Log-Canonical Coordinates for Poisson Brackets

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Outline

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Remark: For $f \in P$, the map $\mathrm{ad}_f = \{f, -\}$ is a derivation of the associative and Lie products.

Poisson Manifolds

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A manifold M is a *Poisson manifold* if $C^\infty(M)$ is a Poisson algebra.

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$M = \mathbb{R}^2$ with coordinates x, y, with bracket defined by

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Definition

For each $f \in C^\infty(M)$, since $\mathrm{ad}_f = \{f, -\}$ is a derivation of $C^\infty(M)$, it induces a vector field X_f. We call this the "Hamiltonian vector field" of f.

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$$\begin{aligned}\{f,g\} &= \sum_{i,j} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\varphi_{ij} \\ &= \sum_{i<j}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right)\varphi_{ij}\end{aligned}$$

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$M = \mathbb{R}^{2n}$ with coordinates $x_1, y_1, \ldots, x_n, y_n$, and

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Symplectic Manifolds (continued...)

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Poisson Bracket

Define a bracket on $C^\infty(M)$ by:

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Darboux's Theorem

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These are called *Darboux coordinates* or *canonical coordinates*.

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So...in this particular example, the answer is "yes".

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Cluster Algebras: Any cluster algebra $\mathcal{A}$ has a natural Poisson bracket for which the cluster variables form a log-canonical coordinate system.

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So...the bracket on R_ω is already "as simple as possible".

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Lemma

$\{x^I, x^J\} = \omega(I, J) x^{I+J}$

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But...ω is skew-symmetric, so $I + J = 0 \implies \omega(I, J) = 0$.

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This contradicts **Theorem 1**.

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This contradicts **Theorem 1**. So all ad_{f_i} must be identically zero.

Thank You!

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