# POLY-FREE CONSTRUCTIONS FOR RIGHT-ANGLED ARTIN GROUPS

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Abstract. We show that every right-angled Artin group $A\Gamma$ defined by a graph $\Gamma$ of finite chromatic number is poly-free with poly-free length bounded between the clique number and the chromatic number of $\Gamma$. Further, a characterization of all right-angled Artin groups of poly-free length 2 is given, namely the group $A\Gamma$ has poly-free length 2 if and only if there exists an independent set of vertices $D$ in $\Gamma$ such that every cycle in $\Gamma$ meets $D$ at least twice. Finally, it is shown that $A\Gamma$ is a semidirect product of 2 free groups of finite rank if and only if $\Gamma$ is a finite tree or a finite complete bipartite graph. All of the proofs of the existence of poly-free structures are constructive.

## 1. Introduction

A group $G$ is *poly-free* if there exists a finite tower of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_N = G$$

for which each quotient $G_{i+1}/G_i$ is a free group. The least natural number $N$ for which such a tower exists is the *poly-free length* of $G$, denoted $\mathsf{pfl}(G)$. A group $G$ is poly-finitely-generated-free, or *poly-fg-free*, if there exists a tower of this form with the additional property that each of the quotients is a finitely generated free group. Since every map onto a free group splits, a poly-free group can be realized as an iterated semidirect product of free groups [15].

Examples of poly-fg-free groups include certain subgroups of Artin groups. Let $\Gamma$ be a finite simplicial graph; throughout the text we will assume that such graphs do not have loops or multiple edges. If the edges of $\Gamma$ are labeled by integers greater than one, the associated *Artin group* $A\Gamma$ has generators corresponding to the vertices, and relations

$$\underbrace{aba\cdots}_{n \text{ letters}} = \underbrace{bab\cdots}_{n \text{ letters}}$$

where $\{a, b\}$ is an edge of the graph labeled $n$. If, in addition, relations are added making each generator of order 2, the resulting quotient is a Coxeter group. *Braid groups* are the Artin groups whose Coxeter quotients are the symmetric groups. When the Coxeter quotient is finite, the Artin group is

*Date*: June 15, 2007.
*Key words and phrases.* Poly-free, right-angled Artin group, free group.
[1]Supported under NSF grant no. DMS-0071037

said to be of *finite type*. *Pure* Artin groups are subgroups of Artin groups which are the kernel of the homomorphism onto the corresponding Coxeter group.

Pure braid groups are examples of poly-fg-free groups [1], as are pure finite type Artin groups whose Coxeter quotients are of type $B_n$, $D_n$, $I_2(p)$, and $F_4$ [5]. If the graph associated to an Artin group is a tree, Hermiller and Meier [11] have shown that the Artin group is an extension of a finitely generated free group by the integers, and hence is poly-fg-free. Recently, Bestvina [3] has asked if all Artin groups of finite type, or indeed all Artin groups of any type, are virtually poly-free.

In this paper we investigate the poly-free properties of the class of *right-angled Artin groups*, which are the Artin groups for which the defining graph has every edge labeled 2. That is, for a simplicial graph $\Gamma$, the right-angled Artin group $A\Gamma$ is the group with generators in one-to-one correspondence with the set $V(\Gamma)$ of vertices of $\Gamma$, and relations $[v, w] = vwv^{-1}w^{-1}$, for each edge between vertices $v$ and $w$ of $\Gamma$. These groups are also known in the literature as graph groups, or free partially commutative groups. (See [7], [9], [10], [16] for information on normal forms for right-angled Artin groups and further references.)

Our main results are as follows.

**Theorem A.** *Let $\Gamma$ be a finite graph or, more generally, a graph of finite chromatic number $\mathsf{chr}(\Gamma)$ (and hence finite clique number $\mathsf{clq}(\Gamma)$). The right-angled Artin group $A\Gamma$ is poly-free. Moreover,*

$$\mathsf{clq}(\Gamma) \leq \mathsf{pfl}(A\Gamma) \leq \mathsf{chr}(\Gamma),$$

*and there exists a poly-free tower for $A\Gamma$ of length $\mathsf{chr}(\Gamma)$.*

During the preparation of the text, W. Dicks has pointed out to us that J. Howie [12] has established $|V(\Gamma)|$ as an upper bound for the poly-free length of a right-angled Artin group defined by a finite graph $\Gamma$. Thus the above result is an improvement in the case of finite graphs and a generalization to a class of infinite graphs.

A graph $\Gamma$ is said to have the *doubly breakable cycle property* if $\Gamma$ is not totally disconnected and there exists a vertex subset $D \subseteq V(\Gamma)$ such that the full subgraph of $\Gamma$ induced by $D$ is totally disconnected, and such that every cycle in $\Gamma$ contains at least two vertices in $D$. For a graph $\Gamma$ with this property, the full subgraph generated by the vertices in $V(\Gamma) - D$ is a forest; if each of the trees in this forest is collapsed to a point in $\Gamma$, the resulting graph is bipartite. Moreover, no vertex in $D$ is connected by an edge to more than one vertex in each tree of the forest $V(\Gamma) - D$. See Figure 1 for an example of a graph with the doubly breakable cycle property. In this example $D$ can be taken to be $D = \{d_1, d_2, d_3, d_4\}$.

**Theorem B.** *Let $\Gamma$ be a graph. The right-angled Artin group $A\Gamma$ is poly-free of length 2 if and only if the graph $\Gamma$ has the doubly breakable cycle property.*

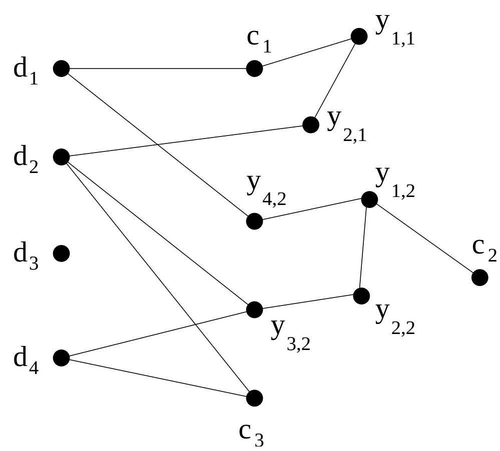


FIGURE 1. An example of a graph with the doubly breakable cycle property

We note that Theorem B is valid in the context of both finite and infinite graphs.

Theorem B can be used to show that each of the bounds in Theorem A is realized, as illustrated in the following two examples. First, consider the graph $C_5$ given by a 5-cycle (i.e. a pentagon; see Figure 2). This graph has chromatic number $\mathsf{chr}(C_5) = 3$, so Theorem A shows that the group $AC_5$ is poly-free with poly-free length at most 3. However, the pentagon satisfies the doubly breakable cycle property (for example, one can take $D = \{a, c\}$). Thus Theorem B improves this bound to $\mathsf{pfl}(AC_5) \leq 2$. Indeed, since $\mathsf{clq}(C_5) = 2$, this group contains $\mathbb{Z}^2$ as a subgroup and is not free, so $\mathsf{pfl}(AC_5) = 2$. Hence the lower bound on $\mathsf{pfl}(A\Gamma)$ given by the clique number in Theorem A is achieved in this example. Next, suppose that $P_5$ is a pentagonal prism (see Figure 2). In this case we have the same clique number and the same chromatic number as for the pentagon, i.e. $\mathsf{clq}(P_5) = 2$ and $\mathsf{chr}(P_5) = 3$, but $P_5$ does not satisfy the doubly breakable cycle property. Indeed, in a graph that satisfies the doubly breakable cycle property exactly two non-neighboring vertices must be selected from each 4-cycle to be in the independent set of vertices $D$ breaking the cycles. Thus if we choose $a$ in $D$, then we must also have $b'$, $c$, $d'$ and $e$ in $D$ (the vertices indicated by squares in the diagram of $P_5$ in Figure 2). But $a$ and $e$ are neighbors, so they cannot both be in $D$. This shows that $a$ cannot be in $D$ and, by symmetry, no element can be in $D$. Since $D$ cannot be empty $P_5$ does not satisfy the doubly breakable cycle property. Theorems A and B show that $\mathsf{pfl}(AP_5) \leq 3$ and $\mathsf{pfl}(AP_5) \neq 2$. Thus $\mathsf{pfl}(AP_5) = 3$ and the chromatic number upper bound is achieved for this second example.

If a graph $\Gamma$ satisfies the doubly breakable cycle property, the graph $\Gamma$ can be colored using three colors, one for the vertices in $D$, and two more for the vertices in $V(\Gamma) - D$ since their full subgraph is a forest. Consequently,

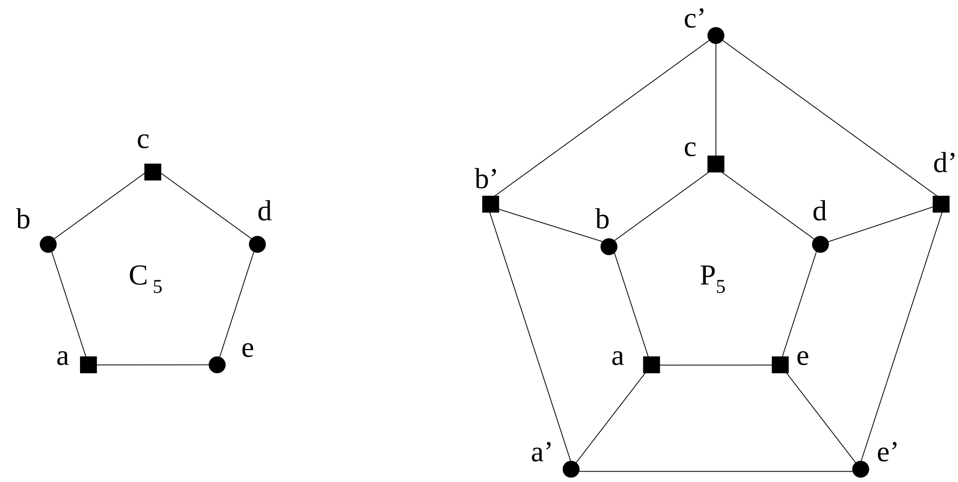


FIGURE 2. A pentagon and a pentagonal prism

Theorem B implies that whenever a right-angled Artin group $A\Gamma$ is poly-free of length 2, then the defining graph $\Gamma$ must have chromatic number at most three.

**Theorem C.** *A right-angled Artin group* $A\Gamma$ *is poly-fg-free of length 2 if and only if* $\Gamma$ *is a finite tree or a finite complete bipartite graph.*

For the right-angled Artin group $AC_5$ discussed above, Theorem C implies that the group $AC_5$ is not poly-fg-free (of any length) even though it is poly-free of length 2. Indeed, by the results of D. Meier from [14] (see also [8]) the poly-fg-free length of a poly-fg-free group is equal to its rational homological dimension. Since the homological dimension of $AC_5$ is 2 (more on this later), $AC_5$ can only be poly-fg-free of length 2. However, $C_5$ is neither a tree nor a complete bipartite graph and therefore $AC_5$ is not poly-fg-free.

**Organization.** Section 2 is a brief review of poly-free groups, right-angled Artin groups, and graph theoretic terminology. In Section 3 we prove one direction of Theorem B, that every right-angled Artin group with poly-free length 2 has the doubly breakable cycle property, utilizing results of [16] and [4] on finiteness properties of subgroups of right-angled Artin groups. Section 4 contains the proof of Theorem C, utilizing a comparison of Euler characteristics for poly-free groups and right-angled Artin groups, together with the results of Section 3. In Section 5 an arbitrary right-angled Artin group is exhibited as a split extension of a free group by a right-angled Artin group on a subgraph, including an explicit description of the action, proving Theorem A. Finally, in Section 6 we prove that for any graph with the doubly breakable cycle property, the corresponding right-angled Artin group $A\Gamma$ is a semidirect product of two free groups, using a refinement of the techniques of the previous section to construct the action. This result completes the proof of Theorem B.

## 2. Background

2.1. **Groups.** Throughout the text, $g^a$ denotes the conjugate $a^{-1}ga$.

Let $G = \langle S \rangle$ be a group generated by $S$. A word $w$ of length $k$ over $S \cup S^{-1}$ is a geodesic word if no word over $S \cup S^{-1}$ of length strictly less than $k$ represents the same element in $G$ as $w$ does. A total order defined on $S \cup S^{-1}$ induces a total order, called shortlex order, on all words over $S \cup S^{-1}$ in which shorter words always precede the longer ones and the words of the same length are ordered lexicographically according to the order defined on $S \cup S^{-1}$. A shortlex representative of an element $g \in G$ is the smallest word in the shortlex order that represents $g$. Such a representative is, by definition, geodesic.

2.2. **Graphs.** Throughout the paper, we assume that every graph is a simplicial graph; that is, a simple undirected graph. Therefore a graph $\Gamma$ is an ordered pair $\Gamma = (V, E)$ in which the set $V = V(\Gamma)$ is a set of vertices and $E = E(\Gamma)$ is a set of edges, which is a set of two element subsets of $V$. An edge $\{a, b\}$ has the vertices $a$ and $b$ as its endpoints. Two vertices $x$ and $y$ in $V$ are neighbors (are adjacent) if $\{x, y\}$ is an edge in $E$ (so no vertex is its own neighbor). A *cycle* in $\Gamma$ is a path of length at least 3 in which no vertex is repeated except for the initial and terminal one, which coincide. The *clique number* $\mathsf{clq}(\Gamma)$ of a graph $\Gamma$ is the largest size of a complete subgraph of $\Gamma$. Thus $\mathsf{clq}(\Gamma)$ is the largest size of a subset $Q$ of $V$ for which every 2-element subset $\{a, b\} \subseteq Q$ is an edge in $E$. A proper coloring of a graph $\Gamma$ by $C$ is a labelling $\ell : C \to V$ of the vertices in $V$ by symbols from a set of colors $C$ in such a way that no two neighbors in $\Gamma$ are colored in the same color. Thus if $\{a, b\} \in E$ then $\ell(a) \neq \ell(b)$. The *chromatic number* of a graph $\Gamma$ is the smallest size of a set $C$ for which there exists a proper coloring of $\Gamma$ by $C$. A set of vertices $D$ is independent if it can be colored by the same color in some proper coloring of $\Gamma$. In other words, no two vertices in $D$ are adjacent.

2.3. **Right-angled Artin groups.** We freely use the following well known observation. If $\Gamma'$ is a subgraph of $\Gamma$ induced by a set of vertices $X \subseteq V$, then the subgroup of the right-angled Artin group $A\Gamma$ generated by the elements of $X$ is $A\Gamma'$.

Throughout the text, given any homomorphism $\phi : A\Gamma \to G$ from a right-angled Artin group to a group $G$, the set $D := \{v \in V(\Gamma) \mid \phi(v) = 1\}$ is called the set of *dead vertices*, the set $L := V(\Gamma) - D$ is the set of *living vertices*, and the full subgraph $\Gamma_L$ generated by $L$ is the *living subgraph* of $\Gamma$, with respect to $\phi$.

**Lemma 2.1.** [2, 9] *Every geodesic representative of an element $t \in A\Gamma$ can be obtained from any other representative of $t$ by finite number of applications of the following operations:*

1. *Eliminate a subword of the form $xx^{-1}$ or $x^{-1}x$ with $x \in V(\Gamma)$.*
2. *If $x, y \in V(\Gamma)$ are adjacent in $\Gamma$, replace a single occurrence of $x^{\pm}y^{\pm}$ by $y^{\pm}x^{\pm}$.*

*In particular, every geodesic representative of an element $t \in A\Gamma_L$ can be obtained from any other geodesic representative of $t$ by finite number of applications of operation 2.*

2.4. **Poly-free groups.** Throughout the text, when $G$ is a semidirect product of two free groups, we will write $G = F_k \rtimes F_q$ with associated canonical homomorphism $\phi : G \to F_q$, so that the rank of the kernel $\ker(\phi)$ is $k$ and the rank of the associate quotient is $q$ (we allow infinite ranks).

**Proposition 2.2.** *If $G$ is poly-free with length $N$ and $H \leq G$, then $H$ is poly-free with length $\leq N$.*

*Proof.* Given a poly-free tower $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_N = G$ for $G$, then the tower $1 = G_0 \cap H \trianglelefteq G_1 \cap H \trianglelefteq \cdots \trianglelefteq G_N \cap H = H$ is a poly-free tower for $H$. □

**Proposition 2.3.** *If $G$ has a normal free subgroup $H$ and the quotient $G/H$ is poly-free with $\mathsf{pfl}(G/H) = N$, then $G$ is poly-free with $\mathsf{pfl}(G) \leq N + 1$.*

*Proof.* Let $\phi : G \to G/H$ be the canonical homomorphism. Given a poly-free tower $1 = Q_0 \trianglelefteq Q_1 \trianglelefteq \cdots \trianglelefteq Q_N = G/H$ for $G/H$, then the tower $1 \trianglelefteq \phi^{-1}(Q_0) = H \trianglelefteq \phi^{-1}(Q_1) \trianglelefteq \cdots \trianglelefteq \phi^{-1}(Q_N) = G$ is a poly-free tower for $G$. □

## 3. Poly-freeness of length 2 implies the doubly breakable cycle property

Before proving the statement of the title of this section in Proposition 3.5, we begin with a few lemmas.

**Lemma 3.1.** *The group $\mathbb{Z}^n$ has poly-free length equal to $n$.*

*Proof.* Since $\mathbb{Z}^n = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n}$ is an iterated direct product of $n$ free groups, $\mathbb{Z}^n$ is poly-free with length at most $n$.

To show that $\mathbb{Z}^n$ cannot have poly-free length less than $n$, assume that $\mathbb{Z}^n = (\ldots(F_{n_1} \rtimes F_{n_2}) \rtimes \ldots) \rtimes F_{n_k}$, for some (finite or infinite) $n_i$, $i = 1, \ldots, k$. Since $\mathbb{Z}^n$ is abelian, we must have $n_1 = n_2 = \cdots = n_k = 1$. Thus $\mathbb{Z}^n$ can be generated by $k$ elements (one for each $F_{n_i}$). However, $Z^n$ cannot be generated by fewer than $n$ elements, which shows that $n \leq k$. □

Note that Lemma 3.1 shows that the right-angled Artin group whose graph $\Gamma$ is a triangle cannot be poly-free of length 2.

**Lemma 3.2.** *Let $\Gamma$ be a graph and let $\phi : A\Gamma \to F_q$ be a homomorphism with $F_q$ a free group of finite or infinite rank. Let $\Gamma_L$ be the corresponding living subgraph. If $\Gamma_\alpha$ is a connected subgraph of $\Gamma_L$, then the subgroup $\langle\{\phi(u)|u \in \Gamma_\alpha\}\rangle$ of $F_q$ is isomorphic to $\mathbb{Z}$.*

*Proof.* Fix a connected subgraph $\Gamma_\alpha$ in $\Gamma_L$. The images $\phi(u)$, $u \in \Gamma_\alpha$ are nontrivial elements in the free group $F_q$. Using the fact that $\Gamma_\alpha$ is connected and [13, Prop. I.2.18] that says that the commuting relation is an equivalence relation on the set of nontrivial elements in a free group, we conclude that the group $\langle\{\phi(u)|u \in \Gamma_\alpha\}\rangle$ is abelian. However, the only abelian subgroup of $F_q$ is $\mathbb{Z}$ and the conclusion follows. □

**Lemma 3.3.** *Let $\Gamma$ be a graph with $A\Gamma = F_k \rtimes F_q$ for free groups $F_k$ and $F_q$ of finite or infinite rank. Let $\phi : A\Gamma \to F_q$ be the canonical homomorphism and let $D := \{v \in V(\Gamma) \mid \phi(v) = 1\}$ be the set of dead vertices. If $d \in D$ and $a_1, ..., a_n \in L = V(\Gamma) - D$ are all adjacent to $d$, then the subgroup $\langle\phi(a_1), ..., \phi(a_n)\rangle$ of $F_q$ is free of rank $n$.*

*Proof.* Suppose that there exists a nontrivial word $u_1 \cdots u_m$ with each $u_i \in \{a_1, ..., a_n\}^{\pm 1}$ and $\phi(u_1) \cdots \phi(u_m) = 1$. Then $u_1 \cdots u_m \in \ker(\phi)$, and since each $a_i$ is adjacent to $d$, the subgroup $\langle d, u_1 \cdots u_m\rangle$ of $\ker(\phi)$ is abelian. By hypothesis the poly-free length of $A\Gamma$ is less than 3, and so Lemma 3.1 and Proposition 2.2 show that the graph $\Gamma$ cannot contain a triangle. Hence there are no adjacencies among the vertices $a_i$ adjacent to $d$, and thus the word $u_1 \cdots u_m$ represents a nontrivial element of $A\Gamma$. Then in the abelianization $A\Gamma^{ab}$, the elements $d$ and $u_1 \cdots u_m$ generate a free abelian subgroup of rank two, and therefore the abelian subgroup $\langle d, u_1 \cdots u_m\rangle$ of $\ker(\phi)$ is also isomorphic to $\mathbb{Z}^2$. This contradicts the hypothesis that $\ker(\phi) = F_k$ is free. □

**Lemma 3.4.** *Let $\Gamma$ be a finite graph, let $\phi : A\Gamma \to Z$ be any group homomorphism from $A\Gamma$ to an infinite cyclic group $Z = \langle z\rangle$, and let $\rho : A\Gamma \to Z$ be the homomorphism defined by $\rho(v) := z$ for a vertex $v \in V(\Gamma)$ if $\phi(v) \neq 1$ and $\rho(v) := 1$ if $\phi(v) = 1$. Then* $\ker(\phi)$ *is free if and only if* $\ker(\rho)$ *is free.*

*Proof.* Let $D := \{w \in V(\Gamma) \mid \phi(w) = 1\}$ and for each $v \in V(\Gamma) - D$, let $n_v \in \mathbb{Z}$ be the unique integer such that $z^{n_v} = \phi(v)$ in $Z$.

First suppose that the group $\ker(\rho)$ is not free. Let $N$ be the least common multiple of the numbers $|n_v|$ for $v \in V(\Gamma) - D$. Then for each $v \in V(\Gamma) - D$, there exists an integer $a_v$ such that $n_v a_v = N$. Define $a_w := 1$ for each $w \in D$. Let $\mu : Z \to Z$ be the homomorphism $z^k \mapsto z^{Nk}$ given by taking the $N$-th power in $Z$. The composition $\mu \circ \rho : A\Gamma \to Z$ has kernel $\ker(\mu \circ \rho) = \ker(\rho)$. Define $\Theta : A\Gamma \to A\Gamma$ by $\Theta(v) := v^{a_v}$ for all $v \in V(\Gamma)$; this defines a homomorphism of groups. Moreover, the compositions $\phi \circ \Theta = \mu \circ \rho$. Hence

$\ker(\rho) = \ker(\mu \circ \rho) = \ker(\phi \circ \Theta)$.

$$\begin{array}{ccc} A\Gamma & \xrightarrow{\Theta} & A\Gamma \\ \rho \downarrow & & \downarrow \phi \\ Z & \xrightarrow{\mu} & Z. \end{array}$$

Suppose that $1 \neq g \in \ker(\Theta)$. Put a total ordering on $V(\Gamma)$, and let $g =_{A\Gamma} v_1^{j_1} v_2^{j_2} \cdots v_k^{j_k}$ be the shortlex least representative of $g$ with each $v_i \in V(\Gamma)$, $j_i \in \mathbb{Z}$, and $v_i \neq v_{i+1}$. Then $\Theta(g) =_{A\Gamma} v_1^{j_1 a_{v_1}} v_2^{j_2 a_{v_2}} \cdots v_k^{j_k a_{v_k}} =_{A\Gamma} 1$. Since $g \neq 1$, the word $v_1^{j_1} v_2^{j_2} \cdots v_k^{j_k}$ is not empty, so there must be commutation relations such that for some indices $i < m$ we have $v_i = v_m$, $j_i j_m < 0$, and $v_i$ commutes with $v_n$ for all $i < n < m$, so that cancellation occurs between $v_i^{j_i a_{v_i}}$ and $v_m^{j_m a_{v_m}}$. However, in that case cancellation must also be possible in the normal form word $v_1^{j_1} v_2^{j_2} \cdots v_k^{j_k}$, giving a contradiction. Therefore $\ker(\Theta) = 1$.

We now have that $\Theta$ is a monomorphism, and $\Theta(\ker(\rho)) = \Theta(\ker(\mu \circ \rho)) = \Theta(\ker(\phi \circ \Theta)) \leq \ker(\phi)$. Hence $\ker(\rho)$ is isomorphic to a subgroup of $\ker(\phi)$. Using the Nielsen-Schreier Subgroup Theorem, that every subgroup of a free group is free, together with the hypothesis that $\ker(\rho)$ is not free, we have that $\ker(\phi)$ cannot be a free group.

Next suppose that $\ker(\phi)$ is not free. Define the function $\Psi : A\Gamma \to A\Gamma$ by $\Psi(v) := v^{n_v}$ for $v \in V(\Gamma) - D$ and $\Psi(w) := w$ for $w \in D$. Then $\phi = \rho \circ \Psi$. An argument similar to the proof above for $\Theta$ shows that $\Psi$ is a monomorphism of groups, so $\Psi$ restricts to an isomorphism from $\ker(\phi)$ to a subgroup of $\ker(\rho)$. Since $\ker(\phi)$ is not free, then $\ker(\rho)$ also cannot be free. □

Note that Lemma 3.4 is the only lemma in this section for which the graph must be finite. In the proof of Theorem B, this lemma is applied only to a cycle $\Gamma'$ in $\Gamma$, which is finite, and hence Theorem B is valid for both finite and infinite graphs $\Gamma$.

**Proposition 3.5.** *If $\Gamma$ is a graph and the right-angled Artin group $A\Gamma$ is poly-free of length 2, then the graph $\Gamma$ has the doubly breakable cycle property.*

*Proof.* The group $A\Gamma$ can be written as a semidirect product $A\Gamma = F_k \rtimes F_q$ where $F_k$ and $F_q$ are free groups of ranks $k$ and $q$, respectively, and $k, q \in \mathbb{N} \cup \{\infty\}$. Let $\phi : A\Gamma \to F_q$ be the canonical surjection, with $\ker(\phi) = F_k$. Let $D$ be the set of vertices $v$ in $\Gamma$ with $\phi(v) = 1$; that is, $v \in ker(\phi)$.

If $d_1$ and $d_2$ are vertices in $D$, then the subgroup $\langle d_1, d_2 \rangle < A\Gamma$ generated by $d_1$ and $d_2$ must also be contained in $\ker(\phi)$. Since $\ker(\phi)$ is free, the subgroup $\langle d_1, d_2 \rangle$ of $\ker(\phi)$ is free. Hence $d_1$ and $d_2$ cannot be joined by an edge in $\Gamma$. Thus the subgraph induced by $D$ is totally disconnected, i.e., $D$ is an independent set of vertices in $\Gamma$.

Suppose that $\Gamma'$ is a cycle in the graph $\Gamma$. Then $\Gamma'$ contains at least 3 vertices. The subgroup $A\Gamma'$ of $A\Gamma$ is again a right-angled Artin group, and Proposition 2.2 says that $A\Gamma'$ is poly-free with length at most 2. Lemma 3.1

says that $\mathbb{Z}^3$ is not poly-free with length less than 3, so the cycle $\Gamma'$ cannot have length 3. Thus $\Gamma'$ contains at least 4 vertices.

If $V(\Gamma') \cap D$ contains less than two vertices, then the full subgraph of $\Gamma'$ generated by the vertices in $V(\Gamma') - D$ is connected. In this case Lemma 3.2 shows that the restriction $\phi|_{A\Gamma'} : A\Gamma' \to F_q$ has range $\langle f \rangle = \mathbb{Z}$ for some $f \in F_q$. Since $\ker(\phi|_{A\Gamma'}) < \ker(\phi)$, then $\ker(\phi|_{A\Gamma'})$ is also a free group.

If $V(\Gamma') \cap D = \emptyset$, then $\phi(v) \neq f^0$ for all $v \in V(\Gamma')$. Define the function $\rho : A\Gamma' \to \mathbb{Z}$ by $\rho(v) = f$ for all $v \in V(\Gamma')$. Then Lemma 3.4 says that $\ker(\rho)$ is free. Since the graph $\Gamma'$ is connected, [16, Theorem 6.3] shows that $\ker(\rho)$ is finitely generated. The cycle $\Gamma'$ is a flag complex, since $\Gamma'$ is not a triangle, and this flag complex is not simply connected. The Main Theorem of [4] shows that $\ker(\rho)$ is not finitely presented. Since a free group cannot be finitely generated but not finitely presented, we have a contradiction.

If there is exactly one vertex $d$ in $V(\Gamma') \cap D$, then the images $\phi(a)$ and $\phi(b)$ of the two neighbors $a$ and $b$ of $d$ in the cycle $\Gamma'$ must generate a free group of rank 2 in $F_q$ (by Lemma 3.3). On the other hand, we already established that the range of $\phi|_{A\Gamma'} : A\Gamma' \to F_q$ is cyclic, resulting again in a contradiction. □

## 4. Poly-fg-freeness of length 2

In this section we prove Theorem C using an analysis of Euler characteristics of right-angled Artin groups and poly-free groups.

**Lemma 4.1.** *Let $\Gamma$ be a finite graph with $A\Gamma = F_k \rtimes F_q$ for free groups $F_k$ and $F_q$ of finite rank. Let $\phi : A\Gamma \to F_q$ be the canonical homomorphism, let $D := \{v \in V(\Gamma) \mid \phi(v) = 1\}$ be the set of dead vertices, and for each $d \in D$, let $N_d$ denote the set of vertices adjacent to $d$ in $\Gamma$. Then the image of the subgroup generated by $N_d$ under the map $\phi$ has finite index in $F_q$, and $k \geq \sum_{d \in D} [F_q : \phi(\langle N_d \rangle)]$ .*

*Proof.* We begin by finding a presentation for the subgroup $K := \ker(\phi) = F_k$ of $A\Gamma$ using the Reidemeister-Schreier procedure, following the notation in [13, Proposition II.4.1]. Let $F(V(\Gamma))$ be the free group on the vertices of $\Gamma$, let $\alpha : F(V(\Gamma)) \to A\Gamma$ be the canonical epimorphism, and let $\widetilde{K} := \alpha^{-1}(K)$. Then $F(V(\Gamma))/\widetilde{K} \cong F_q$. Choose a Schreier transversal $T$ for $\widetilde{K}$ in $F(V(\Gamma))$.

For any element $y \in F(V(\Gamma))$, let $\overline{y}$ denote the element of $T$ for which $\widetilde{K}y = \widetilde{K}\overline{y}$. For $t \in T$ and $v \in V(\Gamma)$, define $\gamma(t, v) := tv\overline{tv}^{-1}$. Given $d \in D$ and $t \in T$, then $\overline{td} = t$. The element $d_t := tdt^{-1} = \gamma(t, d)$ is a conjugate of a nontrivial element in $A\Gamma$, so $d_t$ itself is not trivial. Given $a \in L := V(\Gamma) - D$ and $t \in T$, define $a_t := ta\overline{ta}^{-1} = \gamma(t, a)$ as well. The subset $S$ of nontrivial elements in the set

$$S' := \{d_t \mid t \in T,\ d \in D\ \} \cup \{a_t \mid t \in T,\ a \in L\ \}$$

generates $K$.

For $t \in T$ and $v \in V(\Gamma)$, also define $\gamma(t, v^{-1}) := tv^{-1}(\overline{tv^{-1}})^{-1}$. If $t \in T$, $d \in D$, and $a \in L$, then $\gamma(t, d^{-1}) = d_t^{-1}$ and $\gamma(t, a^{-1}) = (a_{\overline{ta^{-1}}})^{-1}$. Given any word $v = v_1 \cdots v_m$ with each $v_i \in V(\Gamma)^{\pm 1}$, define

$$\tau(v) := \gamma(1, v_1)\gamma(\overline{v_1}, v_2) \cdots \gamma(\overline{v_1 \cdots v_{m-1}}, v_m).$$

Note that for each $t \in T$, since every prefix of $t$ is also in $T$ we have that the element $\tau(t)$ is a product of trivial elements in $S'$, and hence $\tau(t) = 1$. For each relator $r = [u, v] \in R$ it follows that

$$\tau(trt^{-1}) = \gamma(t, u)\gamma(\overline{tu}, v)\gamma(\overline{tuv}, u^{-1})\gamma(\overline{tuvu^{-1}}, v^{-1}) \ .$$

The latter words form the defining relators of the presentation for $K = \ker(\phi)$. In particular, a defining set of relations is given by

$$\begin{aligned} R :=& \{d_t a_t = a_t d_{\overline{ta}} \mid t \in T,\ d \in D,\ a \in L,\ \{d, a\} \in E(\Gamma)\ \} \cup \\ & \{a_t b_{\overline{ta}} = b_t a_{\overline{tb}} \mid t \in T,\ a, b \in L,\ \{a, b\} \in E(\Gamma)\ \} \end{aligned}$$

and the group $K$ is presented by $\langle\ S \mid R\ \rangle$.

Abelianizing this presentation yields a presentation for $K_{ab} = \mathbb{Z}^k$. The subgroup $H$ of $K_{ab}$ generated by the elements of

$$D_T := \{\ d_t \mid d \in D,\ t \in T\ \}$$

is a free abelian direct factor of $K_{ab}$ presented by

$$H = \langle\ D_T \mid \{d_t = d_{\overline{ta}} \mid t \in T,\ d \in D,\ a \in N_d\ \}\ \rangle_{ab} \ .$$

Since all relations in this presentation are equalities between generators, the rank of $H$ is the number of equivalence classes of generators. Note that two generators $d_t$ and $d_s$ are equal in $H$ if and only if there exists a sequence of relations $d_t = d_{\overline{ta_1}} = d_{\overline{ta_1 a_2}} = \cdots = d_{\overline{ta_1 \cdots a_m}} = d_s$ with $s = \overline{ta_1 \cdots a_{m+1}}$ and each $a_i \in N_d^{\pm 1}$. This holds if and only if $Kta_1 \cdots a_{m+1} = Ks$, which is satisfied if and only if $\phi(t)\phi(a_1) \cdots \phi(a_{m+1}) = \phi(s)$. Then $d_t$ and $d_s$ are equal in $H$ if and only if $\phi(t)$ and $\phi(s)$ are in the same coset of $\phi(\langle N_d \rangle)$ in $F_q$. Therefore the rank of $H$ is equal to the sum of indices $\sum_{d \in D}[F_q : \phi(\langle N_d \rangle)]$. Since the rank of $K_{ab}$ is $k$ we have

$$k \geq rank(H) = \sum_{d \in D}[F_q : \phi(\langle N_d \rangle)].$$

In particular, the index $[F_q : \phi(\langle N_d \rangle)]$ is finite for all $d \in D$. □

**Theorem C.** *A right-angled Artin group $A\Gamma$ is poly-fg-free of length 2 if and only if $\Gamma$ is either a finite tree or a finite complete bipartite graph.*

*Proof.* If $\Gamma$ is the complete bipartite graph $K_{k,q}$, then $A\Gamma$ is the direct product $F_k \times F_q$ of free groups of ranks $k$ and $q$. On the other hand, if $\Gamma$ is a tree on $n$ vertices, the Artin group $A\Gamma$ is a semidirect product $F_{n-1} \rtimes \mathbb{Z}$ [11, Proposition 4.6].

Conversely, assume for the rest of this proof that $A\Gamma$ is a poly-fg-free group of length 2. Since $A\Gamma$ is finitely generated, the graph $\Gamma$ must be finite (since the abelianization of $A\Gamma$ is $\mathbb{Z}^V$, any generating set of $A\Gamma$ has at least $|V|$ elements).

There exists a split short exact sequence

$$1 \to F_k \to A\Gamma \xrightarrow{\phi} F_q \to 1,$$

such that the ranks $k$ and $q$ of the free groups are finite and positive. By Proposition 3.5, the graph $\Gamma$ must satisfy the doubly breakable cycle property. Moreover, the proof of that proposition shows that the set of dead vertices $D := \{v \in V(\Gamma) | \phi(v) = 1\}$ associated to $\phi$ is an independent set of vertices in $\Gamma$ such that every cycle in $\Gamma$ meets $D$ at least twice. Since $A\Gamma$ is not poly-fg-free of length 1, the living subgraph $\Gamma_L$ is not empty.

Since semidirect products of free groups are torsion free, the Euler characteristic of the semidirect product $A\Gamma = F_k \rtimes F_q$ is given by $\chi(A\Gamma) = \chi(F_k)\chi(F_q)$ (see [6, Proposition IX.7.3(d)]). Therefore, given that the Euler characteristic of a free group $F_r$ of rank $r$ is $\chi(F_r) = 1 - r$,

$$\chi(A\Gamma) = (k-1)(q-1)\ . \tag{1}$$

The Euler characteristic of a right-angled Artin group can be computed using a $K(A\Gamma, 1)$ space. The doubly breakable cycle property implies that $\Gamma$ contains edges but does not contain any triangles. In this case the standard 2-complex $X$ associated to the standard presentation (from Section 1) of $A\Gamma$ is a $K(A\Gamma, 1)$ [16, Theorem 7.3]. Thus the Euler characteristic of $A\Gamma$ is also

$$\chi(A\Gamma) = \chi(X) = 1 - v + e\ . \tag{2}$$

Denote the connected components of the nonempty graph $\Gamma_L$ by $C_1, \dots, C_c$. Each of these components is a tree with $n_j > 0$ vertices, and hence $n_j - 1$ edges, for $1 \le j \le c$. Let $\delta := |D|$ and denote the degrees of the vertices $d_1, \dots d_\delta$ in $D$ by $g_1, \dots, g_\delta$. Rewriting Equation (2) yields

$$\begin{aligned} \chi(A\Gamma) &= 1 - (\delta + \sum_{j=1}^{c} n_j) + (\sum_{j=1}^{c}(n_j - 1) + \sum_{i=1}^{\delta} g_i) = \\ &= 1 - \delta - c + \sum_{i=1}^{\delta} g_i = 1 - c + \sum_{i=1}^{\delta}(g_i - 1). \end{aligned} \tag{3}$$

For each $1 \le i \le \delta$, let $N_i$ denote the set of vertices adjacent to $d_i$. Lemma 3.3 says that the subgroup $\phi(\langle N_i \rangle)$ of $F_q$ is free of rank equal to the degree $g_i$ of $d_i$. Using the Schreier Formula, $(rank(F_q) - 1)[F_q : \phi(\langle N_i \rangle)] = (rank(\phi(\langle N_i \rangle) - 1)$, so

$$g_i - 1 = (q-1)[F_q : \phi(\langle N_i \rangle)].$$

According to Lemma 4.1, $\sum_{i=1}^{\delta}[F_q : \phi(\langle N_{d_i} \rangle)] \le k$. Thus, taking into account the non-negativity of $q - 1$,

$$\sum_{i=1}^{\delta}(g_i - 1) = (q-1)\sum_{i=1}^{\delta}[F_q : \phi(\langle N_{d_i} \rangle)] \le k(q-1).$$

Combining this with Equation (1) and Equation (3), then

$$(k-1)(q-1) \overset{(1)}{=} \chi(\Gamma) \overset{(3)}{=} 1-c+\sum_{i=1}^{\delta}(g_i-1) \ \leq \ 1-c+k(q-1) \ ,$$

which implies that $c \leq q$ .

Lemma 3.2 says that for each component $C_j$, there exists an element $f_j$ in $F_q$ such that all vertices from the component $C_j$ are mapped by $\phi$ to a power of $f_j$. Since $\phi$ is onto and the dead vertices in $D$ are mapped to the identity in $F_q$, $f_1, \ldots, f_c$ generate $F_q$, which implies that $q \leq c$. Using the inequality at the end of the previous paragraph, then

$$q = c. \tag{4}$$

If $q = c = 1$, then the living subgraph $\Gamma_L$ of $\Gamma$ is a single tree. Since the kernel $\ker(\phi) = F_k$ is finitely generated, [16, Theorem 6.1] says that every dead vertex in $D$ must be adjacent to a vertex in $\Gamma_L$. The doubly breakable cycle property says that there cannot be a cycle in $\Gamma$ that meets a dead vertex in $D$ only once, which implies that for each $d \in D$, $d$ cannot be the endpoint of two different edges whose other endpoints lie in $\Gamma_L$. Therefore in this case the graph $\Gamma$ is also a tree.

Finally suppose that $q = c \geq 2$. As in the previous paragraph, the doubly breakable cycle property says that for each $d_i \in D$, $d_i$ cannot be the endpoint of two different edges whose other endpoints lie in the same component $C_j$ of $\Gamma_L$. Hence each degree $g_i \leq c$, so $\sum_{i=1}^{\delta} g_i \leq c\delta$. Using Equation (4), Equation (1), and Equation (3), then

$$\begin{aligned}(k-1)(c-1) \ &\overset{(4)}{=} \ (k-1)(q-1) \overset{(1)}{=} \chi(A\Gamma) \overset{(3)}{=} 1-\delta-c+\sum_{i=1}^{\delta} g_i \\ &\leq \ 1-\delta-c+c\delta = (c-1)(\delta-1) \ .\end{aligned}$$

Since $c \geq 2$, we obtain

$$k \leq \delta \ . \tag{5}$$

Note that equality holds if and only if each vertex in $D$ has degree $c$; i.e., there exists a single edge between each vertex in $D$ and each component of $\Gamma_L$.

Since $A\Gamma = F_k \rtimes F_q$, the group $A\Gamma$ can be generated by $k+q$ elements. However, the minimal number of generators for the right-angled Artin group $A\Gamma$ is the number $v = \delta + \sum_{j=1}^{c} n_j$ of vertices in $\Gamma$, which in turn is at least as large as $\delta + c$. Thus

$$\delta + c \leq k + q \ . \tag{6}$$

Note that equality in this case is possible only if $\delta + c = v = \delta + \sum_{j=1}^{c} n_j = k+q$, and so $c = \sum_{j=1}^{c} n_j$. Thus equality implies that each $n_j = 1$; i.e., each component $C_j$ of $\Gamma_L$ is a single vertex.

Using the fact that $q = c$, Inequality (5) and Inequality (6) imply that $\delta = k$, so equality holds both in (5) and in (6). Therefore $D$ and $L$ are each independent sets of vertices in $\Gamma$, and there is an edge between each vertex in $D$ and each vertex in $L$. Thus in this case $\Gamma$ is a complete bipartite graph. □

## 5. Every right-angled Artin group is poly-free

Given a graph $\Gamma$ with finite chromatic number greater than one, let $D$ be the set of vertices in one of the colors and let $L := V(\Gamma) - D$ be the vertices in the other colors. Let $\Gamma_L$ be the full subgraph of $\Gamma$ induced by $L$. If we define a homomorphism $\phi : A\Gamma \to A\Gamma_L$ by $\phi(d) = 1$ for $d \in D$ and $\phi(a) = a$ for $a \in L$, then $D$ is the set of dead vertices and $\Gamma_L$ is the living subgraph associated to this homomorphism. In the following proof we construct a free group $F$ (isomorphic to $\ker(\phi)$) and an action of $A\Gamma_L$ on $F$, and exhibit directly that $A\Gamma$ is isomorphic to the semidirect product $F \rtimes A\Gamma_L$ of a free group with $A\Gamma_L$.

**Theorem A.** *Let $\Gamma$ be a finite graph or, more generally, a graph of finite chromatic number $\mathsf{chr}(\Gamma)$ and finite clique number $\mathsf{clq}(\Gamma)$. The right-angled Artin group $A\Gamma$ is poly-free. Moreover,*

$$\mathsf{clq}(\Gamma) \leq \mathsf{pfl}(A\Gamma) \leq \mathsf{chr}(\Gamma),$$

*and there exists a poly-free tower for $A\Gamma$ of length $\mathsf{chr}(\Gamma)$.*

*Proof.* To prove poly-freeness and the upper bound on the poly-free length, we induct on $\mathsf{chr}(\Gamma)$. If $\mathsf{chr}(\Gamma) = 1$, then $\Gamma$ is totally disconnected, so $A\Gamma$ is free, and hence poly-free of length 1. Next suppose that $\mathsf{chr}(\Gamma) \geq 2$, and that for every graph $\Gamma'$ with $\mathsf{chr}(\Gamma') < \mathsf{chr}(\Gamma)$, the group $A\Gamma'$ is poly-free and has a poly-free tower of length $\mathsf{chr}(\Gamma')$.

Choose a coloring of $\Gamma$ in $\mathsf{chr}(\Gamma)$ colors, one of which is gray. Let $D$ be the set of vertices in $V = V(\Gamma)$ colored in gray, $L = V - D$ be the set of vertices colored in a different color, $\Gamma_L$ be the subgraph of $\Gamma$ induced by $L$ and $A\Gamma_L$ be the corresponding right-angled Artin group. Then $\mathsf{chr}(\Gamma_L) = \mathsf{chr}(\Gamma) - 1$ and the inductive assumption implies that there exists a poly-free tower for $A\Gamma_L$ of length $\mathsf{chr}(\Gamma) - 1$.

In the discussion that follows, a geodesic representative of an element $t \in A\Gamma_L$ means a geodesic word in the alphabet $L^{\pm 1}$. For any vertex $v \in V(\Gamma)$, denote by $N_v$ the set of vertices adjacent to $v$; i.e., the neighbors of $v$. For each $d \in D$, define a set of symbols

$$T_d := \{ \ d_t \mid t \in A\Gamma_L, \text{ no geodesic rep. of } t \text{ starts with a letter in } N_d^{\pm 1} \ \}.$$

Let $F(T_d)$ be the free group over $T_d$ and let $F$ be the free group

$$F := *_{d \in D} F(T_d).$$

For each generator $a \in L$, define an endomorphism $\alpha_a : F \to F$ by

$$\alpha_a(d_t) := \begin{cases} d_{ta}, & d_{ta} \in T_d \\ d_t, & d_{ta} \notin T_d \end{cases}, \tag{7}$$

for all $d \in D$ and $d_t \in T_d$. Since $F$ is a free group, this definition of $\alpha_a$ on the generators of $F$ extends to an endomorphism on $F$. In order to show that $a \mapsto \alpha_a$ extends to an action of $A\Gamma_L$ on $F$, we first need to consider when the conditions $d_t \in T_d$ and $d_{ta} \notin T_d$ occur simultaneously.

Assume that $d_t \in T_d$. If $d_{ta} \notin T_d$, then there exists a geodesic representative $w$ of $ta$ that begins with a letter in $N_d^{\pm 1}$. Consider the word $wa^{-1}$ representing $t$. Since $d_t \in T_d$ the word $wa^{-1}$ cannot be geodesic. By Lemma 2.1 we can write $w$ as $u_1 a u_2$ where $a$ commutes with all the letters in $u_2$. The word $u_1 u_2$ is a geodesic representative of $t$. As such, it cannot start in $N_d^{\pm 1}$. Thus the geodesic word $w = u_1 a u_2$ representing $ta$ and the geodesic word $u_1 u_2$ representing $t$ start in a different letter. This is possible only when $u_1$ is empty. Thus $w = au_2$ is a geodesic representative of $ta$ that starts in $N_d^{\pm 1}$ and $a$ commutes with all letters in $u_2$, i.e., $a$ commutes with $d$ and all the letters in $u_2$. However, $u_2$ is a geodesic representative of $t$. Since any other geodesic representative of $t$ can be obtained from $u_2$ by commuting letters we conclude that $a$ commutes with $d$ and all the letters in any geodesic representative of $t$. A similar proof shows that, conversely, if $a$ commutes with $d$ and all letters in any geodesic representative of $t$ then $d_{ta}$ cannot be in $T_d$. Therefore

- **($*$):** If $d_t \in T_d$ then $d_{ta} \notin T_d$ if and only if $d$ and all of the symbols in any geodesic representative of $t$ are adjacent to $a$ in $\Gamma$.

For each $a \in L$ and $d_t \in T_d$, define another endomorphism $\alpha_{a^{-1}} : F \to F$ by replacing $a$ by $a^{-1}$ in Equation 7. Then

$$\alpha_a(\alpha_{a^{-1}}(d_t)) = \alpha_a\left(\begin{cases} d_{ta^{-1}}, & d_{ta^{-1}} \in T_d \\ d_t, & d_{ta^{-1}} \notin T_d \end{cases}\right) = \begin{cases} d_t, & d_{ta^{-1}} \in T_d,\ d_{ta} \in T_d \\ d_{ta^{-1}}, & d_{ta^{-1}} \in T_d,\ d_{ta} \notin T_d \\ d_{ta}, & d_{ta^{-1}} \notin T_d,\ d_{ta} \in T_d \\ d_t, & d_{ta^{-1}} \notin T_d,\ d_{ta} \notin T_d \end{cases}.$$

As a consequence of ($*$) from the previous paragraph, for every $a \in L$, $d \in D$, and $d_t \in T_d$, we have $d_{ta} \in T_d$ if and only if $d_{ta^{-1}} \in T_d$, and so the middle two cases in last expression of the equation above cannot occur. Therefore $\alpha_a(\alpha_{a^{-1}}(d_t)) = d_t$, and similarly $\alpha_{a^{-1}}(\alpha_a(d_t)) = d_t$. Thus the maps $\alpha_a$ and $\alpha_{a^{-1}}$ are automorphisms of $F$ which are inverse to each other.

Finally, for each $a, b \in L$ that are adjacent in $\Gamma$ and each $d \in D$ and $d_t \in T_d$, the equivalence in ($*$) shows that the condition $d_{tab} \in T_d$ is equivalent to the conjunction of the conditions $d_{ta} \in T_d$ and $d_{tb} \in T_d$, so we have

$$\alpha_b(\alpha_a(d_t)) = \begin{cases} d_{tab}, & d_{ta} \in T_d,\ d_{tab} \in T_d \\ d_{ta}, & d_{ta} \in T_d,\ d_{tab} \notin T_d \\ d_{tb}, & d_{ta} \notin T_d,\ d_{tb} \in T_d \\ d_t, & d_{ta} \notin T_d,\ d_{tb} \notin T_d \end{cases} = \begin{cases} d_{tab}, & d_{ta} \in T_d,\ d_{tb} \in T_d \\ d_{ta}, & d_{ta} \in T_d,\ d_{tb} \notin T_d \\ d_{tb}, & d_{ta} \notin T_d,\ d_{tb} \in T_d \\ d_t, & d_{ta} \notin T_d,\ d_{tb} \notin T_d \end{cases}.$$

Therefore, by symmetry, $\alpha_b(\alpha_a(d_t)) = \alpha_a(\alpha_b(d_t))$. Thus $\alpha_a\alpha_b = \alpha_b\alpha_a$ whenever $a$ and $b$ are adjacent in $\Gamma$, which implies that Equation (7) defines a homomorphism $\alpha : A\Gamma_L \to Aut(F)$, given by $a \mapsto \alpha_a$, and an action of $A\Gamma_L$ on the free group $F$.

Let $G := F \rtimes A\Gamma_L$ be the semidirect product defined by this action. Next we show that $G \cong A\Gamma$. A presentation for $G$ is given by

$$G = \langle L \cup (\cup_{d\in D} T_d) \mid R_L \cup (\cup_{d\in D} R_d)\rangle,$$

where $R_L$ is the set of commutation relations defining $A\Gamma_L$ (induced by the edges of $\Gamma_L$) and for each $d \in D$,

$$R_d := \{\ d_t^a = d_{ta} \mid d_t \in T_d,\ d_{ta} \in T_d\ \} \cup \{\ d_t^a = d_t \mid d_t \in T_d,\ d_{ta} \notin T_d\ \}.$$

Next apply Tietze transformations to simplify this presentation. Given an element $d_t \in T_d$, let $\eta_t$ be a geodesic representative of $t$. For each prefix $u$ of $\eta_t$, then $d_{\overline{u}} \in T_d$ as well, so the relations of the type $d_t^a = d_{ta}$ in $R_d$ can be used to show that $d_t = d_1^{\eta_t}$ in $G$. For any other geodesic representative $w$ of $t \in A\Gamma_L$, the relation $d_1^{\eta_t} = d_1^w$ is a consequence of the relations in $R_L$. If we denote $d = d_1$ for $d \in D$, then the presentation of $G$ is Tietze equivalent to

$$\langle L \cup D \mid R_L \cup (\cup_{d\in D} R'_d)\rangle,$$

where

$$R'_d := \{\ d^{ta} = d^t \mid d_t \in T_d,\ d_{ta} \notin T_d\ \}.$$

Note that the relation $d^a = d$ occurs in $R'_d$ if $d_a \notin T_d$, and $d_a \notin T_d$ if and only if $a$ is adjacent to $d$ in $\Gamma$. Thus the relations in $R'_d$ include all the defining relations in $A\Gamma$ involving $d$. For each relation $d^{ta} = d^t$ in $R'_d$ with $t$ a nontrivial element of $A\Gamma_L$, we have $d_{ta} \notin T_d$, which implies by $(*)$ that $a$ is adjacent to $d$ and to all of the symbols in any geodesic for $t$. This shows that the relation $d^{ta} = d^t$ is a consequence of the relation $d^a = d$ and the relations in $R_L$. Thus the presentation for $G$ is Tietze equivalent to

$$\langle L \cup D \mid R_L \cup (\cup_{d\in D} R''_d)\rangle,$$

where

$$R''_d := \{\ d^a = d \mid d_a \notin T_d\ \} = \{\ d^a = d \mid a \text{ is adjacent to } d \text{ in } \Gamma\ \},$$

which is exactly the defining presentation of $A\Gamma$.

Therefore $A\Gamma \cong G = F \rtimes A\Gamma_L$. By induction $A\Gamma_L$ has a poly-free tower of length $\mathsf{chr}(\Gamma) - 1$, so the proof of Proposition 2.3 completes the proof that $A\Gamma$ has a poly-free tower of length $\mathsf{chr}(\Gamma)$ and hence $\mathsf{pfl}(A\Gamma) \le \mathsf{chr}(\Gamma)$.

Next consider the lower bound on the poly-free length. Let $m = \mathsf{clq}(\Gamma)$ and let $\widetilde{\Gamma}$ be a clique of $\Gamma$ with $m$ vertices. Then $\widetilde{\Gamma}$ is a complete graph, and the subgroup $A\widetilde{\Gamma}$ corresponding to $\widetilde{\Gamma}$ is isomorphic to $\mathbb{Z}^m$. Lemma 3.1 says that $m = \mathsf{pfl}(A\widetilde{\Gamma})$, and Proposition 2.2 shows that $\mathsf{pfl}(A\widetilde{\Gamma}) \leq \mathsf{pfl}(A\Gamma)$. □

## 6. Poly-freeness of length 2

In this section we prove the converse of the main result in Section 3, that every graph with the doubly breakable cycle property induces a poly-free right-angled Artin group of length 2. Together with the main result in Section 3, this completes the proof of Theorem B.

Given a graph $\Gamma$ together with a corresponding set $D$ for which $\Gamma$ has the doubly breakable cycle property, let $L := V(\Gamma) - D$ and let $\Gamma_L$ be the full subgraph of $\Gamma$ induced by $L$. Denote by $C$ a set of representatives from $L$ of the components of $\Gamma_L$, and let $F(C)$ be the free group on $C$. If we define a homomorphism $\phi : A\Gamma \to F(C)$ by $\phi(d) = 1$ for $d \in D$ and $\phi(y) = c$ whenever $y \in L$ and $c$ is the generator of $F(C)$ corresponding to the component of $\Gamma_L$ containing $y$, then $D$ is the set of dead vertices and $\Gamma_L$ is the living subgraph associated to this homomorphism. In the proof below, our approach follows the same lines as the proof of Theorem A. We construct a free group $F$ (isomorphic to $\ker(\phi)$) and an action of $F(C)$ on $F$, in order to show explicitly that $A\Gamma$ is isomorphic to a semidirect product $F \rtimes F(C)$ of two free groups.

**Proposition 6.1.** *If $\Gamma$ is a graph with the doubly breakable cycle property, then the right-angled Artin group $A\Gamma$ is poly-free of length 2.*

*Proof.* Fix a set $D$ of independent vertices in $\Gamma$ such that every cycle in $\Gamma$ meets $D$ at least twice. Let $L$ be the complementary set of vertices in $\Gamma$ and let $\Gamma_L$ be the full subgraph of $\Gamma$ induced by $L$. Each of the connected components of the graph $\Gamma_L$ is a tree. Select one vertex from each component of $\Gamma_L$, and denote the set of these vertices by $C$. Define $F(C) := A\Gamma_C$ to be the subgroup of $A\Gamma$ corresponding to the subgraph $\Gamma_C$ induced by $C$. Since $\Gamma_C$ is totally disconnected, the group $F(C)$ is also the free group on $C$.

For every vertex $y \in L$, there exists a unique element $c \in C$ such that $y$ and $c$ are in the same component of $\Gamma_L$, and since this component is a tree, there exists a unique vertex path $(y^{(n)}, \ldots, y^{(2)}, y^{(1)}, c)$ connecting $y = y^{(n)}$ and $c$ that lies inside the component of $c$ and is of minimal length. We call $c$ the *component representative of* $y$ and denote it by $r_y$. For each $d \in D$, let $N_d$ denote the set of vertices in $\Gamma$ adjacent to (i.e. neighbors of) $d$ and let $\mathcal{R}N_d$ be the set of component representatives of the vertices in $N_d$. For every element $c$ in $\mathcal{R}N_d$ there exists a unique vertex $y \in N_d$ that is contained in the connected component containing $c$; if $y \neq c$, denote this neighbor of $d$ by $x(d, c)$. For example, for the graph in Figure 1, $r_{y_{i,j}} = c_j$, for all $i$ and $j$. The element $x(d, c)$ is defined only in the following four cases: $x(d_1, c_2) = y_{4,2}$, $x(d_2, c_1) = y_{2,1}$, $x(d_2, c_2) = y_{3,2}$, and $x(d_4, c_2) = y_{3,2}$.

In the following, the normal form of an element $t \in F(C)$ refers to the freely reduced word over $C^{\pm 1}$ corresponding to $t$. Define $X := L - C$. For each $x \in X$, let $\hat{\mathbf{x}}$ be a copy of $x$ and let $\hat{\mathbf{X}} = \{\hat{\mathbf{x}} | x \in X\}$ be the set of such copies. For each $\hat{\mathbf{x}} \in \hat{\mathbf{X}}$, define a set of symbols

$$T_{\hat{\mathbf{x}}} \quad := \quad \{\ \hat{\mathbf{x}}_t \ \mid \ t \in F(C), \text{ the normal form of } t \text{ does not start with a letter in } \{r_x^{\pm 1}\}\ \}.$$

For each $d \in D$, define a set of symbols

$$T_d \quad := \quad \{\ d_t \ \mid \ t \in F(C), \text{ the normal form of } t \text{ does not start with a letter in } \mathcal{R}N_d^{\pm 1}\ \}.$$

For any $z \in \hat{\mathbf{X}} \cup D$, let $F(T_z)$ be the free group over $T_z$ and let $F$ be the free group

$$F := (*_{\hat{\mathbf{x}} \in \hat{\mathbf{X}}} F(T_{\hat{\mathbf{x}}})) * (*_{d \in D} F(T_d)).$$

Given any $c \in C$, define an endomorphism $\alpha_c$ of the free group $F$ by defining $\alpha_c$ on the generators of $F$ as

(8)

$$\alpha_c(\hat{\mathbf{x}}_t) := \begin{cases} \hat{\mathbf{x}}_{tc}, & t \neq 1 \text{ or } c \neq r_x \\ [(\hat{\mathbf{x}}_1^{(n-1)})^{-1}\hat{\mathbf{x}}_1^{(n)}] \cdots [(\hat{\mathbf{x}}_1^{(2)})^{-1}\hat{\mathbf{x}}_1^{(3)}][(\hat{\mathbf{x}}_1^{(1)})^{-1}\hat{\mathbf{x}}_1^{(2)}]\hat{\mathbf{x}}_1^{(1)}, & t = 1 \text{ and } c = r_x \end{cases}$$

for $\hat{\mathbf{x}} \in \hat{\mathbf{X}}$ and $t \in T_{\hat{\mathbf{x}}}$, where $(x^{(n)}, \ldots, x^{(2)}, x^{(1)}, c)$ is the minimal length path from $x$ to $c$ inside the component of $c$, and

$$\alpha_c(d_t) := \begin{cases} d_{tc}, & t \neq 1 \text{ or } c \notin \mathcal{R}N_d \\ d_1, & t = 1 \text{ and } c \in \mathcal{R}N_d \cap N_d \\ d_1^{\hat{\mathbf{x}}(d,c)_1}, & t = 1 \text{ and } c \in \mathcal{R}N_d - N_d \end{cases} \tag{9}$$

for $d \in D$ and $t \in T_d$.

Similarly, for $c \in C$, we also define an endomorphism $\alpha_{c^{-1}} : F \to F$ by

(10)

$$\alpha_{c^{-1}}(\hat{\mathbf{x}}_t) = \begin{cases} \hat{\mathbf{x}}_{tc^{-1}}, & t \neq 1 \text{ or } c \neq r_x \\ \hat{\mathbf{x}}_1^{(1)}[\hat{\mathbf{x}}_1^{(2)}(\hat{\mathbf{x}}_1^{(1)})^{-1}][\hat{\mathbf{x}}_1^{(3)}(\hat{\mathbf{x}}_1^{(2)})^{-1}] \cdots [\hat{\mathbf{x}}_1^{(n)}(\hat{\mathbf{x}}_1^{(n-1)})^{-1}], & t = 1 \text{ and } c = r_x \end{cases},$$

for $\hat{\mathbf{x}} \in \hat{\mathbf{X}}$ and $t \in T_{\hat{\mathbf{x}}}$, where $(x^{(n)}, \ldots, x^{(2)}, x^{(1)}, c)$ is the minimal length path from $x$ to $c$ inside the component of $c$, and

$$\alpha_{c^{-1}}(d_t) = \begin{cases} d_{tc^{-1}}, & t \neq 1 \text{ or } c \notin \mathcal{R}N_d \\ d_1, & t = 1 \text{ and } c \in \mathcal{R}N_d \cap N_d, \\ d_1^{[\alpha_{c^{-1}}(\hat{\mathbf{x}}(d,c)_1)]^{-1}}, & t = 1 \text{ and } c \in \mathcal{R}N_d - N_d \end{cases}, \tag{11}$$

for $d \in D$ and $t \in T_d$.

As in the proof of Theorem A, it is straightforward to check that the composition of the endomorphisms $\alpha_c$ and $\alpha_{c^{-1}}$ in either order is equal to the identity on the generating set of $F$. (The claim easily follows from $\alpha_c(x_1^{(i)}(x_1^{(i-1)})^{-1}) = (x_1^{(i-1)})^{-1}x_1^{(i)}$ and $\alpha_{c^{-1}}((x_1^{(i-1)})^{-1}x_1^{(i)}) = x_1^{(i)}(x_1^{(i-1)})^{-1}$, which hold for any adjacent pair of vertices $x^{(i)}$ and $x^{(i-1)}$ in the component of $c$ at distance $i$ and $i-1$ from $c$, respectively, as measured within the component of $c$.) Therefore $\alpha_c$ and $\alpha_{c^{-1}}$ are mutually inverse automorphisms of $F$.

Since the group $F(C)$ is free, the map $c \mapsto \alpha_c$ can be extended to a homomorphism $\alpha : F(C) \to \mathsf{Aut}(F)$. Therefore (8) and (9) define an action of $F(C)$ on $F$.

Let $G := F \rtimes F(C)$ be the associated semidirect product. A presentation for $G$ can be given by

$$G = \langle C \cup (\cup_{\hat{\mathbf{x}} \in \hat{\mathbf{X}}} T_{\hat{\mathbf{x}}}) \cup (\cup_{d \in D} T_d) \mid (\cup_{\hat{\mathbf{x}} \in \hat{\mathbf{X}}} R_{\hat{\mathbf{x}}}) \cup (\cup_{d \in D} R_d) \rangle,$$

where, for $d \in D$ and $\hat{\mathbf{x}} \in \hat{\mathbf{X}}$,

$$\begin{aligned} R_{\hat{\mathbf{x}}} :=& \{ \hat{\mathbf{x}}_t^c = \hat{\mathbf{x}}_{tc} \mid c \in C,\ \hat{\mathbf{x}}_t \in T_{\hat{\mathbf{x}}},\ t \neq 1 \text{ or } c \neq r_x \} \cup \\ & \{ \hat{\mathbf{x}}_1^c = [(\hat{\mathbf{x}}_1^{(n-1)})^{-1} \hat{\mathbf{x}}_1^{(n)}] \cdots [(\hat{\mathbf{x}}_1^{(2)})^{-1} \hat{\mathbf{x}}_1^{(3)}][(\hat{\mathbf{x}}_1^{(1)})^{-1} \hat{\mathbf{x}}_1^{(2)}] \hat{\mathbf{x}}_1^{(1)} \mid c = r_x \}, \end{aligned}$$

where $(x^{(n)}, \ldots, x^{(2)}, x^{(1)}, c)$ is the minimal length path from $x$ to $c$ inside the component of $c$ and

$$\begin{aligned} R_d :=& \{ d_t^c = d_{tc} \mid c \in C,\ d_t \in T_d,\ t \neq 1 \text{ or } c \notin \mathcal{R}N_d \} \cup \\ & \{ d_1^c = d_1 \mid c \in \mathcal{R}N_d \cap N_d \} \cup \\ & \{ d_1^c = d_1^{\hat{\mathbf{x}}(d,c)_1} \mid c \in \mathcal{R}N_d - N_d \} . \end{aligned}$$

Next apply Tietze transformations to simplify this presentation.

First note that if $\hat{\mathbf{x}}_t \in T_{\hat{\mathbf{x}}}$, then every prefix $u$ of the normal form of $t$ does not start with a letter in $\{r_x^{\pm 1}\}$, so $\hat{\mathbf{x}}_{\overline{u}} \in T_{\hat{\mathbf{x}}}$ as well, and similarly for $d_t \in T_d$. Using this fact and repeatedly applying the relations of the type $\hat{\mathbf{x}}_t^c = \hat{\mathbf{x}}_{tc}$ and $d_t^c = d_{tc}$ in $R_{\hat{\mathbf{x}}}$ and $R_d$, respectively, shows that in $G$ we have $\hat{\mathbf{x}}_t = \hat{\mathbf{x}}_1^t$ for all $t \in T_{\hat{\mathbf{x}}}$ and $d_t = d_1^t$ for all $t \in T_d$. If we denote $\hat{\mathbf{x}} = \hat{\mathbf{x}}_1$ and $d = d_1$ for each $\hat{\mathbf{x}} \in \hat{\mathbf{X}}$ and $d \in D$, then the presentation for $G$ is Tietze equivalent to

$$\langle C \cup \hat{\mathbf{X}} \cup D \mid (\cup_{\hat{\mathbf{x}} \in \hat{\mathbf{X}}} R'_{\hat{\mathbf{x}}}) \cup (\cup_{d \in D} R'_d) \rangle,$$

where

$$R'_{\hat{\mathbf{x}}} := \{ \hat{\mathbf{x}}^{r_x} = [(\hat{\mathbf{x}}^{(n-1)})^{-1} \hat{\mathbf{x}}^{(n)}] \cdots [(\hat{\mathbf{x}}^{(2)})^{-1} \hat{\mathbf{x}}^{(3)}][(\hat{\mathbf{x}}^{(1)})^{-1} \hat{\mathbf{x}}^{(2)}] \hat{\mathbf{x}}^{(1)} \}$$

and

$$R'_d := \{ d^c = d \mid c \in \mathcal{R}N_d \cap N_d \} \cup \{ d^c = d^{\hat{\mathbf{x}}(d,c)} \mid c \in \mathcal{R}N_d - N_d \}.$$

Second, for every $\hat{\mathbf{x}} \in \hat{\mathbf{X}}$ introduce a single new generator $x$ and a relation $\hat{\mathbf{x}} = x^{-1} r_x$ in the presentation for $G$. Use these new relations to eliminate the generators $\hat{\mathbf{x}} \in \hat{\mathbf{X}}$ from the above presentation and obtain a Tietze equivalent presentation

$$\langle C \cup X \cup D \mid (\cup_{x \in X} R''_x) \cup (\cup_{d \in D} R''_d) \rangle,$$

where

$$R''_x := \{ x^{-1} r_x = [x^{(n-1)} (x^{(n)})^{-1}] \cdots [x^{(2)} (x^{(3)})^{-1}][x^{(1)} (x^{(2)})^{-1}] r_x (x^{(1)})^{-1} \}$$

and

$$R''_d := \{\ d^c = d \mid c \in \mathcal{R}N_d \cap N_d\ \} \ \cup \{\ d = d^{x(d,c)} \mid c \in \mathcal{R}N_d - N_d\ \}.$$

The relations in $R''_d$ say that each $d \in D$ commutes with all $c \in C$ and $x \in X$ that are its neighbors in $\Gamma$, just as in the standard presentation of $A\Gamma$. For each $c \in C$ and each vertex $x$ adjacent to $c$ in $\Gamma_L$, we have $c = r_x$ and $x = x^{(1)}$. The corresponding relation in $R''_x$ is $x^{-1}r_x = r_x x^{-1}$, which implies that $x$ and $c$ commute. When the path length in $\Gamma_L$ from $x$ to $c$ is 2, with a vertex path $(x^{(2)}, x^{(1)}, c)$ from $x = x^{(2)}$ to $c = r_x$, the corresponding relation is $x^{-1}r_x = x^{(1)}x^{-1}r_x(x^{(1)})^{-1}$. Since $x^{(1)}$ and $r_x$ commute, this implies that $x$ and $x^{(1)}$ commute. Continuing in the same fashion we see that all such relations together imply that each generator $x$ whose distance to $r_x$ in $\Gamma_L$ is $n$ commutes with the generator that is the neighbor of $x$ on the unique length minimal path from $x$ to $c = r_x$ inside the component of $r_x$. Thus, the standard relations in $A\Gamma$ can be recreated from the relations in $R''_x$ and $R''_d$. Conversely, each relation $x^{-1}r_x = [x^{(n-1)}(x^{(n)})^{-1}] \cdots [x^{(2)}(x^{(3)})^{-1}][x^{(1)}(x^{(2)})^{-1}]r_x(x^{(1)})^{-1}$ in $R''_x$ is a corollary of the defining relations in $A\Gamma$. Thus the last presentation above is Tietze equivalent to the standard presentation of $A\Gamma$.

Therefore $A\Gamma \cong G = F \rtimes F(C)$ has poly-free length at most 2. The doubly breakable cycle property implies that $\Gamma$ is not totally disconnected, so the group $A\Gamma$ contains a $\mathbb{Z}^2$ subgroup and cannot be free. Thus the poly-free length of $A\Gamma$ is exactly 2. □

The free group automorphisms $\alpha_a$ constructed in the proof of Theorem A permute the basis elements of the free group. Although the free group automorphism $\alpha_c$ of $F$ in the proof above does not have the same property, the automorphism of $F^{ab} = F/[F, F]$ induced by $\alpha_c$ permutes the basis elements of this free abelian group.

We conclude with a fully worked example illustrating a length 2 poly-free structure of a right-angled Artin group defined by a graph with the doubly breakable cycle property, following the proof of Theorem B. Let $\Gamma$ be the

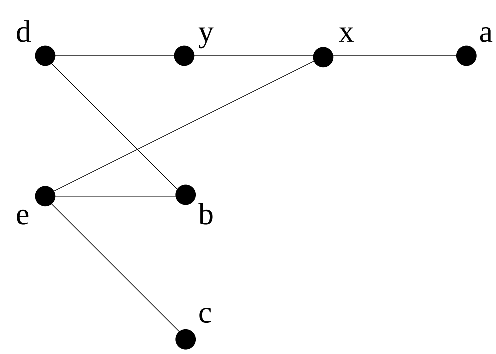


FIGURE 3. Graph with the doubly breakable cycle property

graph in Figure 3. Set $D := \{d, e\}$. The living subgraph has 3 components and the chosen representatives are the elements of $C = \{a, b, c\}$. The only vertices in $X = L - C$ are $x$ and $y$. Denote $F_q := F(a, b, c)$. We have

$$T_d = \{d_1\} \cup \{\ d_t \mid t \in F_q,\ \text{the normal form of } t \text{ starts with } c^{\pm 1}\ \},$$
$$T_e = \{e_1\},$$

and, for $\hat{\mathbf{z}} \in \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$,

$$T_{\hat{\mathbf{z}}} = \{\hat{\mathbf{z}}_1\} \cup \{\ \hat{\mathbf{z}}_t \mid t \in F_q,\ \text{the normal form of } t \text{ starts with } b^{\pm 1} \text{ or } c^{\pm 1}\ \}.$$

Denote $F_k := F(T_d \cup T_e \cup T_{\hat{\mathbf{x}}} \cup T_{\hat{\mathbf{y}}})$. Then $A\Gamma = F_k \rtimes F_q$, where the action of $F_q$ on $F_k$ is given by Table 1.

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $d_1$ | $d_1^{\hat{\mathbf{y}}_1}$ | $d_1$ | $d_c$ |
| $d_t$ | $d_{ta}$ | $d_{tb}$ | $d_{tc}$ |
| $e_1$ | $e_1^{\hat{\mathbf{x}}_1}$ | $e_1$ | $e_1$ |
| $\hat{\mathbf{x}}_1$ | $\hat{\mathbf{x}}_1$ | $\hat{\mathbf{x}}_b$ | $\hat{\mathbf{x}}_c$ |
| $\hat{\mathbf{x}}_t$ | $\hat{\mathbf{x}}_{ta}$ | $\hat{\mathbf{x}}_{tb}$ | $\hat{\mathbf{x}}_{tc}$ |
| $\hat{\mathbf{y}}_1$ | $(\hat{\mathbf{x}}_1)^{-1}\hat{\mathbf{y}}_1\hat{\mathbf{x}}_1$ | $\hat{\mathbf{y}}_b$ | $\hat{\mathbf{y}}_c$ |
| $\hat{\mathbf{y}}_t$ | $\hat{\mathbf{y}}_{ta}$ | $\hat{\mathbf{y}}_{tb}$ | $\hat{\mathbf{y}}_{tc}$ |

Table 1. Action of $F_q$ on $F_k$

In Table 1, the entry in the row labeled on the left by the letter $\sigma$ and column labeled above by $\tau$ is the conjugate $\sigma^\tau$. In the leftmost column of the table, the $d_t$, $\hat{\mathbf{x}}_t$, and $\hat{\mathbf{y}}_t$ entries range over all symbols in $T_d \setminus \{d_1\}$, $T_{\hat{\mathbf{x}}} \setminus \{\hat{\mathbf{x}}_1\}$, and $T_{\hat{\mathbf{y}}} \setminus \{\hat{\mathbf{y}}_1\}$, respectively.

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