Ideals with free Koszul homologies

Mohsen Gheibi
University of Texas at Arlington

KUMUNUjr, University of Nebraska-Lincoln, 2019

UNL

Ideals with free Koszul homologies

Throughout R is a commutative Noetherian ring with unitary element.

Throughout R is a commutative Noetherian ring with unitary element.

Definition

Let I be an ideal of R, and let $a_1, \cdots, a_n$ minimally generate I. Then the Koszul complex $\mathrm{K}(I)$ of I over R is the complex

$$\mathrm{K}(I):\ 0 \to R_{e_1\wedge\cdots\wedge e_n} \xrightarrow{\partial_n} \bigoplus_{i_1<\cdots<i_{n-1}} R_{e_{i_1}\wedge\cdots\wedge e_{i_{n-1}}} \xrightarrow{\partial_{n-1}} \cdots \to \bigoplus_{i=1}^{n} R_{e_i} \xrightarrow{\partial_1} R \to 0,$$

Where $\partial_r(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \sum_{j=1}^{r}(-1)^{j-1} a_{i_j} e_{i_1} \wedge \cdots e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \cdots \wedge e_{i_r}$.

Ideals with free Koszul homologies

Throughout R is a commutative Noetherian ring with unitary element.

Definition

Let I be an ideal of R, and let $a_1, \cdots, a_n$ minimally generate I. Then the Koszul complex $\mathrm{K}(I)$ of I over R is the complex

$$\mathrm{K}(I):\ 0 \to R_{e_1\wedge\cdots\wedge e_n} \overset{\partial_n}{\to} \bigoplus_{i_1<\cdots<i_{n-1}} R_{e_{i_1}\wedge\cdots\wedge e_{i_{n-1}}} \overset{\partial_{n-1}}{\to} \cdots \to \bigoplus_{i=1}^{n} R_{e_i} \overset{\partial_1}{\to} R \to 0,$$

Where $\partial_r(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \sum_{j=1}^{r}(-1)^{j-1} a_{i_j} e_{i_1} \wedge \cdots e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \cdots \wedge e_{i_r}$.

We denote $\mathrm{H}_i(I) := \mathrm{H}_i(\mathrm{K}(I))$ the ith Koszul homology of I over R.

Ideals with free Koszul homologies

Throughout R is a commutative Noetherian ring with unitary element.

Definition

Let I be an ideal of R, and let $a_1, \cdots, a_n$ minimally generate I. Then the Koszul complex $\mathrm{K}(I)$ of I over R is the complex

$$\mathrm{K}(I): \ 0 \to R_{e_1 \wedge \cdots \wedge e_n} \stackrel{\partial_n}{\to} \bigoplus_{i_1 < \cdots < i_{n-1}} R_{e_{i_1} \wedge \cdots \wedge e_{i_{n-1}}} \stackrel{\partial_{n-1}}{\to} \cdots \to \bigoplus_{i=1}^{n} R_{e_i} \stackrel{\partial_1}{\to} R \to 0,$$

Where $\partial_r(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \sum_{j=1}^{r} (-1)^{j-1} a_{i_j} e_{i_1} \wedge \cdots e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \cdots \wedge e_{i_r}$.

We denote $\mathrm{H}_i(I) := \mathrm{H}_i(\mathrm{K}(I))$ the ith Koszul homology of I over R.
For every ideal I, $\mathrm{H}_i(I)$ is an R/I-module, $i \geq 0$.

Ideals with free Koszul homologies

Throughout R is a commutative Noetherian ring with unitary element.

Definition

Let I be an ideal of R, and let $a_1, \cdots, a_n$ minimally generate I. Then the Koszul complex $\mathrm{K}(I)$ of I over R is the complex

$$\mathrm{K}(I):\ 0 \to R_{e_1 \wedge \cdots \wedge e_n} \xrightarrow{\partial_n} \bigoplus_{i_1 < \cdots < i_{n-1}} R_{e_{i_1} \wedge \cdots \wedge e_{i_{n-1}}} \xrightarrow{\partial_{n-1}} \cdots \to \bigoplus_{i=1}^{n} R_{e_i} \xrightarrow{\partial_1} R \to 0,$$

Where $\partial_r(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \sum_{j=1}^{r} (-1)^{j-1} a_{i_j} e_{i_1} \wedge \cdots e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \cdots \wedge e_{i_r}$.

We denote $\mathrm{H}_i(I) := \mathrm{H}_i(\mathrm{K}(I))$ the ith Koszul homology of I over R.
For every ideal I, $\mathrm{H}_i(I)$ is an R/I-module, $i \geq 0$.
In this talk, we are interested in ideals I such that $\mathrm{H}_i(I)$ is free over R/I, for all i. Let's call such ideals "FKH" ideals.

Ideals with free Koszul homologies

Example

Let $(R, \mathfrak{m})$ be a local ring, and let I be a proper ideal of R.

1. $\mathfrak{m}$ is a FKH ideal.

Ideals with free Koszul homologies

Example

Let $(R, \mathfrak{m})$ be a local ring, and let I be a proper ideal of R.

1. $\mathfrak{m}$ is a FKH ideal.
2. If I is complete intersection (i.e. generated with a regular sequence) then I is FKH.

Ideals with free Koszul homologies

Example

Let $(R, \mathfrak{m})$ be a local ring, and let I be a proper ideal of R.

1. $\mathfrak{m}$ is a FKH ideal.
2. If I is complete intersection (i.e. generated with a regular sequence) then I is FKH.
3. If I is quasi-complete intersection then I is FKH.

Ideals with free Koszul homologies

Example

Let $(R, \mathfrak{m})$ be a local ring, and let I be a proper ideal of R.

1. $\mathfrak{m}$ is a FKH ideal.
2. If I is complete intersection (i.e. generated with a regular sequence) then I is FKH.
3. If I is quasi-complete intersection then I is FKH.

Definition (Avramov, Henriques, Şega)

An ideal I is called *quasi-complete intersection* if $\mathrm{H}_1(I)$ is free R/I-module, and the natural homomorphism

$$\lambda_* : \bigwedge_* \mathrm{H}_1(I) \longrightarrow \mathrm{H}_*(I)$$

of graded R/I-algebras with $\lambda_1 = \mathrm{Id}_{\mathrm{H}_1(I)}$, is bijective.

Ideals with free Koszul homologies

Example

Let $R = \dfrac{k[w,x,y,z]}{(x^2, y^2, w^2, z^2, wz)}$.

Ideals with free Koszul homologies

Example

Let $R = \dfrac{k[w,x,y,z]}{(x^2, y^2, w^2, z^2, wz)}$.

1. The ideal $I = (x, y)$ is a quasi-complete intersection ideal, and hence, it is FKH.

Ideals with free Koszul homologies

Example

Let $R = \dfrac{k[w,x,y,z]}{(x^2,y^2,w^2,z^2,wz)}$.

1. The ideal $I = (x,y)$ is a quasi-complete intersection ideal, and hence, it is FKH.
2. The ideal $J = (w,z)$ is a FKH ideal but not a quasi-complete intersection.

Ideals with free Koszul homologies

Example

Let $R = \dfrac{k[w,x,y,z]}{(x^2,y^2,w^2,z^2,wz)}$.

1. The ideal $I=(x,y)$ is a quasi-complete intersection ideal, and hence, it is FKH.
2. The ideal $J=(w,z)$ is a FKH ideal but not a quasi-complete intersection.

Example

Let $(R,\mathfrak{m})$ be a fiber product and let $\mathfrak{m} = I \oplus J$. Suppose J is a principal ideal and R/J is a regular ring. Then I is FKH.

Ideals with free Koszul homologies

Example

Let $R = \dfrac{k[w, x, y, z]}{(x^2, y^2, w^2, z^2, wz)}$.

1. The ideal $I = (x, y)$ is a quasi-complete intersection ideal, and hence, it is FKH.
2. The ideal $J = (w, z)$ is a FKH ideal but not a quasi-complete intersection.

Example

Let $(R, \mathfrak{m})$ be a fiber product and let $\mathfrak{m} = I \oplus J$. Suppose J is a principal ideal and R/J is a regular ring. Then I is FKH.

For example, $R = \dfrac{k[w, x, y, z]}{(wx, wy, wz)}$ and $I = (x, y, z)$.

Main Result

Theorem

Let R be a local ring and let I be a FKH ideal of R. Let M be a finite R-module, $g = \min\{\operatorname{grade}_R(I), \operatorname{grade}_R(I, M)\}$ and $n = \nu_R(I)$.

Main Result

Theorem

Let R be a local ring and let I be a FKH ideal of R. Let M be a finite R-module, $g = \min\{\text{grade}_R(I), \text{grade}_R(I, M)\}$ and $n = \nu_R(I)$.

1. *If* $\text{Tor}_i^R(R/I, M) = 0$ *for all* $m \leq i \leq m + n - g$ *then* $\text{Tor}_i^R(R/I, M) = 0$ *for all* $i \geq m$.

Main Result

Theorem

Let R be a local ring and let I be a FKH ideal of R. Let M be a finite R-module, $g = \min\{\operatorname{grade}_R(I), \operatorname{grade}_R(I, M)\}$ and $n = \nu_R(I)$.

1. *If $\operatorname{Tor}_i^R(R/I, M) = 0$ for all $m \leq i \leq m + n - g$ then $\operatorname{Tor}_i^R(R/I, M) = 0$ for all $i \geq m$.*
2. *If $\operatorname{Tor}_{i \gg 0}^R(R/I, M) = 0$ then $\operatorname{Tor}_i^R(R/I, M) = 0$ for all $i > \operatorname{grade}_R(I) - \operatorname{grade}_R(I, M)$.*

Main Result

Theorem

Let R be a local ring and let I be a FKH ideal of R. Let M be a finite R-module, $g = \min\{\mathrm{grade}_R(I), \mathrm{grade}_R(I, M)\}$ and $n = \nu_R(I)$.

1. *If $\mathrm{Tor}_i^R(R/I, M) = 0$ for all $m \leq i \leq m + n - g$ then $\mathrm{Tor}_i^R(R/I, M) = 0$ for all $i \geq m$.*
2. *If $\mathrm{Tor}_{i \gg 0}^R(R/I, M) = 0$ then $\mathrm{Tor}_i^R(R/I, M) = 0$ for all $i > \mathrm{grade}_R(I) - \mathrm{grade}_R(I, M)$.*
3. *If $\mathrm{Ext}_R^i(R/I, M) = 0$ for all $m \leq i \leq m + n - g$ then $\mathrm{Ext}_R^i(R/I, M) = 0$ for all $i \geq m$.*

Main Result

Theorem

Let R be a local ring and let I be a FKH ideal of R. Let M be a finite R-module, $g = \min\{\mathrm{grade}_R(I), \mathrm{grade}_R(I, M)\}$ and $n = \nu_R(I)$.

1. *If* $\mathrm{Tor}_i^R(R/I, M) = 0$ *for all* $m \leq i \leq m + n - g$ *then* $\mathrm{Tor}_i^R(R/I, M) = 0$ *for all* $i \geq m$.
2. *If* $\mathrm{Tor}_{i \gg 0}^R(R/I, M) = 0$ *then* $\mathrm{Tor}_i^R(R/I, M) = 0$ *for all* $i > \mathrm{grade}_R(I) - \mathrm{grade}_R(I, M)$.
3. *If* $\mathrm{Ext}_R^i(R/I, M) = 0$ *for all* $m \leq i \leq m + n - g$ *then* $\mathrm{Ext}_R^i(R/I, M) = 0$ *for all* $i \geq m$.
4. *If* $\mathrm{Ext}_R^{i \gg 0}(R/I, M) = 0$ *then* $\mathrm{Ext}_R^i(R/I, M) = 0$ *for all* $i > \mathrm{grade}_R(I)$.

Corollaries

Example

Let $R = \dfrac{k[x,y]}{(xy)}$ and $I = (x)$. Then I is a FKH ideal with $\mathrm{grade}_R(I) = 0$.
We have $\mathrm{Tor}_1^R(R/I, R/(y)) = 0$ but $\mathrm{Tor}_2^R(R/I, R/(y)) \neq 0$.

Corollaries

Example

Let $R = \dfrac{k[x,y]}{(xy)}$ and $I = (x)$. Then I is a FKH ideal with $\text{grade}_R(I) = 0$.
We have $\text{Tor}_1^R(R/I, R/(y)) = 0$ but $\text{Tor}_2^R(R/I, R/(y)) \neq 0$.

Corollary

Let R be a Cohen-Macaulay local ring and let I be a FKH ideal of R. Then I is a Cohen-Macaulay ideal of R (i.e. R/I is Cohen-Macaulay ring).

Corollaries

Example

Let $R = \dfrac{k[x,y]}{(xy)}$ and $I = (x)$. Then I is a FKH ideal with $\operatorname{grade}_R(I) = 0$.
We have $\operatorname{Tor}_1^R(R/I, R/(y)) = 0$ but $\operatorname{Tor}_2^R(R/I, R/(y)) \neq 0$.

Corollary

Let R be a Cohen-Macaulay local ring and let I be a FKH ideal of R. Then I is a Cohen-Macaulay ideal of R (i.e. R/I is Cohen-Macaulay ring).

Proof.

By replacing R with $\widehat{R}$ we may assume that R admits the canonical module ω_R. By Theorem, we have $\operatorname{Ext}_R^i(R/I, \omega_R) = 0$ for all $i \neq \operatorname{grade}_R(I)$. Then the Local Duality Theorem implies that $\operatorname{H}_{\mathfrak{m}}^i(R/I) = 0$ for all $i \neq \dim R - \operatorname{grade}_R(I) = \dim R/I$. Therefore $\dim R/I = \operatorname{depth} R/I$. □

Corollaries

Definition

Let R be a local ring, and let M be a finitely generated R-module. Then the *complete intersection dimension* of M over R is defined by

$$\text{CI-dim}_R(M) = \inf\{\text{pd}_Q(M \otimes_R R') \mid R \to R' \leftarrow Q \text{ is a quasi-deformation of} R\}$$

Corollaries

Definition

Let R be a local ring, and let M be a finitely generated R-module. Then the *complete intersection dimension* of M over R is defined by

$$\text{CI-dim}_R(M) = \inf\{\text{pd}_Q(M \otimes_R R') | R \to R' \leftarrow Q \text{ is a quasi-deformation of} R\}$$

Definition

An R-module M is called *totally reflexive* (or of *G-dimension zero*) if the evaluation homomorphism $M \to M^{**}$ is an isomorphism and $\text{Ext}^i_R(M, R) = 0 = \text{Ext}^i_R(M^*, R)$ for all $i > 0$.

Corollaries

Definition

Let R be a local ring, and let M be a finitely generated R-module. Then the *complete intersection dimension* of M over R is defined by

$$\text{CI-dim}_R(M) = \inf\{\text{pd}_Q(M \otimes_R R') | R \to R' \leftarrow Q \text{ is a quasi-deformation of} R\}$$

Definition

An R-module M is called *totally reflexive* (or of *G-dimension zero*) if the evaluation homomorphism $M \to M^{**}$ is an isomorphism and $\text{Ext}^i_R(M, R) = 0 = \text{Ext}^i_R(M^*, R)$ for all $i > 0$.

The infimum positive integer n for which there exists an exact sequence

$$0 \to G_n \to \cdots \to G_0 \to M \to 0,$$

of R-modules with each G_i totally reflexive, is called the *Gorenstein dimension* of M denoted $\text{G-dim}(M)$.

Corollaries

Definition

A proper ideal I of R is called *quasi-Gorenstein* if

$$\mathrm{Ext}^i_R(R/I, R) \cong \begin{cases} R/I & i = \mathrm{grade}_R(I), \\ 0 & \text{otherwise.} \end{cases}$$

Corollaries

Definition

A proper ideal I of R is called *quasi-Gorenstein* if

$$\operatorname{Ext}^i_R(R/I, R) \cong \begin{cases} R/I & i = \operatorname{grade}_R(I), \\ 0 & \text{otherwise.} \end{cases}$$

If I is quasi-Gorenstein then $\operatorname{G-dim}_R(I) < \infty$.

Definition

A proper ideal I of R is called *quasi-Gorenstein* if

$$\operatorname{Ext}^i_R(R/I, R) \cong \begin{cases} R/I & i = \operatorname{grade}_R(I), \\ 0 & \text{otherwise.} \end{cases}$$

If I is quasi-Gorenstein then $\operatorname{G-dim}_R(I) < \infty$.

Theorem (Avramov, Henriques, Şega)

1. *If* $\mathrm{H}_1(I)$ *is free* R/I*-module and* $\operatorname{CI-dim}_R(I) < \infty$ *then* I *is a quasi-complete intersection ideal. In particular if* R *is complete intersection then* I *is quasi-complete intersection.*

Corollaries

Definition

A proper ideal I of R is called *quasi-Gorenstein* if

$$\operatorname{Ext}^i_R(R/I, R) \cong \begin{cases} R/I & i = \operatorname{grade}_R(I), \\ 0 & \text{otherwise.} \end{cases}$$

If I is quasi-Gorenstein then $\text{G-dim}_R(I) < \infty$.

Theorem (Avramov, Henriques, Şega)

1. *If* $\mathrm{H}_1(I)$ *is free* R/I*-module and* $\text{CI-dim}_R(I) < \infty$ *then* I *is a quasi-complete intersection ideal. In particular if* R *is complete intersection then* I *is quasi-complete intersection.*
2. *If* I *is a quasi-complete intersection then* I *is a quasi-Gorenstein ideal.*

Corollaries

Definition

A proper ideal I of R is called *quasi-Gorenstein* if

$$\operatorname{Ext}^i_R(R/I, R) \cong \begin{cases} R/I & i = \operatorname{grade}_R(I), \\ 0 & \text{otherwise.} \end{cases}$$

If I is quasi-Gorenstein then $\operatorname{G-dim}_R(I) < \infty$.

Theorem (Avramov, Henriques, Şega)

1. *If* $\mathrm{H}_1(I)$ *is free* R/I*-module and* $\operatorname{CI-dim}_R(I) < \infty$ *then* I *is a quasi-complete intersection ideal. In particular if* R *is complete intersection then* I *is quasi-complete intersection.*
2. *If* I *is a quasi-complete intersection then* I *is a quasi-Gorenstein ideal.*

The proof of (1) uses Gulliksen's Theorem.

Corollaries

Theorem (Gulliksen)

If $\mathrm{H}_1(I)$ *is free* R/I*-module and* $\mathrm{pd}_R(I) < \infty$ *then* I *is complete intersection.*

Corollaries

Theorem (Gulliksen)

If $\mathrm{H}_1(I)$ *is free* R/I*-module and* $\mathrm{pd}_R(I) < \infty$ *then* I *is complete intersection.*

Theorem

Let I *be a FKH ideal of* R*.*

1. *If* $\text{G-dim}_R(I) < \infty$ *then* I *is a quasi-Gorenstein ideal. In particular, if* R *is Gorenstein then* R/I *is a Gorenstein ring.*

Corollaries

Theorem (Gulliksen)

If $\mathrm{H}_1(I)$ *is free* R/I*-module and* $\mathrm{pd}_R(I) < \infty$ *then* I *is complete intersection.*

Theorem

Let I *be a FKH ideal of* R*.*

1. *If* $\text{G-dim}_R(I) < \infty$ *then* I *is a quasi-Gorenstein ideal. In particular, if* R *is Gorenstein then* R/I *is a Gorenstein ring.*
2. *If* $\text{G-dim}_R(I) < \infty$ *then there exists an isomorphism* $\mathrm{H}_i(I) \cong \mathrm{H}_{n-i}(I)$ *where* $n = \nu_R(I) - \mathrm{grade}_R(I)$ *for all* $i \geq 0$*.*

Corollaries

Theorem (Gulliksen)

If $\mathrm{H}_1(I)$ is free R/I-module and $\mathrm{pd}_R(I) < \infty$ then I is complete intersection.

Theorem

Let I be a FKH ideal of R.

1. *If $\text{G-dim}_R(I) < \infty$ then I is a quasi-Gorenstein ideal. In particular, if R is Gorenstein then R/I is a Gorenstein ring.*
2. *If $\text{G-dim}_R(I) < \infty$ then there exists an isomorphism $\mathrm{H}_i(I) \cong \mathrm{H}_{n-i}(I)$ where $n = \nu_R(I) - \mathrm{grade}_R(I)$ for all $i \geq 0$.*

Theorem

Let I be a FKH ideal of R. If $\mathrm{Tor}^R_{i \gg 0}(I, I) = 0$ then I is complete intersection.

Corollaries

Question

Assume $\mathrm{H}_1(I)$ *is free over* R/I *then is* $\mathrm{H}_i(I)$ *free* R/I*-module for all* i*?*

Corollaries

Question

Assume $\mathrm{H}_1(I)$ *is free over* R/I *then is* $\mathrm{H}_i(I)$ *free* R/I*-module for all* i*?*

Proposition

Let R *be a ring and let* I *be a FKH ideal of* R *with* $\text{G-dim}_R(I) < \infty$*. Let* M *be a finite* R*-module. Then* $\mathrm{Tor}^R_{i \gg 0}(R/I, M) = 0$ *if and only if* $\mathrm{Ext}_R^{i \gg 0}(R/I, M) = 0$*.*

Corollaries

Question

Assume $\mathrm{H}_1(I)$ *is free over* R/I *then is* $\mathrm{H}_i(I)$ *free* R/I*-module for all* i*?*

Proposition

Let R *be a ring and let* I *be a FKH ideal of* R *with* $\text{G-dim}_R(I) < \infty$*. Let* M *be a finite* R*-module. Then* $\mathrm{Tor}^R_{i \gg 0}(R/I, M) = 0$ *if and only if* $\mathrm{Ext}_R^{i \gg 0}(R/I, M) = 0$*.*

Corollary

Let $(R, \mathfrak{m})$ *be a Gorenstein local ring and let* M *be a finite* R*-module. Then* $\mathrm{pd}_R M < \infty$ *if and only if* $\mathrm{id}_R M < \infty$*.*

Corollaries

Question

Assume $\mathrm{H}_1(I)$ *is free over* R/I *then is* $\mathrm{H}_i(I)$ *free* R/I*-module for all* i*?*

Proposition

Let R *be a ring and let* I *be a FKH ideal of* R *with* $\operatorname{G-dim}_R(I) < \infty$*. Let* M *be a finite* R*-module. Then* $\operatorname{Tor}^R_{i \gg 0}(R/I, M) = 0$ *if and only if* $\operatorname{Ext}^{i \gg 0}_R(R/I, M) = 0$*.*

Corollary

Let $(R, \mathfrak{m})$ *be a Gorenstein local ring and let* M *be a finite* R*-module. Then* $\operatorname{pd}_R M < \infty$ *if and only if* $\operatorname{id}_R M < \infty$*.*

Proof.

Set $I = \mathfrak{m}$ in the Proposition. □

Thank You!