Finiteness of Frobenius Test Exponents

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Conventions and Notation

Throughout, $(R, \mathfrak{m})$ will be a local ring of dimension d and of prime characteristic $p > 0$. (Local will include the hypothesis Noetherian.) A *parameter ideal* in R will be an ideal generated by a full system of d parameters.

Let $x_1, \cdots, x_t \in R$ be a sequence of elements. The symbol $\underline{x}$ will be used to indicate the list of elements, for example $J = (\underline{x}) = (x_1, \cdots, x_t)$. If a sequence (i.e. order matters) of generators of J is mentioned, then $J_i = (x_1, \cdots, x_i)$.

Similarly, letting $n_1, \cdots, n_t \in \mathbb{Z}^t$, we may use $\underline{n}$ for the list and $\underline{x}^{\underline{n}} = x_1^{n_1}, \cdots, x_t^{n_t}$. In particular, if $n_1 = \cdots = n_t = n$ for some $n \in \mathbb{N}$, we write $\underline{x}^n = x_1^n, \cdots, x_t^n$.

Frobenius closure of ideals

Definition:

Let $I \subset R$ be an ideal. The **Frobenius closure of** I is:

$$I^F = \left\{ x \in R \mid x^{p^e} \in I^{[p^e]} \quad \forall e \gg 0 \right\}.$$

Note $\bullet^F$ is a closure operation, i.e. $I \subset I^F$, and $(I^F)^F = I^F$.

Since R is Noetherian, there must be an $e_0 \in \mathbb{N}$ such that $(I^F)^{[p^{e_0}]} = I^{[p^{e_0}]}$. Call the smallest such e_0 the **Frobenius test exponent for** I, denoted $\mathrm{Fte}(I)$. Then, if $e = \mathrm{Fte}(I)$, $x \in I^F$ if and only if $x^{p^e} \in I^{[p^e]}$.

Frobenius test exponents

It would be nice to know that there is a fixed bound for $\mathrm{Fte}(I)$ depending only on R. Unfortunately, this is not the case – Brenner [Bre06] showed that one cannot expect this property even in nice rings of low dimension.

However Katzman and Sharp showed that if R is Cohen-Macaulay and if we restrict our attention to parameter ideals, we can find a **uniform Frobenius test exponent for (parameter ideals of)** R, denoted $\mathrm{Fte}(R)$.

Finiteness results, special cases

It is natural to ask when $\text{Fte}(R) < \infty$. The following results are known.

- Katzman and Sharp [KS06] Cohen-Macaulay rings
- Huneke, Katzman, Sharp, and Yao [HKSY06] Generalized Cohen-Macaulay rings
- Quy [Quy18] Generalized Cohen-Macaulay rings (simplified proof), weakly F-nilpotent rings
- - [Mad18] Generalized weakly F-nilpotent rings

All known results depend on uniform vanishing or nilpotence of local cohomology modules.

Regular and filter regular sequences

Recall: A sequence of elements $\underline{x} = x_1, \cdots, x_t$ is a regular sequence if and only if $((x_1, \cdots, x_i) :_R x_{i+1})/(x_1, \cdots, x_i) = 0$ for each i.

Definition:

A sequence of elements $\underline{x} = x_1, \cdots, x_t$ is a **filter regular sequence** if $((x_1, \cdots, x_i) :_R x_{i+1})/(x_1, \cdots, x_i)$ is finite length as an R-module for each i.

Filter regular systems of parameters allow us to mimic proofs that are trivial in Cohen-Macaulay rings when your ring is not Cohen-Macaulay.

Filter regular sequences

An equivalent characterization of filter regularity is that $\mathrm{Ass}_R(((x_1, \cdots, x_i) :_R x_{i+1})/(x_1, \cdots, x_i)) \subset \{\mathfrak{m}\}$, or that for each i, $x_i \notin \mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Ass}_R(R/(x_1, \cdots, x_{i-1})) \setminus \{\mathfrak{m}\}$. Powers of filter regular sequences are also filter regular, and in particular if $\underline{x} = x_1, \cdots, x_t$ is filter regular, then for any $e \in \mathbb{N}$ $\underline{x}^{p^e}$ is a filter regular sequence.

Proposition:

If $\mathfrak{q} \subset R$ is a parameter ideal, then we can choose a filter regular system of parameters to generate $\mathfrak{q}$.

Proof.

Carefully avoid the necessary primes. □

Filter regular sequences

Proposition:

Let $\underline{x} = x_1, \cdots, x_t$ be a filter regular sequence and $J = (\underline{x})$ and let $I \subset R$ be any ideal. Then for any $0 \leq i \leq t$ and any $j > 0$ we have:

$$H_I^j(R/(J_{i-1} :_R x_i)) \simeq H_I^j(R/J_{i-1})$$

Consequently, for any $0 < i + j < t$ we have an exact sequence:

$$\cdots \longrightarrow H_I^j(R/J_{i-1}) \xrightarrow{\cdot x_i} H_I^j(R/J_{i-1}) \longrightarrow H_I^j(R/J_i) \longrightarrow H_I^{j+1}(R/J_{i-1}) \xrightarrow{\cdot x_i} \cdots$$

Most useful case: $\underline{x} = x_1, \cdots, x_d$ a filter regular system of parameters, $\mathfrak{q} = (\underline{x})$, $I = \mathfrak{m}$.

Frobenius actions on modules

Definition:

Let M and N be R-modules, and let $\varphi : M \to N$ be a $\mathbb{Z}$-linear map. Say φ is p^e**-linear** for some $e \in \mathbb{N}$ if $\varphi(xm) = x^{p^e}\varphi(m)$ for all $x \in R$, $m \in M$. Clearly when $e = 0$, φ is R-linear. Call a p-linear endomorphism $f : M \to M$ a **Frobenius action on** M.

A Frobenius action on M can also be viewed as a left module structure over a non-commutative ring $R[F]$, which is freely generated on the left as an R-module by $\{X^e\}_{e\in\mathbb{N}}$ but on the right, $Xr = r^pX$.
The dual notion of right module structures over $R[F]$ are called Cartier actions and are also well-studied.

Examples of Frobenius actions

Examples:

- $F^e : R \to R$ Frobenius is p^e-linear.
- $f_R : R/J \to R/J^{[p]}$ by $r + J \mapsto r^p + J^{[p]}$ is p-linear

Example:

Consider the Čech cocomplex on any set of generators $\underline{x} = x_1, \cdots, x_t \in R$ with $I = (\underline{x})$. Each entry of the cocomplex

$$\check{C}^j(\underline{x}, R) = \oplus R_{x_{i_1} \cdots x_{i_j}}$$

has a canonical Frobenius action and the differential is given by signed linear combinations of the entries, which is preserved by p^eth powers. Hence the cohomology $H^j_I(R)$ has a canonical Frobenius action.

HSL numbers

Definition:

Let M be an R-module with a Frobenius action f. Then we define the **Frobenius orbit closure of zero** to be the R-submodule:

$$0_M^f = \bigcup_{e \in \mathbb{N}} \ker(f^e : M \to M).$$

The **Hartshorne-Speiser-Lyubeznik number of** M is:

$$\mathrm{HSL}(M) = \inf\{e \in \mathbb{N} \mid 0_M^f = \ker(f^e)\} \in \mathbb{N} \cup \{\infty\}.$$

Warning – does not agree with Frobenius closure of $I \subset R$. In this sense, the Frobenius orbit closure of I is $\sqrt{I}$.

HSL numbers

If M is Noetherian, it's clear that $\mathrm{HSL}(M) < \infty$. Another vital case is known:

Theorem: [HS77],[Lyu97],[Sha07]

Let A be an Artinian R-module with a Frobenius action. Then $\mathrm{HSL}(A) < \infty$.

Remark:

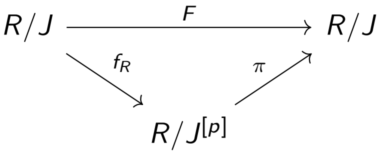
If we (re)define:

$$\mathrm{HSL}(R) := \max\{\mathrm{HSL}\left(H^j_{\mathfrak{m}}(R)\right) \mid 0 \leq j \leq d\},$$

then the theorem shows $\mathrm{HSL}(R) < \infty$ for any local ring R.

Relative Frobenius actions on local cohomology

The Frobenius endomorphism $F : R/J \to R/J$ can be factored as follows:



where $f_R(x + J) = x^p + J^{[p]}$. This p-linear map is the **relative Frobenius map**, which induces a p-linear map for all $I \subset R$:

$$f_R : H^j_I(R/J) \to H^j_I\left(R/J^{[p]}\right)$$

called the **relative Frobenius action on local cohomology**.

Recovers Frobenius closure:

$$\cup_e \ker(f_R^e : R/J \to R/J^{[p^e]}) = J^F/J.$$

Relative Frobenius actions on local cohomology

Definition:

Let $I, J \subset R$ be ideals and let $f_R^e : H_I^j(R/J) \to H_I^j(R/J^{[p^e]})$ be the relative Frobenius action. We define the **relative Frobenius closure of** 0 to be:

$$0^{f_R}_{H_I^j(R/J)} = \bigcup_{e \in \mathbb{N}} \ker \left(f_R^e : H_I^j(R/J) \to H_I^j\left(R/J^{[p^e]}\right)\right),$$

and we define the **relative Hartshorne-Speiser-Lyubeznik number** of $H_I^j(R/J)$ to be:

$$\mathrm{HSL}_R\left(H_I^j(R/J)\right) = \inf\left\{ e \in \mathbb{N} \mid f_R^e\left(0^{f_R}_{H_I^j(R/J)}\right) = 0 \right\}.$$

Relative Frobenius actions on local cohomology

Observation:

Let $\mathfrak{q} \subset R$ be a parameter ideal. Then, $\mathrm{Fte}(\mathfrak{q}) = \mathrm{HSL}_R(H^0_{\mathfrak{m}}(R/\mathfrak{q}))$.

Proof.

Since $\mathfrak{q}$ is $\mathfrak{m}$-primary, $H^0_{\mathfrak{m}}(R/\mathfrak{q}) = R/\mathfrak{q}$. Then $0^{f_R}_{R/\mathfrak{q}} = \mathfrak{q}^F/\mathfrak{q}$. □

This proposition, together with the long exact sequence:

$$\cdots \longrightarrow H^j_{\mathfrak{m}}\left(R/\mathfrak{q}^{[p^e]}_{i-1}\right) \xrightarrow{\cdot x_i^{p^e}} H^j_{\mathfrak{m}}\left(R/\mathfrak{q}^{[p^e]}_{i-1}\right) \xrightarrow{\alpha_e} H^j_{\mathfrak{m}}\left(R/\mathfrak{q}^{[p^e]}_{i}\right) \xrightarrow{\beta_e} H^{j+1}_{\mathfrak{m}}\left(R/\mathfrak{q}^{[p^e]}_{i-1}\right) \xrightarrow{\cdot x_i^{p^e}} \cdots$$

will help us uniformly control $\mathrm{Fte}(\mathfrak{q})$.

Using the filter regular long exact sequence

Proposition:

Let $\mathfrak{q} \subset R$ be a parameter ideal and let $\underline{x} = x_1, \cdots, x_d$ be a filter regular system of parameters such that $\mathfrak{q} = (\underline{x})$. Then, for any $0 \leq i + j \leq d$, the following diagram of long exact sequences in local cohomology commutes:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^j_{\mathfrak{m}}(R/\mathfrak{q}_{i-1}) & \xrightarrow{\cdot x_i} & H^j_{\mathfrak{m}}(R/\mathfrak{q}_{i-1}) & \xrightarrow{\alpha_0} & H^j_{\mathfrak{m}}(R/\mathfrak{q}_i) & \xrightarrow{\beta_0} & H^{j+1}_{\mathfrak{m}}(R/\mathfrak{q}_{i-1}) & \xrightarrow{\cdot x_i} & \cdots \\
& \downarrow f_R^e & & \downarrow f_R^e & & \downarrow f_R^e & & \downarrow f_R^e & & \\
\cdots \longrightarrow & H^j_{\mathfrak{m}}\left(R/\mathfrak{q}_{i-1}^{[p^e]}\right) & \xrightarrow{\cdot x_i^{p^e}} & H^j_{\mathfrak{m}}\left(R/\mathfrak{q}_{i-1}^{[p^e]}\right) & \xrightarrow{\alpha_e} & H^j_{\mathfrak{m}}\left(R/\mathfrak{q}_i^{[p^e]}\right) & \xrightarrow{\beta_e} & H^{j+1}_{\mathfrak{m}}\left(R/\mathfrak{q}_{i-1}^{[p^e]}\right) & \xrightarrow{\cdot x_i^{p^e}} & \cdots
\end{array}$$

Now since α_e commutes with Frobenius, we have:

$$0^{f_R}_{H^j_{\mathfrak{m}}\left(R/\mathfrak{q}_{i-1}^{[p^e]}\right)} \subset \alpha_e^{-1}\left(0^{f_R}_{H^j_{\mathfrak{m}}\left(R/\mathfrak{q}_i^{[p^e]}\right)}\right).$$

A sufficient condition for finiteness

Theorem [-]:

Suppose there is an $e_0 \in \mathbb{N}$ such that for any $e \geq e_0$ and any parameter ideal $\mathfrak{q}$ generated by a filter regular system of parameters $\underline{x} = x_1, \cdots, x_d$, we have the map:

$$\alpha_e : H^j_{\mathfrak{m}}\left(R/\mathfrak{q}^{[p^e]}_{i-1}\right) \to H^j_{\mathfrak{m}}\left(R/\mathfrak{q}^{[p^e]}_{i}\right)$$

has the property:

$$\alpha_e^{-1}\left(0^{f_R}_{H^j_{\mathfrak{m}}\left(R/\mathfrak{q}^{[p^e]}_{i}\right)}\right) = 0^{f_R}_{H^j_{\mathfrak{m}}\left(R/\mathfrak{q}^{[p^e]}_{i-1}\right)}$$

for all $i + j < d$. Then, $\mathrm{Fte}(R) \leq e_0 + \sum_{k=0}^{d} \binom{d}{k} \mathrm{HSL}(H^k_{\mathfrak{m}}(R))$.

Cases satisfying the condition:

Definitions:

- R is **generalized Cohen-Macaulay (gCM)** if $H^j_{\mathfrak{m}}(R)$ is finite length for all $0 \leq j < d$.
- R is **weakly F-nilpotent (wFn)** if $H^j_{\mathfrak{m}}(R) = 0^F_{H^j_{\mathfrak{m}}(R)}$ for each $0 \leq j < d$.
- Say R is **generalized weakly F-nilpotent (gwFn)** if $H^j_{\mathfrak{m}}(R)/0^F_{H^j_{\mathfrak{m}}(R)}$ is finite length for all $0 \leq j < d$.

Notice in each case we have a sort of uniform vanishing in the lower local cohomology modules.

A hierarchy

Finiteness results are known in the following hierarchy of rings in characteristic p:

$$\begin{array}{ccc} \{\text{CM}\} & \subset & \{\text{wFn}\} \\ \cap & & \cap \\ \{\text{gCM}\} & \subset & \{\text{gwFn}\} \end{array}$$

Generalized weakly F-nilpotent rings

Lemma [-]:

Suppose R is generalized weakly F-nilpotent. Then there is an $e_0 \in \mathbb{N}$ such that for any $e \geq e_0$, any parameter ideal $\mathfrak{q}$ generated by a filter regular system of parameters $\underline{x} = x_1, \cdots, x_d$, and any $0 \leq i + j < d$ we have:

$$\mathfrak{q}^{[p^e]} H^j_{\mathfrak{m}}\left(R/\mathfrak{q}_i^{[p^e]}\right) \subset 0^{f_R}_{H^j_{\mathfrak{m}}\left(R/\mathfrak{q}_i^{[p^e]}\right)}.$$

This lemma will help us prove the sufficient condition is satisfied for generalized weakly F-nilpotent rings.

Generalized weakly F-nilpotent rings

Theorem [-]:

Suppose R is generalized weakly F-nilpotent. Then $\mathrm{Fte}(R) < \infty$.

Need to chase this diagram:

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^j_{\mathfrak{m}}(R/\mathfrak{q}_{i-1}) & \xrightarrow{\cdot x_i} & H^j_{\mathfrak{m}}(R/\mathfrak{q}_{i-1}) & \xrightarrow{\alpha_0} & H^j_{\mathfrak{m}}(R/\mathfrak{q}_i) & \xrightarrow{\beta_0} & H^{j+1}_{\mathfrak{m}}(R/\mathfrak{q}_{i-1}) & \xrightarrow{\cdot x_i} & \cdots \\
 & & \downarrow{\scriptstyle f_R^e} & & \downarrow{\scriptstyle f_R^e} & & \downarrow{\scriptstyle f_R^e} & & \downarrow{\scriptstyle f_R^e} & & \\
\cdots & \longrightarrow & H^j_{\mathfrak{m}}\left(R/\mathfrak{q}_{i-1}^{[p^e]}\right) & \xrightarrow{\cdot x_i^{p^e}} & H^j_{\mathfrak{m}}\left(R/\mathfrak{q}_{i-1}^{[p^e]}\right) & \xrightarrow{\alpha_e} & H^j_{\mathfrak{m}}\left(R/\mathfrak{q}_i^{[p^e]}\right) & \xrightarrow{\beta_e} & H^{j+1}_{\mathfrak{m}}\left(R/\mathfrak{q}_{i-1}^{[p^e]}\right) & \xrightarrow{\cdot x_i^{p^e}} & \cdots
\end{array}
$$

This recaptures all previously known finiteness results.

Further questions

Question:

For which other classes of rings can we show $\operatorname{Fte}(R) < \infty$? In particular, can we find methods other than uniform vanishing of local cohomology to study finite test exponents?

Question:

Do generalized weakly F-nilpotent rings have an interesting characteristic 0 analog?

Question:

Is the sufficient condition from before also necessary?

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Thank you!