Commutative algebra and representations of finite groups

Srikanth Iyengar

University of Nebraska, Lincoln

Genoa, 7th June 2012
Describe a bridge between two, seemingly different, worlds:

Representation theory of finite groups in char $p > 0$
and
Modules over polynomial rings

This has had, and continues to have, an impact on both fields:
Representation theory and commutative algebra.

Another connection: Algebraic topology (via classifying spaces)
I will not touch upon this today.
Outline

Part I - Groups to polynomial rings
- Modular representations of finite groups
- Elementary abelian groups in char 2 - Koszul duality
- Elementary abelian groups in char $p \geq 3$

Part II - Applications
- Testing projectivity
- Classifying modular representations
Based primarily on the following articles:

Homology of perfect complexes

Stratifying modular representations of finite groups
Throughout, $G$ will be a finite group and $k$ will be a field.

A representation of $G$ is a $k$-vectorspace $V$ with a $G$-action: a map $G \times V \mapsto V$ such that for $g, h \in G$, $u, v \in V$, and $c \in K$

$$g(u + v) = gu + gv \text{ and } g(cv) = cg(v)$$

$$g(hv) = (gh)v \text{ and } 1(v) = v$$

Equivalently, there is a homomorphism $G \rightarrow \text{Gl}_k(V)$ of groups.

The sum of representations $V, W$ is the vectorspace $V \oplus W$ with

$$g(v + w) = gv + gw \text{ for } g \in G, v \in V \text{ and } w \in W.$$
I will focus on finite dimensional representations: \( \text{rank}_k V < \infty \).

Evidently, given a representation \( V \) of \( G \), one can decompose it as

\[
V = \bigoplus_{i=1,\ldots,s} W_i^{e_i} \quad \text{with } e_i \geq 1
\]

where the \( W_i \) are indecomposable and \( W_i \not\cong W_j \) for \( i \neq j \).

This decomposition of \( V \) is essentially unique:

**Theorem (Krull-Remak-Schmidt; Schur in char 0)**

The \( W_i \) that appear and their multiplicities, \( e_i \), depend only on \( V \).

**Up shot:** One can focus on indecomposable representations.
Consider the $k$-vectorspace $\bigoplus_{h \in G} kh$ with $G$ action defined by
\[
g\left(\sum_{h \in G} c_h h\right) = \sum_{h \in G} c_h gh\]
This is the regular representation of $G$.

**Theorem (Maschke)**

*If char $k = 0$, then any indecomposable representation of $G$ is (isomorphic to) a direct summand of the regular representation.*

This result holds also when char $k$ does not divide the order of $G$.

**Corollary**

*G has only finitely many indecomposable representations, in char 0.*

Character theory gives an efficient way to describe them all. In char 0, representation theory thus has a combinatorial flavor.
Modular case

All this fails when char $k$ divides $|G|$; this is the **modular** case.

The **trivial representation** of $G$ is $k$ with $g(c) = c$ for all $g \in G$.

**Example**

$G := \mathbb{Z}/2 = \langle g \mid g^2 = 1 \rangle$ and char $k = 2$. Then the trivial representation is not a direct summand of the regular one.

Reason: The $G$-action on the regular representation, $k \oplus kg$, is given by

$$g \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

If $k$ were a direct summand, this matrix would be diagonalizable, which it is not.

In the example above, the only indecomposable representations are $k$ and the regular one.

This is not the case with the next one.
Kronecker group

\[G = \langle g_1, g_2 \mid g_1^2 = 1 = g_2^2, g_1 g_2 = g_2 g_1 \rangle; \text{ thus } G \cong \mathbb{Z}/2 \times \mathbb{Z}/2.\]

Let \(k\) algebraically closed of char 2.

Kronecker classified all the indecomposable representations of \(G\):

- The trivial representation.
- For each odd integer \(n \geq 3\), there are two indecomposable representation of dimension \(n\).
- For each even integer \(n \geq 2\), there is a family of indecomposable representations parameterized by \(k\).

The two dimensional representations are given by:

\[g_1 \mapsto \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad g_2 \mapsto \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \quad \lambda \in k.\]

Note: For \(G = (\mathbb{Z}/2)^3\), the indecomposables cannot be classified.
In char $p > 0$, it is not possible to classify indecomposable representations; nor is it a meaningful endeavor, in general.

In this context, representation theory has a different flavor (module theoretic aspects dominate), and requires new tools: cohomology.

The group algebra of $G$, over $k$, is the $k$-vectorspace

$$kG := \bigoplus_{g \in G} kg$$

with multiplication induced from $G$.

Observe: This multiplication is commutative $\iff G$ is abelian.

**Example**

For $G = \langle g \mid g^d = 1 \rangle$, the cyclic group of order $d$, one has

$$kG = \bigoplus_{i=0}^{d-1} kg^i \cong k[x]/(x^d - 1)$$
Let $V$ be a representation of $G$. The $G$ action on $V$ extends to $kG$:

$$\left(\sum_{g \in G} c_g g\right) v = \sum_{g \in G} c_g gv$$

In this way, $V$ becomes a module over $kG$.

Representation of $G$ are the same as modules over $kG$.

Regular representation becomes $kG$ viewed as a module over itself.

**Definition**

A $kG$-module $P$ is **projective** if it is a direct summand of a free $kG$-module, i.e. some $(kG)^n$.

Maschke’s theorem can be phrased as: When char $k$ does not divide $|G|$, the order of $G$, every $kG$-module is projective.
Projective modules

In general, Krull-Remak-Schmidt implies:

1. An indecomposable is projective \iff it is a summand of \( kG \).
2. There are only finitely many indecomposable projectives.

**Lemma (char \( k = p > 0 \))**

If \( |G| = p^n \), then \( kG \) is the only indecomposable projective.

In summary: The projectives are the “well-understood” part of modular representation theory.

There are more indecomposables than projectives; recall Kronecker!

There is a useful construction that allows one to focus on the rest. This is the stable module category, and it is the main actor in modular representation theory.
Let $G$ be a finite group and $k$ a field; henceforth char $k$ divides $|G|$. Give $kG$-modules $M$ and $N$, we write:

$\text{Hom}_{kG}(M, N)$ for the $kG$-linear maps $f : M \to N$, and

$\text{PHom}_{kG}(M, N)$ for the $f$ that factor through a projective module:

![Diagram](attachment:image.png)

with $P$ a projective module. This is a $k$-subspace of $\text{Hom}_{kG}(M, N)$. Note: $\text{PHom}_{kG}(M, N) = \text{Hom}_{kG}(M, N)$ if $M$ or $N$ is projective.
The stable module category of $kG$, denoted $\text{stmod} kG$ has for
Objects: all $kG$-modules (finite dimensional over $k$)
Morphisms: Given $kG$-modules $M, N$, set
\[ \text{Hom}_{kG}(M, N) = \text{Hom}_{kG}(M, N)/\text{PHom}_{kG}(M, N) \]
These are the stable maps from $M$ to $N$.

**Lemma**

A $kG$-module $M$ is 0 in $\text{stmod} kG$ $\iff$ $M$ is projective.

In any category an object $X$ is 0 $\iff$ id$_X$, the identity map on $X$ is 0.

**Proof of Lemma.**

id$_M = 0$ in $\text{stmod} kG$ $\iff$ as a map of $kG$-modules it factors as

\[
\begin{array}{c}
\text{P} \\
\downarrow \\
\text{id}_M \\
\downarrow \\
M \\
\rightarrow \\
\downarrow \\
M
\end{array}
\]

with $P$ projective, $\iff$ $M$ is a direct summand of a projective.
Passage to elementary abelian groups

**Definition**

Groups of the form $(\mathbb{Z}/p)^c$ are called **elementary abelian $p$-groups**; the integer $c$ is its **rank**.

Let $H \leq G$ be a subgroup and $M$ be a $kG$-module.

$M \downarrow_H :=$ the $kH$-module obtained by restricting the $G$ action to $H$.

Quillen (1971), Alperin and Evens (1981): The properties of a $kG$-module $M$ are controlled by those of $M \downarrow_E$, where $E$ spans over the elementary abelian $p$-subgroups of $G$; here $p = \text{char } k$.

There are precise results; here is a easily stated special case.

**Theorem (Chouinard, 1972)**

A $kG$-module $M$ is projective if and only if $M \downarrow_E$ is projective for every elementary abelian $p$-subgroup $E \leq G$. 
Group algebra of an elementary abelian

Set \( k[x_1, \ldots, x_c] := \) the ring of polynomials in variables \( x_1, \ldots, x_c \).

**Example**

\( E := \mathbb{Z}/2 \) and char \( k = 2 \). Then \( kE = k \oplus kg \) with \( g^2 = 1 \), so

\[
kE \cong k[x]/(x^2 - 1) \cong k[z]/(z^2) \quad \text{where } z = x - 1
\]

**Fact:** If char \( k = p > 0 \), and \( R \) is a commutative \( k \)-algebra, then

\[
(a + b)^p = a^p + b^p \quad \text{for all } a, b \in R
\]

**Example**

For \( E := (\mathbb{Z}/p)^c \) and char \( k = p > 0 \), one has

\[
kE \cong k[x_1, \ldots, x_c]/(x_1^p - 1, \ldots, x_c^p - 1) \cong k[z_1, \ldots, z_c]/(z_1^p, \ldots, z_c^p)
\]

where \( z_i = x_i - 1 \).
$E = (\mathbb{Z}/2)^c$ and $k$ a field of characteristic 2; thus

$$kE \cong k[z_1, \ldots, z_c]/(z_1^2, \ldots, z_c^2)$$

This is a Koszul algebra. Note that $|z_i| = 0$ for $i = 1, \ldots, c$.

The Koszul dual of $kE$ is the algebra

$$\text{Ext}^*_{kE}(k, k) = k[x_1, \ldots, x_c]$$

with $|x_i| = 1$ for $i = 1, \ldots, c$.

The collection of $kE$-modules and of modules over this polynomial ring are different ways of looking at the “same thing”.

To make this precise, one has to bring in one further ingredient.
Observe that \( \text{Proj}(k[x_1, \ldots, x_c]) = \mathbb{P}^{c-1} \).

A differential sheaf on \( \mathbb{P}^{c-1} \) is a coherent sheaf \( \mathcal{F} \) with a \( \mathcal{O}_{\mathbb{P}^{c-1}} \) linear map \( d: \mathcal{F} \to \mathcal{F}(1) \) such that \( d^2 = 0 \).

\( \text{Diff} \mathbb{P}^{c-1} \) is an abelian category, with obvious notion of morphisms.


There is an equivalence of categories

\[
\text{stmod} kE \xrightarrow{\cong} D^b(\text{Diff} \mathbb{P}^{c-1})
\]

Under this equivalence, \( k \) maps to \( \mathcal{O}_{\mathbb{P}^{c-1}} \).

- The equivalence can be made explicit.
- \( \text{stmod} kE \) involves all \( kE \)-modules; not only the graded-ones.
- This is why we get \( D^b(\text{Diff} \mathbb{P}^{c-1}) \) and not \( D^b(\text{Coh} \mathbb{P}^{c-1}) \).
Elementary abelian groups: general case

\[ E = (\mathbb{Z}/p)^c \text{ and } k \text{ a field of characteristic } p, \text{ so} \]

\[ kE \cong k[z_1, \ldots, z_c]/(z_1^p, \ldots, z_c^p) \]

This is not Koszul algebra if \( p \geq 3 \).

Set \( \mathcal{X} = \text{Proj } k[y_1, \ldots, y_c] \), with \( |y_i| = 2 \) for \( i = 1, \ldots, c \).

\( \mathcal{X} \cong \mathbb{P}^{c-1} \) as spaces, but the grading is important.

**Theorem (Avramov, Buchweitz, - , Miller, 2010)**

There is a functor \( F : \text{stmod} kE \to D^b(\text{Diff } \mathcal{X}) \) such that

\[
F(k) \cong \bigoplus_{i=0}^{c} O_{\mathcal{X}}(-i)^{c \choose i}.
\]

The functor \( F \) is not an equivalence, but comes close.
For example, for any $kE$-module $M$, the following are equivalent:

- $M = 0$ in $\text{stmod}kE$; i.e. $M$ is projective.
- $F(M) = 0$ in $D^b(\Diff X)$; that is, $H(F(M)) = 0$.

This says that the functor $F$ is faithful.

Basic idea: Study modules over $kE$ by passing to $X$, using $F$.

Premise: Sheaves over $X$ are easier to understand.

Remark: One is really dealing with modules over $k[y_1, \ldots, y_c]$!

I will illustrate this technique on a fundamental question:

How to test whether a $kG$-module is projective.

Recall Chouinard: A $kG$ module $M$ is projective $\iff M\downarrow_E$ is projective for each elementary abelian $p$-subgroup $E \leq G$.

So we can focus on modules over elementary abelian groups.

Henceforth: $k$ will be algebraically closed, of char $p > 0$. 
Testing projectivity

\[ kE = k[z_1, \ldots, z_c]/(z_1^p, \ldots, z_c^p). \] For each \( a \in \mathbb{P}^{c-1} \), set

\[ z_a = \sum_{i=1}^{c} a_iz_i \quad \text{where} \quad a = [a_1, \ldots, a_c]. \]

This is well-defined, up to a scalar in \( k \).

**Lemma (Dade, 1978)**

A \( kE \)-module \( M \) is projective \( \iff \) it is projective as a module over the sub-algebra \( k[z_a] \) for each \( a \in \mathbb{P}^{c-1} \).

**Proof.**

\( M \) is projective \( \iff \) it is 0 in \( \text{stmod}kE \).

Recall \( \mathcal{X} = \text{Proj}(k[y_1, \ldots, y_c]) \) and that \( \mathcal{X} \cong \mathbb{P}^{c-1} \). Using \( F \), the desired statement translates to a standard fact:

A sheaf \( F \) on \( \mathbb{P}^{c-1} \) is zero if and only if its restriction, \( F_a \), to each point \( a \in \mathbb{P}^{c-1} \) is zero.

- The proof is something of a gross over-simplification!
- It is not Dade’s original proof.
$kE = k[z_1, \ldots, z_c]/(z_1^p, \ldots, z_c^p)$. Fix $a = [a_1, \ldots, a_c] \in \mathbb{P}^{n-1}$

Since $(z_a)^p = (\sum_i a_i z_i)^p = \sum_i a_i^p z_i^p = 0$, as $k$-algebras

$$k[z_a] \cong k[t]/(t^p)$$

Note that $z_a$ defines a $k$-linear map on $M$. It is not hard to show:

$M$ free over $k[z_a] \iff z_a$ has maximal rank, $\frac{(p-1)}{p} \text{rank}_k M$

**Example (Kronecker algebra)**

$kE = k[z_1, z_2]/(z_1^2, z_2^2)$. For each $\lambda \in k$, module $M_\lambda$ with action

$$z_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix}$$

Matrix representing $z_a = a_1 z_1 + a_2 z_2$ is

$$\begin{bmatrix} 0 & 0 \\ a_1 + \lambda a_2 & 0 \end{bmatrix}$$

This has maximal rank (i.e. 1) exactly when $[a_1, a_2] \neq [\lambda, -1]$. 
Varieties for modules over $kE$

Dade’s theorem is the starting point of Carlson’s theory of ‘rank varieties’ for modules over $kE$: For any $kE$-module $M$, set

$$
\mathcal{V}_{kE}(M) = \{ a \in \mathbb{P}^{c-1} \mid M \text{ not projective over } k[z_a] \}
$$

By Dade, this is a closed of $\mathbb{P}^{c-1}$ under the Zariski topology.

Example (Kronecker algebra)

$kE = k[z_1, z_2]/(z_1^2, z_2^2)$. Then, by the previous computation

$$
\mathcal{V}_{kE}(M_\lambda) = \{[\lambda, 1] \in \mathbb{P}^1\} \quad \text{for any } \lambda \in k
$$

In particular, $M_\lambda \not\cong M_{\lambda'}$, for $\lambda \neq \lambda'$.

Dade’s Lemma can be strengthened, with the same proof, to

$$
\mathcal{V}_{kE}(M) = \text{Supp } H(F(M))
$$

where Supp $\mathcal{F}$ denotes the support of a (coherent) sheaf $\mathcal{F}$. 
Classifying modular representations

Fix a $kG$-module $M$, where $G$ is a finite group.

Question: What $kG$-modules can one build out of $M$, in $\text{stmod}kG$?

One is allowed the following operations:

1. Finite direct sums: if $N_1, \ldots, N_t$ are built out of $M$, so is $\oplus_i N_i$.
2. Direct summands: if $L \oplus N$ is built out of $M$, so are $L$ and $N$.
3. If $L$ and $N$ are built out of $M$ and $f : L \to N$ is a $kE$-linear map, pick a projective cover $\pi : P \to N$ and consider the exact sequence

$$0 \to \text{Ker}(f, \pi) \to L \oplus P \xrightarrow{(f, \pi)} N \to 0$$

Then $\text{Ker}(f, \pi)$ is built out of $M$; this is the mapping fiber of $f$.

1 and 3 define make $\text{stmod}kG$ a triangulated category.

Definition

$\text{Thick}(M) : = \text{subcategory of } \text{stmod}kG \text{ of modules built out of } M$.

Properties of $M$ are inherited by modules in $\text{Thick}(M)$. 
Let $E$ be an elementary abelian $p$-group, and $k$ a field with $\text{char } k = p$.

**Theorem (Benson, Carlson, Rickard, 1997)**

If $\mathcal{V}_{kE}(M) \subseteq \mathcal{V}_{kE}(N)$, then $M$ is in $\text{Thick}(N)$.

**Proof:** Use $F$ to pass to $D^b(\text{Diff } \mathcal{X})$ and apply the result below. Here $\mathcal{X} = \text{Proj } S$ and $\mathcal{F}, \mathcal{G} \in D^b(\text{Diff } \mathcal{X})$, differential sheaves on $\mathcal{X}$.

**Theorem (Hopkins, 1984; Carlson, Iyengar, 2012)**

If $\text{Supp } H(\mathcal{F}) \subseteq \text{Supp } H(\mathcal{G})$, then $\mathcal{F} \in \text{Thick}(\mathcal{G})$.

- Again, I am over-simplifying but the proof is new and simpler.
- By Quillen theory, one can extend the BCR theorem to any $G$.
- This is remarkable, for it means that properties of $kG$-modules are (to a large extent) controlled by their support.
A more refined picture

\[ kE \cong k[z_1, \ldots, z_c]/(z_1^p, \ldots, z_c^p) \]

\[ S = k[y_1, \ldots, y_c], \text{ with } |y_i| = 2 \text{ for } i = 1, \ldots, c \]

\[ \mathcal{X} = \text{Proj } S \]

The functor \( F \) is induced by a functor \( \tilde{F} \):

\[
\begin{array}{ccc}
\text{stmod}(kE) & \xleftarrow{\text{ }} & D^b(kE) \\
\downarrow F & & \downarrow \tilde{F} \\
D^b(\text{Diff } \mathcal{X}) & \xleftarrow{\text{ }} & D^b(S)
\end{array}
\]

\( D^b(kE) := \text{derived category of finite dimensional } kE\text{-modules} \)

\( D^b(S) := \text{derived category of finitely generated dg } S\text{-modules} \)

The horizontal arrows are quotient functors (a la Verdier):

the top one is a quotient by projective \( kE \)-modules;

the lower one is a quotient by dg modules with finite homology.
A closer look at $\tilde{F}$

$K :=$ Koszul complex on $z_1, \ldots, z_c$ over $kE$, as a dg algebra.

$\Lambda :=$ exterior algebra (over $k$) on $\zeta_1, \ldots, \zeta_c$ with $|\zeta_i| = -1$, viewed as a dg algebra with $d^\Lambda = 0$.

$$
\begin{array}{ccc}
  kE & \xrightarrow{i} & K \\
  \downarrow^q & & \downarrow \simeq \\
  \Lambda & \xleftarrow{\text{Koszul pair}} & S
\end{array}
$$

These give rise to exact functors of triangulated categories:

$$
\begin{array}{ccc}
  D^b(kE) & \xrightarrow{K \otimes kE} & D^b(K) \\
  \phantom{\xrightarrow{K \otimes kE}} & \equiv & \phantom{\equiv} \\
  D^b(\Lambda) & \equiv & D^b(S)
\end{array}
$$

$S$ is also viewed as a dg algebra with $d^S = 0$.

$D^b(K) :=$ derived category of dg $K$-modules with finitely generated cohomology.

$D^b(\Lambda)$ and $D^b(S)$, the corresponding derived categories.
The functors $F$ and $\tilde{F}$ are shadows of one on a larger category:

$K(\text{Free } kE) := \text{homotopy category of complexes of free } kE\text{-modules}$

$D^b(S) := \text{full derived category of dg } S\text{-modules}$

Then there is a commutative diagram:

$$
\begin{array}{c}
\text{stmod}(kE) \leftrightarrow D^b(kE) \leftrightarrow K(\text{Free } kE) \\
\downarrow F \quad \downarrow \tilde{F} \quad \\
D^b(Diff X) \leftrightarrow D^b(S) \leftrightarrow D(S)
\end{array}
$$

For deeper applications, one needs the rightmost functor.

Final remark: $kE$ is a (very special) complete intersection ring.

The diagram above remains valid when $kE$ is replaced by an arbitrary local complete intersection ring.


