Secant Ideals of Plücker-embedded Grassmannians

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March 30, 2019

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We let $\mathrm{Sec}_1(I) = I$ and for $r \geq 2$, $\mathrm{Sec}_r(I) = I \star \mathrm{Sec}_{r-1}(I)$. These are called the *secant ideals*.

Geometric Picture

Geometrically, we have $V(I), V(J) \subset \mathbf{k}^n$, then

$$V(I \star J) = \overline{\{x + y \mid x \in V(I),\ y \in V(J)\}}.$$

Interesting choices for I

Ideal of Veronese Embedding
Fix a finite dimensional vector space $\mathbf{V}$,

$$\mathbf{P}(\mathbf{V}) \overset{\mathcal{O}(d)}{\hookrightarrow} \mathbf{P}(\mathrm{Sym}^d(\mathbf{V}))$$

Given by taking the dth tensor power.

Interesting choices for I

Ideal of Segre Embedding
Fix finite dimensional vector spaces $\mathbf{V}_1, \ldots, \mathbf{V}_n$,

$$\mathbf{P}(\mathbf{V}_1) \times \cdots \times \mathbf{P}(\mathbf{V}_n) \stackrel{\mathcal{O}(1,1,\ldots,1)}{\hookrightarrow} \mathbf{P}(\mathbf{V}_1 \otimes \cdots \otimes \mathbf{V}_n).$$

where the map is given by taking the tensor product.

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Ideal of Plücker Embedding

$$\mathbf{Gr}(d,n) \hookrightarrow \mathbf{P}(\bigwedge^d \mathbf{k}^n) \cong \mathbf{P}^N,$$

given by,

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Example

Most basic example that is not just projective space is when $d=2$ and $n=4$, then if we take $v_1 = e_1 + e_2$ and $v_2 = e_3$,

$$\operatorname{span}(e_1+e_2, e_3) \mapsto [e_1 \wedge e_3 + e_2 \wedge e_3] \longleftrightarrow [0:1:0:1:0:0].$$

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The corresponding ideal is defined by a single equation, the Klein quadric. If $x_{i,j}$ are coordinates with $i \neq j$ and $i < j$,

$$x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3}.$$

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1. $r = 1$ (Classical) : Segre embedding is defined by degree 2 equations.

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4. $r = 4$ (Strassen 1983) : need equations of degree 9.

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What can we say about the generators for these ideals and their secant varieties in general?

Some Positive Results

Theorem (Draisma-Kuttler 2011)

Fix $r \geq 1$*, then there exists a constant* $C(r)$ *such that the* r^{th} *secant variety of a Segre embedding of projective spaces is set-theoretically defined by equations of* $\deg \leq C(r)$*.*

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Both of these results rely heavily on topology and hence can only tell us about the ideal up to set-theoretic generators.

Ideal-Theoretic Result

Theorem (Sam 2017)

Over a field of characteristic 0*. Fix* $r \geq 1$*, then there exists a constant* $C(r)$ *such that the* r^{th} *secant variety of a Veronese embedding of projective space is ideal-theoretically defined by equations of* $\deg \leq C(r)$*.*

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Are there ideal theoretic versions of the first two results?

Main Result

Theorem (L. 2018)

Assume $\mathrm{char}(k) = 0$*, fix* $r \geq 1$*, then there exists a constant* $C(r)$ *such that the* r^{th} *secant variety of any Plücker embedded Grassmannian,* $\mathbf{Gr}(d, n) \hookrightarrow \mathbf{P}(\bigwedge^d \mathbf{k}^n)$*, is ideal-theoretically defined by equations of* $\deg \leq C(r)$*.*

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General proof technique:

1. Reduce to considering $\mathbf{Gr}(d, (r+1)d)$ for fixed $r \geq 1$ (Manivel-Michałek 2015).

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3. Argue the collection of all Plücker ideals forms an ideal.

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One can make the same dimension reduction argument. However, it is hard to find the correct operations to place on $\bigoplus_{n,d\geq 0} \mathrm{Sym}^n(\mathbf{k}^{\otimes d})$ that preserve the collection of Segre ideals and make all ideals finitely generated.

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The current techniques rely heavily on a symmetrization argument in which we must be able to divide.

Thank you!!