd} A^\eta \arrow[r] & M_l \arrow[r] & 0 \\ & M_{l'} \arrow[u] & \end{tikzcd}$$

By tensoring with k, we obtain a commutative diagram:

$$\begin{tikzcd} k^\eta \arrow[r, "\simeq"] & k \otimes_A M_l \arrow[r] & 0 \\ & k \otimes_A M_{l'} \arrow[u, "\simeq"'] & \end{tikzcd}$$

which shows that $k^\eta \to k \otimes_A M_{l'}$ is an isomorphism, thus that $A^\eta \to M_{l'}$ is a surjection. Thus we have a commutative diagram:

$$\begin{tikzcd} & 0 & \\ A^\eta \arrow[r] & M_l \arrow[r] \arrow[u] & 0 \\ A^\eta \arrow[r] \arrow[u, equal] & M_{l'} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

There exists an increasing sequence of integers $c(l)$ such that if $\mathfrak{m}$ is the maximal ideal of A, we have $\mathfrak{m}^{c(l)}M_l = 0$. Thus we can factorize the preceeding diagram into a commutative diagram:

$$\begin{tikzcd}
 & 0 & 0 & \\
 & A^\eta/\mathfrak{m}^{c(l)}A^\eta \arrow[u] \arrow[r] & M_l \arrow[u] \arrow[r] & 0 \\
A^\eta \arrow[r] \arrow[d, equal] & & & \\
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 & 0 & \\
A^\eta \arrow[r] & A^\eta/\mathfrak{m}^{c(l)}A^\eta \arrow[u] \arrow[r] & M_l \arrow[u] \arrow[r] & 0 \\
A^\eta \arrow[r] \arrow[u, equal] & A^\eta/\mathfrak{m}^{c(l')}A^\eta \arrow[u] \arrow[r] & M_{l'} \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

Let $K_l = \operatorname{Ker}((A^\eta/\mathfrak{m}^{c(l)}A^\eta) \to M_l)$.

We obtain a projective system of exact sequences:

$$0 \to K_l \to A^\eta/\mathfrak{m}^{c(l)}A^\eta \to M_l \to 0\ .$$

All the modules described here being of finite length, the projective system (k_l) verifies the Mittag-Leffler condition.

Thus we obtain an exact sequence:

$$0 \to \varprojlim K_l \to \varprojlim A^\eta/\mathfrak{m}^{c(l)}A^\eta \to \varprojlim M_l \to 0\ .$$

Thus we have a surjection of $\widehat{A}$-modules $\widehat{A}^\eta \to \varprojlim M_l \to 0$ and the proposition is proved.

As a corollary of the local theorem of finiteness in characteristic $p > 0$, we obtain a global theorem of finiteness in characteristic $p > 0$, stronger than the general conjecture by Grothendieck. This theorem, or a result very similar, is no doubt already proved by R. Hartshorne, by geometric methods.

Theorem 4.16. *Let X be a closed sub-scheme of the projective space $P = \mathbb{P}^n_k$ over a field k of characteristic p. Let X_j be the irreducible components of X, and let $d = \inf \dim X_j$. Suppose that X is S_i with $i \leqslant d$ (i.e. for each $x \in X$, $\operatorname{depth} \mathcal{O}_{X,x} \geqslant \inf(i, \dim \mathcal{O}_{X,x})$).*

Then $\mathrm{H}^s(P - X, \mathcal{F})$ is a k vector space of finite type, for each coherent sheaf $\mathcal{F}$ over $P - X$, and for each integer $s \geqslant n - i$.

Moreover, for the same values of s, $\mathrm{H}^s(P - X, \mathcal{F}(r)) = 0$ for r very big.

This theorem is deduced from theorem (4.9) thanks to propositions (1.7) and (1.8).

5. On the penultimate local cohomology group

Theorem 5.1. *Let R be a complete regular local ring of dimension d, with residue field separably closed. Let U be a complementary open of a closed point in $\operatorname{Spec} R$, and let I the ideal of R, such that $\mathrm{V}(I) \cap U$ is connected of dimension $\geqslant 1$.*

The following conditions are equivalent:

(1) *$\mathrm{H}^s_I(M)$ is an artinian R-module for $s \geqslant d - 1$ and for each R-module of finite type M.*
(2) *$\mathrm{H}^s_I(R)$ is an artinian R-module for $s \geqslant d - 1$.*
(3) *$\mathrm{H}^s_I(M) = 0$ for $s \geqslant d - 1$ and for each R-module M.*
(4) *$\mathrm{H}^s_I(R) = 0$ for $s \geqslant d - 1$.*
(5) *$\mathrm{H}^0(\widehat{U}, \mathcal{O}_{\widehat{U}})$ is finte over R, where $\widehat{U}$ is the completion of U along $V(I) \cap U$.*

(6) *The canonical homomorphism* $R \longrightarrow \mathrm{H}^0(\widehat{U}, \mathcal{O}_{\widehat{U}})$ *is an isomorphism.*

Conditions 1), 2) and 5) are equivalent by (2.3). The equivalence of conditions 3),4) and 6) is a consequence of the exact sequence of (2.2), taking into account the local theorem of Lichtenbaum which asserts here $\mathrm{H}^d_I(\cdot) = 0$. As we obviously have 6) $\Rightarrow$ 5), it remains to prove 5) $\Rightarrow$ 6).

We know that we have:

$$\mathrm{H}^0(\widehat{U}, \mathcal{O}_{\widehat{U}}) = \varprojlim{}_{x \in U \cap \mathrm{V}(I)} \mathcal{O}_{\widehat{U},x} \ . \tag{*}$$

We call that if $x \in U \cap \mathrm{V}(I)$, and if $\mathfrak{q}_x$ is the prime ideal of R corresponding to x, then $\mathcal{O}_{\widehat{U},x}$ is a local ring, the completion of which is $\widehat{R_{\mathfrak{q}_x}}$, completion of the local ring $R_{\mathfrak{q}_x}$ for the ideal $\mathfrak{q}_x R_{\mathfrak{q}_x}$. We deduce from it that all the arrows appearing in the projective limit (*) are injective.

Lemma 5.2. *For each* $x \in U \cap \mathrm{V}(I)$*, the homomorphism* $\mathrm{H}^0(\widehat{U}, \mathcal{O}_{\widehat{U}}) \longrightarrow \mathcal{O}_{\widehat{U},x}$ *is injective.*

Evidently, it suffices to show that if α is a non zero element of $\mathrm{H}^0(\widehat{U}, \mathcal{O}_{\widehat{U}})$, its image in $\mathcal{O}_{\widehat{U},x}$ is non zero for each $x \in U \cap \mathrm{V}(I)$.

As $\alpha \neq 0$, there exists $x \in U \cap \mathrm{V}(I)$ such that the image of α in $\mathcal{O}_{\widehat{U},x}$ is non zero. Let $y \in U \cap \mathrm{V}(I)$ and let $\mathfrak{q}_x$ and $\mathfrak{q}_y$ be prime ideals of R corresponding to x and y. As $U \cap \mathrm{V}(I)$ is connected, there exists $z_1, \ldots, z_s \in U \cap \mathrm{V}(I)$ having for corresponding prime ideals $\mathfrak{q}_{z_1}, \ldots, \mathfrak{q}_{z_s}$ such that if we consider the sequence of prime ideals $\mathfrak{q}_x, \mathfrak{q}_{z_1}, \ldots, \mathfrak{q}_{z_s}, \mathfrak{q}_y$, there is always a inclusion relation between two successive ideals. We deduce from it that if the image of α in $\mathcal{O}_{\widehat{U},x}$ is non zero, then the image of α in $\mathcal{O}_{\widehat{U},y}$ is non zero. Thus the lemma is proved.

Lemma 5.3. $\mathrm{H}^0(\widehat{U}, \mathcal{O}_{\widehat{U}})$ *is a finite integrally closed ring over* R*.*

As for $x \in U \cap \mathrm{V}(I)$, the completion of $\mathcal{O}_{\widehat{U},x}$ is a regular ring, $\mathcal{O}_{\widehat{U},x}$ is intregrally closed. Let α be an element in the fraction field of $\mathrm{H}^0(\widehat{U}, \mathcal{O}_{\widehat{U}})$, integral over $\mathrm{H}^0(\widehat{U}, \mathcal{O}_{\widehat{U}})$. By (4.2),this is an element of the fraction field of $\mathcal{O}_{\widehat{U},x}$, thus $\alpha \in \mathcal{O}_{\widehat{U},x}$ and this for each $x \in U \cap \mathrm{V}(I)$. We deduce from it that $\alpha \in \mathrm{H}^0(\widehat{U}, \mathcal{O}_{\widehat{U}})$. The fact that $\mathrm{H}^0(\widehat{U}, \mathcal{O}_{\widehat{U}})$ is finite over R is nothing other than the hypothesis.

Lemma 5.4. $\operatorname{Spec} \mathrm{H}^0(\widehat{U}, \mathcal{O}_{\widehat{U}})$ *is a etale covering of* $\operatorname{Spec} R$*.*

As R is a regular ring, by the theorem of purity, it suffices to prove that each prime ideal of height 1 of the ring $A = \mathrm{H}^0(\widehat{U}, \mathcal{O}_{\widehat{U}})$ is etale over R. For each $x \in U \cap \mathrm{V}(I)$, let $\mathfrak{q}_x$ be the corresponding prime ideal of R. We set:

$$\mathfrak{p}_x = A \cap \mathfrak{q}_x \mathcal{O}_{\widehat{U},x} = A \cap \mathfrak{q}_x \widehat{R_{\mathfrak{q}_x}} \ .$$

The injections $R_{\mathfrak{q}_x} \hookrightarrow A_{\mathfrak{p}_x} \hookrightarrow \mathcal{O}_{\widehat{U},x} \hookrightarrow \widehat{R_{\mathfrak{q}_x}}$ show that $\mathfrak{p}_x$ is etale over R. As $A = \varprojlim{}_{x \in U \cap \mathrm{V}(I)} \mathcal{O}_{\widehat{U},x}$, we have $A = \cap_{x \in U \cap \mathrm{V}(I)} A_{\mathfrak{p}_x}$.

We deduce from it that each prime ideal of height 1 of A is contained in $\mathfrak{p}_x$. As the collection of points where the morphism $\operatorname{Spec} A \longrightarrow \operatorname{Spec} R$ is etale, is open, we deduce from it that each prime ideal of height 1 of A is etale over R, and the lemma is proved.

But, as R is complete with separably closed residue field, the only connected etale covering of R is R itself, thus the homomorphism $R \longrightarrow \mathrm{H}^0(\widehat{U}, \mathcal{O}_{\widehat{U}})$ is indeed an isomorphism and the theorem is proved.

Corollary 5.5. *Let R be a complete local ring of characteristic $p > 0$, of dimension d, with separably closed residue field. Let U be the complementary open of a closed point of $\operatorname{Spec} R$, and let I an ideal of R such that $\mathrm{V}(I) \cap U$ is connected of dimension $\geqslant 1$. Then:*

1) *The functors $\mathrm{H}_I^{d-1}(\cdot)$ and $\mathrm{H}_I^d(\cdot)$ are zero.*
2) *If $\widehat{U}$ is the formal completion of U along $\mathrm{V}(I) \cap U$, the canonical homomorphism $R \longrightarrow \Gamma(\widehat{U}, \mathcal{O}_{\widehat{U}})$ is an isomorphism.*

By the local theorem of Lichtenbaum $\mathrm{H}_I^d(\cdot) = 0$.

As for each irreducible component Y of $\operatorname{Spec} R/I$ we have $1 < \dim Y$, replacing I be the itersection of prime ideals containing it, we can apply the local theorem of finiteness in characteristic $p > 0$ (4.9), thus $\mathrm{H}_I^{d-1}(R)$ is an artinian R-module. But by the theorem (5.1), this implies $\mathrm{H}_I^d(\cdot) = 0$, as well as property 2) which is nothing but property 6) of (5.1).

Finally, we give the global corollary of this local result.

Corollary 5.6. *Let k be a separably closed field of characteristic $p > 0$. Let X be a closed sub-scheme of dimension $\geqslant 1$ of the projective space $P = \mathbb{P}_k^n$. Then for each quasi-coherent sheaf $\mathcal{F}$ on $P - X$, we have:*

$$\mathrm{H}^{n-1}(P - x, \mathcal{F}) = 0 \ .$$

We already know, by the theorem of Lichtenbaum, that $\mathrm{H}^n(P - x, \mathcal{F}) = 0$. Each quasi-coherent sheaf being an inductive limit of coherent sheaves, it suffices to show that $\mathrm{H}^{n-1}(P - x, \mathcal{F}) = 0$ for each coherent sheaf $\mathcal{F}$ on $P - X$. But this result is deduced from the preceeding local result exactly as the global theorem of finiteness is deduced from the local theorem of finiteness.

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